Homological stability
for configuration spaces
of disconnected submanifolds

Martin Palmer // *Lausanne, 8th July 2013*

Slides also on webpage: zatibq.com

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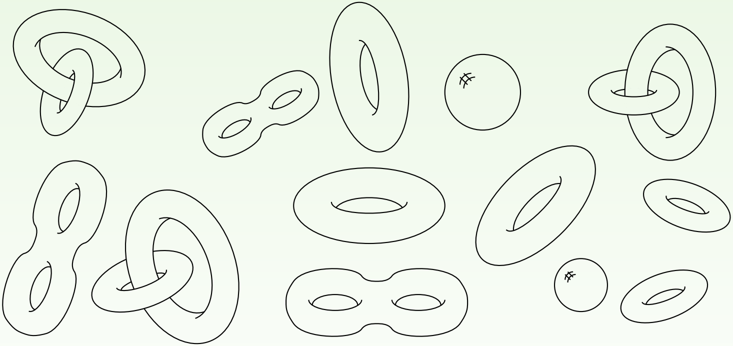

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- There are many examples of this phenomenon from different areas:
  - classical groups
  - mapping class groups
  - automorphism groups of free groups
  - configuration spaces
  - ...

  More detailed examples on the next slide...

A selection of homological stability results (from many more)

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X_n*		$f(n)$		Y	
Σ_n	symmetric groups	$n/2$	[Nakaoka]	$\Omega_0^\infty S^\infty$	[BPQ]
B_n	braid groups	$n/2$	[Arnol'd]	$\Omega_0^2 S^2 \simeq \Omega^2 S^3$	[Segal]
$C_n(\mathbb{R}^d)$	config. spaces on $\mathbb{R}^d$	$n/2$	[Segal]	$\Omega_0^d S^d$	[Segal]
$C_n(M)$	(M conn. and open)	$n/2$	[McDuff,Segal]	$\Gamma_0(\dot{T}M)$	[McDuff]
$GL_n(R)$	(R Dedekind domain)	$(n-1)/4$	[Charney]	$K(GL(R),1)^+$	$\rightsquigarrow$ K-theory...
$Sp_{2n}(R)$	(R Dedekind domain)	$(n-6)/2$	[Charney]	$K(Sp(R),1)$	
$O_n(\mathbb{F})$	$(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$	$n-1$	[Sah]	$K(O(\mathbb{F}),1)$	
Many other families of classical groups...					
$\mathrm{Aut}(F_n)$	(F_n = free group)	$(n-2)/2$	[Hatcher-Vogtmann]	$\Omega_0^\infty S^\infty$	[Galatius]
$\mathrm{Out}(F_n)$		$(n-4)/2$	[Hatcher-Vogtmann]		
MCGs of oriented surfaces		$2g/3$	[Harer,Ivanov,Boldsen]	$\Omega_0^\infty MTSO(2)$	[Madsen-Weiss]
MCGs of nonorientable surfaces		$(g-3)/3$	[Wahl,Randal-Williams]	$\Omega_0^\infty MTO(2)$	[Wahl]
MCGs of 3-dim handlebodies		$(g-2)/2$	[Hatcher-Wahl]†	$\Omega_0^\infty \Sigma^\infty BSO(3)_+$	[Hatcher]
$B\mathrm{Diff}_\partial$ of $\sharp_g(S^1 \times S^2) \smallsetminus \mathring{D}^3$				$\Omega_0^\infty \Sigma^\infty BSO(4)_+$	[Hatcher]
$B\mathrm{Diff}_\partial$ of $\sharp_g(S^n \times S^n) \smallsetminus \mathring{D}^{2n}$ ‡		$(g-4)/2$	[Galatius-Randal-Williams]	$\Omega_0^\infty MTO(2n)^{\langle n \rangle}$	[GRW]

* $X_n := K(G_n, 1)$ if the entry is a group G_n.

Notation: we use $K(G,1)$ when G is a discrete group, and BG when it is a (non-discrete) topological group.

† More generally: MCGs of (compact connected) oriented 3-manifolds.

‡ For $n \geqslant 3$.

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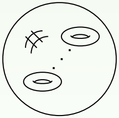
$= \mathsf{MCG}\Big($  $\Big)/$twists

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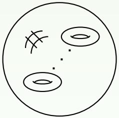

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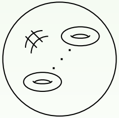

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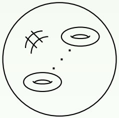

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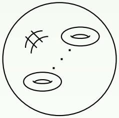

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So $C_{n.S^1}(\mathbb{R}^3)$ cannot be aspherical.

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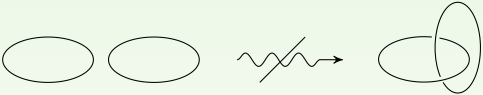

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  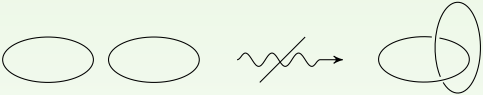

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  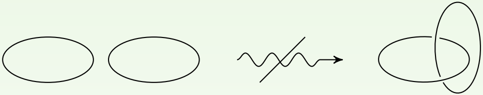

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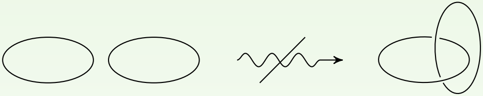

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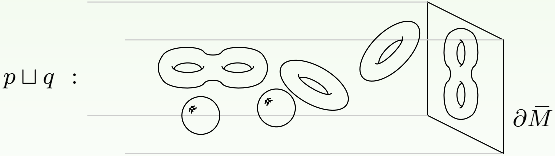

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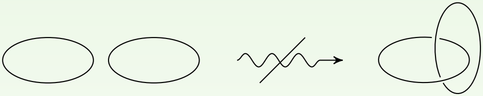

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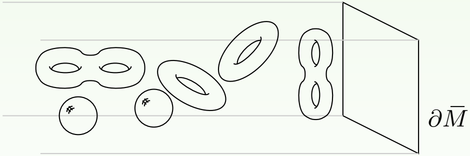

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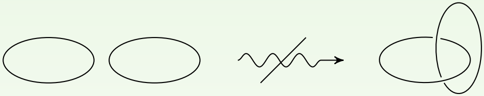

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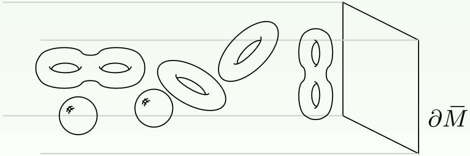

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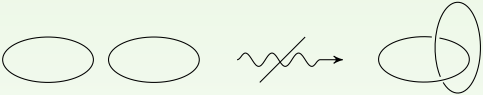

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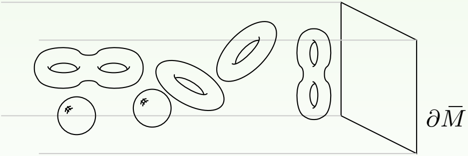

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  Iterating this we get a sequence $C_q(M) \to C_{q+p}(M) \to C_{q+2.p}(M) \to \cdots$

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  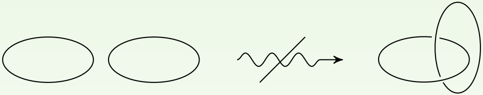

  in $\mathbb{R}^3$

  Define $C_q(M) \coloneqq$ the path-component of $C_Q(M)$ containing $[q]$.
- Define $q + p\colon Q \sqcup P \hookrightarrow M$ to be $e \circ (q \sqcup p)$:

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  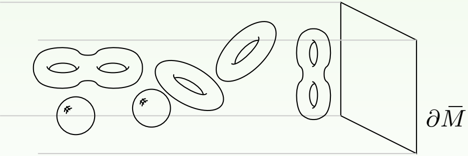

  where $e\colon \bar{M} \hookrightarrow \bar{M}$ is a self-embedding such that $e(\partial\bar{M}) \subset M$.
- This induces a map $C_q(M) \to C_{q+p}(M)$, defined by $q' \mapsto q' + p$.

  Iterating this we get a sequence $C_q(M) \to C_{q+p}(M) \to C_{q+2.p}(M) \to \cdots$
- In particular this defines $C_{n.p}(M)$ $\qquad$ for $n \geqslant 0$.

Configuration spaces of disconnected submanifolds

- Choose:

  M connected, non-compact, $=$ interior$(\bar{M})$ P and Q closed (same dim.)

  An embedding $q\colon Q \hookrightarrow M$ and an embedding “at infinity” $p\colon P \hookrightarrow \partial\bar{M}$
- Define $C_Q(M) \coloneqq \mathrm{Emb}(Q, M)/\mathrm{Diff}(Q)$.

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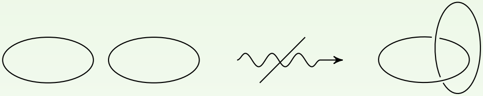

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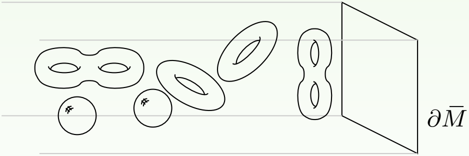

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So homological stability is certainly true when $M = \mathbb{R}^\infty$ (at least when $Q = \varnothing$).

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- Improvement (in progress): Remove the codimension condition altogether… This would then include the case $C_{n.S^1}(\mathbb{R}^3)$.

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- Then $H_*(\mathcal{E}_g(M);\mathbb{Z}) \cong H_*(\mathcal{E}_{g+1}(M);\mathbb{Z})$ in the stable range $* \leqslant (2g-2)/3$.

Thanks for listening ☺