Homology of *symmetric diffeomorphism groups* of manifolds and *diffeomorphism groups of manifolds with singularities*

Martin Palmer — Université Paris XIII

Topology of Manifolds, Lisbon — 29 June 2016

Homological stability for families of spaces

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- Configuration spaces, moduli spaces of manifolds, ...

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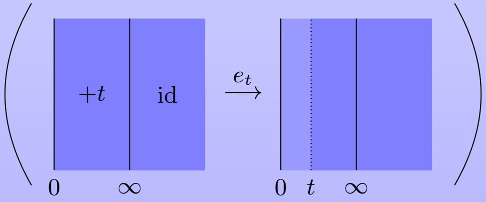

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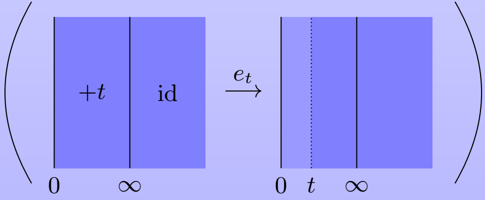

$\rightarrowtail$ one-parameter family of embeddings $[0, \infty) \longrightarrow \mathrm{Emb}(M, M)$

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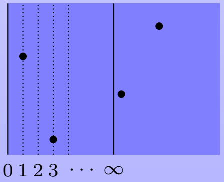

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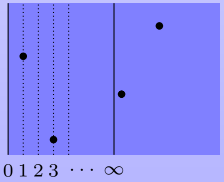

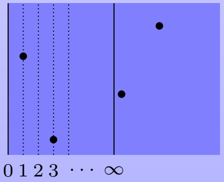

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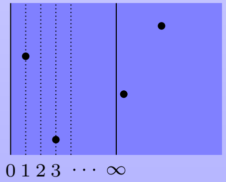

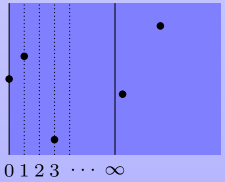

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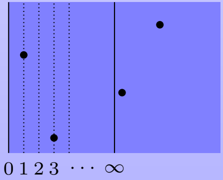

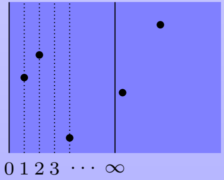

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$$\begin{tikzcd}
\{\text{knots in } D^3\} \arrow[r] \arrow[dr] & \{\text{2-component links in } D^3\} \\
& \{\text{linking number} = 0\} \arrow[u, hook]
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[Cantero-Randal-Williams]

$$\mathrm{conf}_{\Sigma_g}(M) = \mathrm{Emb}(\Sigma_g, M)/\mathrm{Diff}^+(\Sigma_g),$$

when $\pi_1(M) = 0$ and $\dim(M) \geqslant 5$.

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Theorem (P)

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is homologically stable.

Decorated moduli spaces

- May also replace $\mathrm{Diff}(P)$ by $G \leqslant \mathrm{Diff}(P)$

  correspondingly $\mathrm{Diff}(nP) = \mathrm{Diff}(P) \wr \mathfrak{S}_n \rightsquigarrow G \wr \mathfrak{S}_n$

  $\rightarrowtail \mathrm{conf}_{nP}(M; X; G)$
- e.g. $G = \mathrm{Diff}^+(P)$ if P is orientable

  $\rightarrowtail$ moduli space of n oriented copies of P in M
- e.g. $G = \{e\}$

  $\rightarrowtail$ moduli space of n parametrised copies of P in M

Theorem (P)

If $\dim(P) \leqslant \frac{1}{2}(\dim(M) - 3)$ *and* $G \leqslant \mathrm{Diff}(P)$ *is open or trivial then*

$$\mathrm{conf}_P(M; X; G) \longrightarrow \mathrm{conf}_{2P}(M; X; G) \longrightarrow \mathrm{conf}_{3P}(M; X; G) \longrightarrow \cdots$$

is homologically stable.

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Connected sum

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$$\rightarrowtail \qquad X \sharp Y = \left(X \setminus \exp(T_x^{\leqslant \varepsilon} X)\right) \cup_\phi \left(Y \setminus \exp(T_y^{\leqslant \varepsilon} Y)\right)$$

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If $\dim(P) \leqslant \frac{1}{2}(\dim(M) - 3)$ *then this is homologically stable.*

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- Also true for *H-symmetric diffeomorphism groups* (under certain conditions on H)

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- Generalises a theorem of Tillmann

$\longleftrightarrow$ $\qquad P =$ point and the ‘usual’ $\sharp$

Manifolds with Baas-Sullivan singularities

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Manifolds with Baas-Sullivan singularities

Definition (*Q-manifold*)

- space M
- $A \subset M$ *singularity set*

  such that A and $M \smallsetminus A$ are smooth manifolds ($\partial A = \varnothing$)
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If $\dim(P) \leqslant \frac{1}{2}(\dim(M) - 3)$ *then this is homologically stable.*

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Then: $$\Sigma\mathrm{Diff}\Big(M \underset{nP}{\sharp} nT\Big) = \mathrm{Diff}^{\partial T}(\mathbf{N}_n)$$

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$$\cdots \longrightarrow \mathrm{Diff}^{\partial T}(\mathbf{N}_n) \longrightarrow \mathrm{Diff}^{\partial T}(\mathbf{N}_{n+1}) \longrightarrow \cdots$$

- [Perlmutter]: homotopy type of $\mathbf{Cob}_{d+1}^{Q}$

Directions

- Improve the codimension condition $\quad \dim(P) \leqslant \frac{1}{2}(\dim(M) - 3)$

  $\rightsquigarrow \quad S^1$'s in 3-manifolds
- Treat more complicated path-components

  $\rightsquigarrow$ e.g. non-trivial links in $\mathbb{R}^3$

  $\rightsquigarrow$ coro: stability for $\Sigma\mathrm{Diff}$(3-manifolds) w.r.t. iterated Dehn surgery
- More general singularities

  $\rightsquigarrow$ e.g. $A = \Sigma_g$ instead of $A = \{1, \dots, n\}$
- Q: What is the limiting homology?

$$\cdots \longrightarrow \mathrm{Diff}^{\partial T}(\mathbf{N}_n) \longrightarrow \mathrm{Diff}^{\partial T}(\mathbf{N}_{n+1}) \longrightarrow \cdots$$

- [Perlmutter]: $\quad$ homotopy type of $\mathbf{Cob}^{Q}_{d+1}$

$$\mathrm{hocolim}_{n\to\infty}\big(B\mathrm{Diff}^{\partial T}(\mathbf{N}_n)\big) \overset{?}{\longleftrightarrow} \mathbf{Cob}^{\partial T}_{\dim(M)}$$

Thank you for your attention