Twisted homological stability and homological representations of braid groups

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Configuration spaces

Let M be a connected, smooth, open manifold and $n \geqslant 1$ an integer.
Definition. The n^{th} *unordered configuration space* on M is

$$C_n(M) = \{(x_1, \ldots, x_n) \in M^n \mid i \neq j \Rightarrow x_i \neq x_j\}/\mathfrak{S}_n$$

Facts. The space $C_n(\mathbb{R}^2)$ is aspherical and $\pi_1(C_n(\mathbb{R}^2)) \cong B_n$, the braid group on n strands. In other words $C_n(\mathbb{R}^2) \simeq K(B_n, 1)$.
Variants. Instead of quotienting by the symmetric group $\mathfrak{S}_n$ above, we may instead quotient by a subgroup, for example:

	$\{1\}$	*ordered configuration space*	$\widetilde{C}_n(M)$
	A_n	*oriented configuration space*	$C_n^+(M)$
$(n = km)$	$\mathfrak{S}_k \wr \mathfrak{S}_m$	*partitioned configuration space*	$C_{m\vert k}(M)$

Moduli spaces of submanifolds

Generalisation. Let $M = \text{interior}(\overline{M})$ and $P \subseteq \partial\overline{M}$ closed submanifold.
Examples. (a) $P = $ point (b) $P = $ finite set of size k (c) $P = S^1 \subseteq \partial D^3$
Construction. Fix $n \geqslant 1$.
$\text{Emb}(nP, M)$: space of embeddings of n disjoint copies of P into M.
$\text{Diff}(nP)$: topological group of diffeomorphisms of n disjoint copies of P.
$c_0 \in \text{Emb}(nP, M)$: n parallel copies of $P \subseteq \partial\overline{M}$ in a collar neighbourhood.
Definition. The n^{th} *moduli space of submanifolds of M modelled on P* is

$$C_{nP}(M) = \text{the path-component of } \text{Emb}(nP, M)/\text{Diff}(nP) \text{ containing } [c_0]$$

Examples. The examples above define (a) unordered and (b) partitioned configuration spaces. Example (c) is the moduli space of unlinks in $\mathbb{R}^3$.
Note. This may be extended to allow (i) embedded copies of P parametrised modulo a fixed subgroup $G \leqslant \text{Diff}(P)$ and (ii) labels in a bundle $E \to \text{Emb}(P, \overline{M})$.

Untwisted homological stability

Let's abbreviate $C_n = C_{nP}(M)$. By default, H_i means *integral* homology.
Observation. There are natural *stabilisation maps* $C_n \longrightarrow C_{n+1}$.
(Idea: push a new copy of P inwards from the boundary along a collar neighbourhood.)
Theorem A. For any fixed $i \geqslant 0$ the sequence

$$\cdots \longrightarrow H_i(C_{n-1}) \longrightarrow H_i(C_n) \longrightarrow H_i(C_{n+1}) \longrightarrow \cdots \qquad (1)$$

consists of isomorphisms (*stabilises*) for $n \geqslant 2i$, as long as

- P is a point [S₁,M,S₂]
- the dimension of P is at most $\frac{1}{2}(\dim(M) - 3)$ [P₁]
- P is a finite set and M is a surface [PT]
- $P = S^1$ and $\dim(M) = 3$ (the moduli space of unlinks) [Ku]

Question. Is there a space X (depending on M and P) whose homology is the limiting homology (also called *stable homology*) of the sequence (1)?
Partial answer. If P is a single point then we may take X to be any path-component of $\Gamma_c(\dot{T}M)$. E.g. if $M = \mathbb{R}^m$ this space is $\Omega^m S^m$.
Open question. What is X for other manifolds P?

Related results

Some related spaces (or groups) whose homology is also stable as $n \to \infty$.

- The groups $\pi_1(C_{nS^1}(\mathbb{R}^3))$ (*string motion groups*) [HW]
  Note that $C_{nS^1}(\mathbb{R}^3)$ is *not* aspherical, in contrast to $C_n(\mathbb{R}^2)$.
- Moduli spaces $\text{Emb}(S_n, M)/\text{Diff}^+(S_n)$ of embedded surfaces of genus n, assuming $\pi_1(M) = 0$ and $\dim(M) \geqslant 5$. [CRW]
- The sequences $C_{2n}(M)$ and $C_{2n+1}(M)$, when M is *closed* and odd-dimensional. [CP]
  Note: $C_n(M)$ is usually unstable if M is closed, e.g. $H_1(C_n(S^2)) \cong \mathbb{Z}/(2n-2)$.

References

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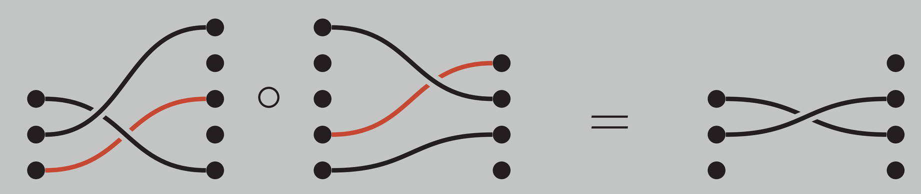
Twisted coefficient systems

Setting. Fix $\overline{M}$ and P as before and let $P_1, P_2, P_3, \ldots$ be pairwise disjoint, parallel copies of P in a collar neighbourhood of the boundary of $\overline{M}$.
Notation. For a set S of positive integers write P_S for the union $\bigcup_{n \in S} P_n$.
Definition. The *partial braid category* $\mathcal{B}_P(M)$ has objects $0, 1, 2, \ldots$ and

$$\text{Hom}(m, n) = \coprod_{k,S,T} \{\text{paths in } C_{kP}(M) \text{ from } P_S \text{ to } P_T\} / \sim$$

where the disjoint union runs over $k \leqslant \min(m, n)$ and k-element subsets $S \subseteq \{1, \ldots, m\}$ and $T \subseteq \{1, \ldots, n\}$, and $\sim$ is endpoint-preserving homotopy.
Composition is defined as suggested by the following example:



The red strands are erased and the black (composable) strands are glued.
Idea. Arrows are *partially-defined braids in $M \times [0,1]$* with *cross-section P*.
Note. The automorphism group of n is $\pi_1(C_{nP}(M))$.

Twisted homological stability

Let $F\colon \mathcal{B}_P(M) \longrightarrow \text{Ab}$ be a functor to the category of abelian groups.
Degree. There are two equivalent definitions of the *degree* of F, using
(1) A *stabilisation endofunctor* $\mathcal{B}_P(M) \longrightarrow \mathcal{B}_P(M)$ and recursion.
(2) A decomposition of $F(n)$ into *cross-effects*.
Question. Since $\pi_1(C_{nP}(M))$ acts on $F(n)$ we may ask if the sequence

$$\cdots \longrightarrow H_i(C_n; F(n)) \longrightarrow H_i(C_{n+1}; F(n+1)) \longrightarrow \cdots \qquad (2)$$

of twisted homology groups stabilises, where we abbreviate $C_n = C_{nP}(M)$.
Theorem B. If the sequence (1) stabilises for all $i \geqslant 0$, and $\deg(F) < \infty$, then the sequence (2) stabilises for all $i \geqslant 0$. [P₂]
In particular, twisted homological stability holds for all four cases on the left.

Corollaries and extensions.

Example. The actions $\pi_1(C_n) \longrightarrow \mathfrak{S}_n \curvearrowright \mathbb{Q}[\mathfrak{S}_n/\mathfrak{S}_{\lambda[n]}]$ for a partition λ assemble into a finite-degree functor $P_\lambda\colon \mathcal{B}_P(M) \longrightarrow \text{Vect}_{\mathbb{Q}} \subset \text{Ab}$.
Corollary. Theorem B for $F = P_\lambda$, together with elementary representation theory, implies that the rational cohomology of the ordered analogue $\widetilde{C}_n$ of C_n satisfies *representation stability*. (First proved for configuration spaces by [C].)
Extension. Twisted homological stability for unordered configuration spaces $C_n(M)$ has been extended by [RWW] and [Kr] to all finite-degree functors $F\colon \mathcal{UB}(M) \longrightarrow \text{Ab}$ defined on another certain category $\mathcal{UB}(M)$.
(There is a functor $\Upsilon\colon \mathcal{UB}(M) \longrightarrow \mathcal{B}_{\text{point}}(M)$ so that $- \circ \Upsilon$ preserves degree.)

More examples — Homological representations of braid groups

Idea. Given a space X with a G-action, construct a G-representation by lifting to a covering X^φ of X and taking the induced action on $H_d(-;\mathbb{Z})$.
This is an action by $\mathbb{Z}[H]$-module automorphisms, where $H = \text{Deck}(X^\varphi)$.
Burau representation. Take $X = D_n = D^2 \smallsetminus \{n \text{ points}\}$.
Its *mapping class group* is $\pi_0(\text{Diff}(D_n, \partial D_n)) \cong B_n = G$.
The quotient $\varphi\colon \pi_1(D_n) = F_n \longrightarrow \mathbb{Z} = H$ determines a covering X^φ.
The induced B_n-representation (with $d = 1$) is the *Burau representation*.
Higher homological (Lawrence) representations. Set $X = C_d(D_n)$.
For $d \geqslant 2$ there is a quotient $\varphi\colon \pi_1(C_d(D_n)) \longrightarrow \mathbb{Z}^2 = H$ defined by:

$$r\colon \pi_1(C_d(D_n)) \longrightarrow B_{d+n} \longrightarrow \mathbb{Z}$$
$$t\colon \pi_1(C_d(D_n)) \longrightarrow B_d \longrightarrow \mathbb{Z} \qquad \varphi = \left(t, \tfrac{1}{2}(r-t)\right)$$

The induced B_n-representation is the *Lawrence representation* at *height* d.
Example C. (Part of joint work in progress with Arthur Soulié.)
These assemble into a functor $\mathcal{L}_d\colon \mathcal{UB}(\mathbb{R}^2) \longrightarrow \mathbb{Z}[\mathbb{Z}^2]$-mod of degree d.
Thus the Lawrence representations are homologically stable.

Further directions / ongoing projects

- What is the stable homology of $C_{nP}(M)$ when P is not a point?
- What is the stable *twisted* homology of $C_n(M)$? (…of $C_{nP}(M)$?)
- Weaken the dimension restriction on P in Theorem A to allow, e.g., moduli spaces of unlinked copies of a *non-trivial* knot in a 3-manifold.
- Construct interesting twisted coefficient systems on $\mathcal{B}_P(M)$ or $\mathcal{UB}(M)$
  - when $P = $ point (or a finite set) and M is a surface.
  - when $P = S^1$ and M is a 3-manifold.

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