

Homological stability for oriented configuration spaces

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Fix a manifold M , space X .

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Abbreviate to $C_n(M)$ and $C_n^+(M)$ since we will keep X fixed throughout.

$$C_n(M) \xrightarrow{s} C_{n+1}(M)$$

- assume $M =$ interior of manifold-with-boundary \bar{M}
- choose boundary-component of \bar{M} : $\partial_0 \bar{M}$
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Stabilisation maps

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$$\text{Oriented case: } C_n^+(M) \longrightarrow C_{n+1}^+(M)$$

2 possible conventions for induced orientations; say

$$\begin{aligned} +s: \begin{bmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{bmatrix} &\mapsto \begin{bmatrix} p'_1 & \cdots & p'_n & b_0 \\ x_1 & \cdots & x_n & x_0 \end{bmatrix} \\ -s: \begin{bmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{bmatrix} &\mapsto \begin{bmatrix} p'_1 & \cdots & p'_{n-1} & b_0 & p'_n \\ x_1 & \cdots & x_{n-1} & x_0 & x_n \end{bmatrix} \end{aligned}$$

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- Rewrite this in terms of n : we want a function $k(n)$ so that

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- In our case k will always be linear, and we can talk about the *slope* of the stability range.

Brief history

	manifold M	space X	slope
Randal-Williams 10/11	[*]	path-conn.	$1/2$

[*] connected, dimension ≥ 2 , interior of a manifold-with-boundary

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- [**] connected, orientable, (finite-type)

Brief history

Answering the same question for $H_*C_n^+(M)$:

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NB: These proofs both involve calculations very specific to their respective situations, so do not generalise naturally.

Main theorem

In this talk we will outline the proof of **homological stability** for $C_n^+(M, X)$, with:

- M the interior of a manifold-with-boundary,
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$$M = \mathbb{R}^\infty \text{ and } X = pt \rightsquigarrow A_n$$

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More generally, taking $M = S =$ interior of a compact connected surface with boundary and $X = pt$ gives H.S. for $A\beta_n^S$.

Taking $X = BG$, we get H.S. for $A\beta_n^S \wr G$.

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We will start by outlining the proof of the *unordered* version of this theorem:

Theorem (Randal-Williams)

$C_n(M) \xrightarrow{s} C_{n+1}(M)$ is

- an isomorphism on H_* for $* \leq n/2 - 1$
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This can be rephrased as

$$H_* R_n(M) = 0 \quad \text{for } * \leq n/2$$

where $R_n(M)$ is the *mapping cone* (homotopy cofibre) of $C_n(M) \xrightarrow{s} C_{n+1}(M)$.

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Strategy — construct a map with target $R_n(M)$, and prove it is

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Factorise s into

$$C_{n-1}(M) \xrightarrow{p} C_{n-1}(M \setminus pt) \xrightarrow{f} C_n(M)$$

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Using excision, can identify its **homotopy cofibre** (up to H_* -isom.) with that of the $\dim(M)$ -fold suspension of

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\tilde{f} : Involved spectral sequence argument, which I won't go into now...

Deferred to the end, if time.

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So we have

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'Ladder trick':

The maps of cofibrations

$$\begin{array}{ccccc} C_{n-1}(M) & \xrightarrow{s} & C_n(M) & \rightarrow & R_{n-1}(M) \\ s \downarrow & 1 & \downarrow s & & \downarrow \tilde{s}_1 \\ C_n(M) & \xrightarrow{s} & C_{n+1}(M) & \rightarrow & R_n(M) \end{array} \quad \text{and} \quad \begin{array}{ccccc} C_{n-1}(M) & \xrightarrow{s} & C_n(M) & \rightarrow & R_{n-1}(M) \\ s \downarrow & (12) & \downarrow s & & \downarrow \tilde{s}_{(12)} \\ C_n(M) & \xrightarrow{s} & C_{n+1}(M) & \rightarrow & R_n(M) \end{array}$$

each induce an **exact ladder** on homology:

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Main theorem – the oriented version

Now we will turn to the *oriented* version of the theorem:

Theorem (P.)

$C_n^+(M) \xrightarrow{s} C_{n+1}^+(M)$ is

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where $R_n^+(M)$ is the **mapping cone** (homotopy cofibre) of $C_n^+(M) \xrightarrow{s} C_{n+1}^+(M)$.

Outline of proof: 4: the oriented version

Note: We defined $s: \begin{bmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{bmatrix} \mapsto \begin{bmatrix} p'_1 & \cdots & p'_n & b_0 \\ x_1 & \cdots & x_n & x_0 \end{bmatrix}$.

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So in order for f and s to commute, we need to define

$$f: \begin{bmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{bmatrix} \mapsto \begin{bmatrix} * & p_1 & \cdots & p_n \\ x_0 & x_1 & \cdots & x_n \end{bmatrix}$$

(* = the missing point in $M \setminus pt$).

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Note: We defined $s: \begin{bmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{bmatrix} \mapsto \begin{bmatrix} p'_1 & \cdots & p'_n & b_0 \\ x_1 & \cdots & x_n & x_0 \end{bmatrix}$.

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$$f: \begin{bmatrix} p_1 & \cdots & p_n \\ x_1 & \cdots & x_n \end{bmatrix} \mapsto \begin{bmatrix} * & p_1 & \cdots & p_n \\ x_0 & x_1 & \cdots & x_n \end{bmatrix}$$

(* = the missing point in $M \setminus pt$).

Upshot: the composition

$$C_{n-1}^+(M) \xrightarrow{P} C_{n-1}^+(M \setminus pt) \xrightarrow{f} C_n^+(M)$$

is a factorisation of $\boxed{(-1)^{n-1}s}$.

Outline of proof: 4: the oriented version

Now f and s commute exactly, and p and s commute up to the (12)-homotopy as before, so we get induced maps \tilde{f} , \tilde{p} on mapping cones, which factorise $\tilde{s}_{(12)}$:

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s is indicated on the vertical arrows from $C_{n-1}^+(M)$ to $C_n^+(M)$ and from $C_n^+(M)$ to $C_{n+1}^+(M)$.

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s is indicated on the vertical arrows from $C_{n-1}^+(M)$ to $C_n^+(M)$ on both sides.

- $\tilde{\mathfrak{S}}_{(12)}$ is surjective on H_* for $* \leq (n-2)/3$.

Problems:

- 1 The homotopy (12) does *not* split, so $\tilde{\mathfrak{S}}_{(12)}$ is not zero on H_* .

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Problems:

- 1 The homotopy (12) does *not* split, so $\tilde{\mathfrak{S}}_{(12)}$ is not zero on H_* .
- 2 The 'ladder trick' depends on knowing (in advance) that s induces injections on H_* ($\forall *$). In the oriented case this is **false**.

Solution to 1

Solution to (1) – Extend ‘further back’ to $R_{n-2}^+(M)$:

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- Show that each \tilde{f}, \tilde{p} is surjective on H_* for $* \leq (n-2)/3$. *This is where the $1/3$ stability slope becomes necessary.*
- The *other* two homotopies, (123) and 1 , split, so we have...

Solution to 2

- $\tilde{S}_{(132)}$ is **surjective** on H_* for $* \leq (n-2)/3$,
- $\tilde{S}_{(123)}$ and \tilde{S}_1 are **zero** on H_* ($\forall *$).

Solution to 2

- $\tilde{\mathfrak{S}}_{(132)}$ is **surjective** on H_* for $* \leq (n-2)/3$,
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This time we can't use the 'ladder trick' to combine these facts to finish the inductive step; instead:

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*The **existence** of a split homotopy (e.g. $\tilde{\mathfrak{s}}_{(123)}$ or $\tilde{\mathfrak{s}}_1$) filling the square implies that **any** homotopy (e.g. $\tilde{\mathfrak{s}}_{(132)}$) filling the square factorises as below.*

$$R_{n-2}^+(M) \longrightarrow \Sigma C_{n-2}^+(M) \dashrightarrow C_{n+1}^+(M) \longrightarrow R_n^+(M)$$

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 & & \Sigma(-s) \uparrow & & \circlearrowleft & & \uparrow s \\
 R_{n-3}^+(M) & \longrightarrow & \Sigma C_{n-3}^+(M) & \dashrightarrow & C_n^+(M) & \longrightarrow & R_{n-1}^+(M)
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Thanks for your attention!

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Sketch of the spectral sequence argument deferred earlier.

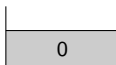
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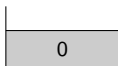
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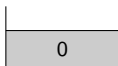
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- On H_* , \tilde{a} is (for a different spectral sequence) the d^1 -differentials in the leftmost column. This spectral sequence $\Rightarrow 0$ in the range of interest, and by the ind. hyp., the E^1 -page looks like

