# Twisted homological stability and homological representations of braid groups

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## Configuration spaces

Let $M$ be a connected, smooth, open manifold and $n \geqslant 1$ an integer.
**Definition.** The $n^{\text{th}}$ *unordered configuration space* on $M$ is

$$C_n(M) = \{(x_1, \ldots, x_n) \in M^n \mid i \neq j \Rightarrow x_i \neq x_j\}/\mathfrak{S}_n$$

**Facts.** The space $C_n(\mathbb{R}^2)$ is aspherical and $\pi_1(C_n(\mathbb{R}^2)) \cong B_n$, the braid group on $n$ strands. In other words $C_n(\mathbb{R}^2) \simeq K(B_n, 1)$.
**Variants.** Instead of quotienting by the symmetric group $\mathfrak{S}_n$ above, we may instead quotient by a subgroup, for example:

| | | |
|---|---|---|
| $\{1\}$ | *ordered configuration space* | $\widetilde{C}_n(M)$ |
| $A_n$ | *oriented configuration space* | $C_n^+(M)$ |
| $(n = km) \quad \mathfrak{S}_k \wr \mathfrak{S}_m$ | *partitioned configuration space* | $C_{m\mid k}(M)$ |

## Moduli spaces of submanifolds

**Generalisation.** Let $M = \mathrm{interior}(\overline{M})$ and $P \subseteq \partial\overline{M}$ closed submanifold.
**Examples.** (a) $P = \text{point}$   (b) $P = \text{finite set of size } k$   (c) $P = S^1 \subseteq \partial D^3$
**Construction.** Fix $n \geqslant 1$.
$\mathrm{Emb}(nP, M)$: space of embeddings of $n$ disjoint copies of $P$ into $M$.
$\mathrm{Diff}(nP)$: topological group of diffeomorphisms of $n$ disjoint copies of $P$.
$c_0 \in \mathrm{Emb}(nP, M)$: $n$ parallel copies of $P \subseteq \partial\overline{M}$ in a collar neighbourhood.
**Definition.** The $n^{\text{th}}$ *moduli space of submanifolds of $M$ modelled on $P$* is

$$C_{nP}(M) = \text{the path-component of } \mathrm{Emb}(nP, M)/\mathrm{Diff}(nP) \text{ containing } [c_0]$$

**Examples.** The examples above define (a) unordered and (b) partitioned configuration spaces. Example (c) is the moduli space of unlinks in $\mathbb{R}^3$.
**Note.** This may be extended to allow (i) embedded copies of $P$ parametrised modulo a fixed subgroup $G \leqslant \mathrm{Diff}(P)$ and (ii) labels in a bundle $E \to \mathrm{Emb}(P, \overline{M})$.

## Untwisted homological stability

Let's abbreviate $C_n = C_{nP}(M)$. By default, $H_i$ means *integral* homology.
**Observation.** There are natural *stabilisation maps* $C_n \longrightarrow C_{n+1}$.
(Idea: push a new copy of $P$ inwards from the boundary along a collar neighbourhood.)
**Theorem A.** For any fixed $i \geqslant 0$ the sequence

$$\cdots \longrightarrow H_i(C_{n-1}) \longrightarrow H_i(C_n) \longrightarrow H_i(C_{n+1}) \longrightarrow \cdots \qquad (1)$$

consists of isomorphisms (*stabilises*) for $n \geqslant 2i$, as long as
- $P$ is a point                                                               [S₁,M,S₂]
- the dimension of $P$ is at most $\frac{1}{2}(\dim(M) - 3)$                    [P₁]
- $P$ is a finite set and $M$ is a surface                                      [PT]
- $P = S^1$ and $\dim(M) = 3$    (the moduli space of unlinks)                  [Ku]

**Question.** Is there a space $X$ (depending on $M$ and $P$) whose homology is the limiting homology (also called *stable homology*) of the sequence (1)?
**Partial answer.** If $P$ is a single point then we may take $X$ to be any path-component of $\Gamma_c(\dot{T}M)$. E.g. if $M = \mathbb{R}^m$ this space is $\Omega^m S^m$.
**Open question.** What is $X$ for other manifolds $P$?

## Related results

Some related spaces (or groups) whose homology is also stable as $n \to \infty$.
- The groups $\pi_1(C_{nS^1}(\mathbb{R}^3))$   (*string motion groups*)          [HW]
  Note that $C_{nS^1}(\mathbb{R}^3)$ is *not* aspherical, in contrast to $C_n(\mathbb{R}^2)$.
- Moduli spaces $\mathrm{Emb}(S_n, M)/\mathrm{Diff}^+(S_n)$ of embedded surfaces   [CRW]
  of genus $n$, assuming $\pi_1(M) = 0$ and $\dim(M) \geqslant 5$.
- The sequences $C_{2n}(M)$ and $C_{2n+1}(M)$,                                    [CP]
  when $M$ is *closed* and odd-dimensional.
  Note: $C_n(M)$ is usually unstable if $M$ is closed, e.g. $H_1(C_n(S^2)) \cong \mathbb{Z}/(2n-2)$.

## References

[CP]   F. Cantero, M. Palmer, *On homological stability for configuration spaces on closed background manifolds*, 2015.
[CRW]  F. Cantero, O. Randal-Williams, *Homological stability for spaces of embedded surfaces*, 2017.
[C]    T. Church, *Homological stability for configuration spaces of manifolds*, 2012.
[HW]   A. Hatcher, N. Wahl, *Stabilization for mapping class groups of 3-manifolds*, 2010.
[Kr]   M. Krannich, *Homological stability of topological moduli spaces*, preprint, 2017.
[Ku]   A. Kupers, *Homological stability for unlinked circles in a 3-manifold*, preprint, 2017.
[M]    D. McDuff, *Configuration spaces of positive and negative particles*, 1975.
[P₁]   M. Palmer, *Configuration spaces and homological stability*, PhD thesis, 2013.
[P₂]   M. Palmer, *Twisted homological stability for configuration spaces*, 2018.
[PT]   M. Palmer, T. Tran, *Homological stability for partitioned surface braid groups*, to appear, 2018.
[RWW]  O. Randal-Williams, N. Wahl, *Homological stability for automorphism groups*, 2017.
[S₁]   G. Segal, *Configuration spaces and iterated loop spaces*, 1973.
[S₂]   G. Segal, *The topology of spaces of rational functions*, 1979.

## Twisted coefficient systems

**Setting.** Fix $\overline{M}$ and $P$ as before and let $P_1, P_2, P_3, \ldots$ be pairwise disjoint, parallel copies of $P$ in a collar neighbourhood of the boundary of $\overline{M}$.
**Notation.** For a set $S$ of positive integers write $P_S$ for the union $\bigcup_{n \in S} P_n$.
**Definition.** The *partial braid category* $\mathcal{B}_P(M)$ has objects $0, 1, 2, \ldots$ and

$$\mathrm{Hom}(m, n) = \coprod_{k,S,T} \{\text{paths in } C_{kP}(M) \text{ from } P_S \text{ to } P_T\}/\sim$$

where the disjoint union runs over $k \leqslant \min(m, n)$ and $k$-element subsets $S \subseteq \{1, \ldots, m\}$ and $T \subseteq \{1, \ldots, n\}$, and $\sim$ is endpoint-preserving homotopy. Composition is defined as suggested by the following example:
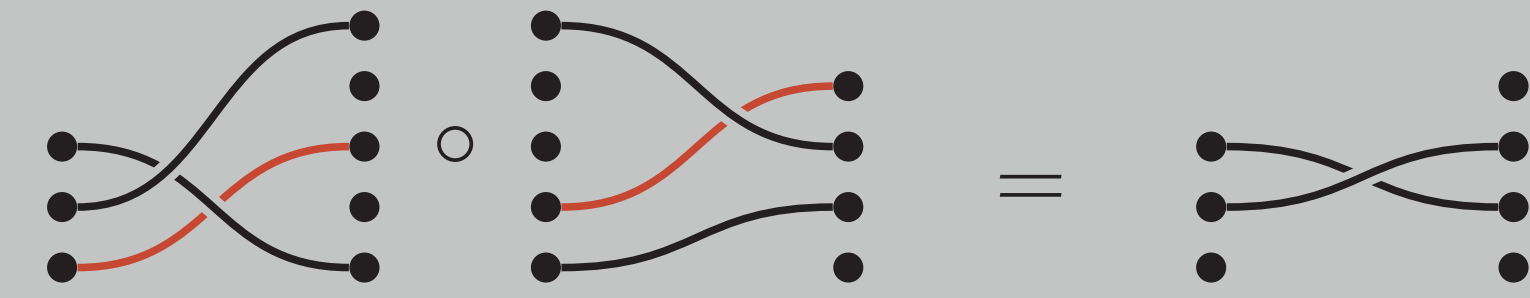


The red strands are erased and the black (composable) strands are glued.
**Idea.** Arrows are *partially-defined braids in $M \times [0, 1]$* with *cross-section $P$*.
**Note.** The automorphism group of $n$ is $\pi_1(C_{nP}(M))$.

## Twisted homological stability

Let $F: \mathcal{B}_P(M) \longrightarrow \mathrm{Ab}$ be a functor to the category of abelian groups.
**Degree.** There are two equivalent definitions of the *degree* of $F$, using
(1) A *stabilisation endofunctor* $\mathcal{B}_P(M) \longrightarrow \mathcal{B}_P(M)$ and recursion.
(2) A decomposition of $F(n)$ into *cross-effects*.
**Question.** Since $\pi_1(C_{nP}(M))$ acts on $F(n)$ we may ask if the sequence

$$\cdots \longrightarrow H_i(C_n; F(n)) \longrightarrow H_i(C_{n+1}; F(n+1)) \longrightarrow \cdots \qquad (2)$$

of twisted homology groups stabilises, where we abbreviate $C_n = C_{nP}(M)$.
**Theorem B.** If the sequence (1) stabilises for all $i \geqslant 0$, and $\deg(F) < \infty$, then the sequence (2) stabilises for all $i \geqslant 0$.                        [P₂]
In particular, twisted homological stability holds for all four cases on the left.

## Corollaries and extensions.

**Example.** The actions $\pi_1(C_n) \longrightarrow \mathfrak{S}_n \curvearrowright \mathbb{Q}[\mathfrak{S}_n/\mathfrak{S}_{\lambda[n]}]$ for a partition $\lambda$ assemble into a finite-degree functor $P_\lambda: \mathcal{B}_P(M) \longrightarrow \mathrm{Vect}_{\mathbb{Q}} \subset \mathrm{Ab}$.
**Corollary.** Theorem B for $F = P_\lambda$, together with elementary representation theory, implies that the rational cohomology of the ordered analogue $\widetilde{C}_n$ of $C_n$ satisfies *representation stability*. (First proved for configuration spaces by [C].)
**Extension.** Twisted homological stability for unordered configuration spaces $C_n(M)$ has been extended by [RWW] and [Kr] to all finite-degree functors $F: \mathcal{UB}(M) \longrightarrow \mathrm{Ab}$ defined on another certain category $\mathcal{UB}(M)$.
(There is a functor $\Upsilon: \mathcal{UB}(M) \longrightarrow \mathcal{B}_{\text{point}}(M)$ so that $- \circ \Upsilon$ preserves degree.)

## More examples — Homological representations of braid groups

**Idea.** Given a space $X$ with a $G$-action, construct a $G$-representation by lifting to a covering $X^\varphi$ of $X$ and taking the induced action on $H_d(-; \mathbb{Z})$. This is an action by $\mathbb{Z}[H]$-module automorphisms, where $H = \mathrm{Deck}(X^\varphi)$.
**Burau representation.** Take $X = D_n = D^2 \smallsetminus \{n \text{ points}\}$.
Its *mapping class group* is $\pi_0(\mathrm{Diff}(D_n, \partial D_n)) \cong B_n = G$.
The quotient $\varphi: \pi_1(D_n) = F_n \longrightarrow \mathbb{Z} = H$ determines a covering $X^\varphi$.
The induced $B_n$-representation (with $d = 1$) is the *Burau representation*.
**Higher homological (Lawrence) representations.** Set $X = C_d(D_n)$.
For $d \geqslant 2$ there is a quotient $\varphi: \pi_1(C_d(D_n)) \longrightarrow \mathbb{Z}^2 = H$ defined by:

$$r: \pi_1(C_d(D_n)) \longrightarrow B_{d+n} \longrightarrow \mathbb{Z}$$
$$t: \pi_1(C_d(D_n)) \longrightarrow B_d \longrightarrow \mathbb{Z} \qquad \varphi = \left(t, \tfrac{1}{2}(r - t)\right)$$

The induced $B_n$-representation is the *Lawrence representation* at *height* $d$.
**Example C.** (Part of joint work in progress with Arthur Soulié.)
These assemble into a functor $\mathcal{L}_d: \mathcal{UB}(\mathbb{R}^2) \longrightarrow \mathbb{Z}[\mathbb{Z}^2]\text{-mod}$ of degree $d$.
Thus the Lawrence representations are homologically stable.

## Further directions / ongoing projects

- What is the stable homology of $C_{nP}(M)$ when $P$ is not a point?
- What is the stable *twisted* homology of $C_n(M)$? (... of $C_{nP}(M)$?)
- Weaken the dimension restriction on $P$ in Theorem A to allow, e.g., moduli spaces of unlinked copies of a *non-trivial* knot in a 3-manifold.
- Construct interesting twisted coefficient systems on $\mathcal{B}_P(M)$ or $\mathcal{UB}(M)$
  - when $P = \text{point}$ (or a finite set) and $M$ is a surface.
  - when $P = S^1$ and $M$ is a 3-manifold.