Homological stability for oriented configuration spaces

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Objects of study

Fix a manifold M, space X.

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Abbreviate to $C_n(M)$ and $C_n^+(M)$ since we will keep X fixed throughout.

Stabilisation maps

$C_n(M) \xrightarrow{s} C_{n+1}(M)$

- assume $M =$ interior of manifold-with-boundary $\bar{M}$
- choose boundary-component of $\bar{M}$: $\partial_0 \bar{M}$
- s : push configuration away from $\partial_0 \bar{M}$ slightly, add a new point b_0 in the newly-vacated neighbourhood of $\partial_0 \bar{M}$ to the configuration.
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Oriented case: $C_n^+(M) \longrightarrow C_{n+1}^+(M)$

2 possible conventions for induced orientations; say

$$+s\colon \left[\begin{smallmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{smallmatrix}\right] \quad \mapsto \quad \begin{bmatrix} p_1' & \cdots & p_n' & b_0 \\ x_1 & \cdots & x_n & x_0 \end{bmatrix}$$

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- In our case k will always be linear, and we can talk about the *slope* of the stability range.

Brief history

	manifold M	space X	slope
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Church 11	$[**]$	pt	1	*with $\mathbb{Q}$-coefficients*

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$[**]$ connected, orientable, (finite-type)

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Answering the same question for $H_* C_n^+(M)$:

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NB: These proofs both involve calculations very specific to their respective situations, so do not generalise naturally.

Main theorem

In this talk we will outline the proof of homological stability for $C_n^+(M, X)$, with:

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$$M = \mathbb{R}^\infty \text{ and } X = pt \quad \rightsquigarrow \quad A_n$$
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Taking $X = BG$, we get H.S. for $A\beta_n^S \wr G$.

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We will start by outlining the proof of the *unordered* version of this theorem:

Theorem (Randal-Williams)

$C_n(M) \xrightarrow{s} C_{n+1}(M)$ is

- *an isomorphism on H_* for $* \leq n/2 - 1$*
- *a surjection on H_* for $* \leq n/2$.*

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- *an isomorphism on* H_* *for* $* \leq n/2 - 1$
- *a surjection on* H_* *for* $* \leq n/2$.

This can be rephrased as

$$H_* R_n(M) = 0 \quad \text{for } * \leq n/2$$

where $R_n(M)$ is the mapping cone (homotopy cofibre) of $C_n(M) \xrightarrow{s} C_{n+1}(M)$.

Outline of proof: 1

Strategy — construct a map with target $R_n(M)$, and prove it is

1. surjective on H_* for $* \leq {}^n/_2$
2. zero on H_* for $* \leq {}^n/_2$.

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Factorise s into

$$C_{n-1}(M) \xrightarrow{p} C_{n-1}(M \setminus pt) \xrightarrow{f} C_n(M)$$

p — push configuration away from $\partial_0 \bar{M}$, remove b_0 from M.

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This is a factorisation of $\tilde{s}_{(12)}$.

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$\tilde{f}$: Involved spectral sequence argument, which I won't go into now...

Deferred to the end, if time.

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So we have

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each induce an exact ladder on homology:

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So we have

- $\tilde{s}_{12}$ surjective on H_* for $* \leq n/2$
- $\tilde{s}_1$ zero on H_* $(\forall *)$.

'Ladder trick':

The maps of cofibrations

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Main theorem – the oriented version

Now we will turn to the *oriented* version of the theorem:

Theorem (P.)

$C_n^+(M) \xrightarrow{s} C_{n+1}^+(M)$ *is*

- *an isomorphism on* H_* *for* $* \leq {}^{(n-2)}/_3 - 1$
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This can be rephrased as

$$H_* R_n^+(M) = 0 \quad \text{for } * \leq {}^{(n-2)}\!/_3$$

where $R_n^+(M)$ is the mapping cone (homotopy cofibre) of $C_n^+(M) \xrightarrow{s} C_{n+1}^+(M)$.

Outline of proof: 4: the oriented version

Note: We defined $s\colon \left[\begin{smallmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{smallmatrix}\right] \mapsto \left[\begin{matrix} p_1' & \cdots & p_n' & b_0 \\ x_1 & \cdots & x_n & x_0 \end{matrix}\right]$.

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So in order for f and s to commute, we need to define

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Upshot: the composition

$$C_{n-1}^{+}(M) \xrightarrow{p} C_{n-1}^{+}(M \setminus pt) \xrightarrow{f} C_n^{+}(M)$$

is a factorisation of $\boxed{(-1)^{n-1}s}$.

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2. The 'ladder trick' depends on knowing (in advance) that s induces injections on H_* ($\forall *$). In the oriented case this is false.

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- The *other* two homotopies, (123) and 1, split, so we have...

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- $\tilde{s}_{(132)}$ is surjective on H_* for $* \leq {}^{(n-2)}\!/_{3}$,
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Thanks for your attention!

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Appendix

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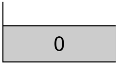

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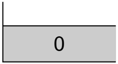

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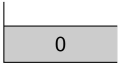

- On H_*, $\tilde{a}$ is (for a different spectral sequence) the d^1-differentials in the leftmost column.This spectral sequence $\Rightarrow 0$ in the range of interest, and by the ind. hyp., the E^1-page looks like