Homology of moduli spaces of submanifolds

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Abstract:

Given a closed manifold L and a connected manifold M, the *moduli space of submanifolds* of M that are diffeomorphic to L is $\mathcal{M}(L,M) = \mathrm{Emb}(L,M)/\mathrm{Diff}(L)$, the space of embeddings $L \hookrightarrow M$ modulo reparametrisation by self-diffeomorphisms of L.

Two natural questions are to understand $\pi_0(\mathcal{M}(L,M))$ and to understand $H_*(\mathcal{M}_{[e]}(L,M))$, where in the latter the subscript $[e]$ means we have restricted to the path-component of the moduli space corresponding to a specific embedding $e\colon L \hookrightarrow M$. The first question has its own very important story (it contains knot theory, for example), but in this talk I will focus on the second question, of understanding the *higher homology of certain path-components of* $\mathcal{M}(L,M)$.

The homology of these spaces has been much studied, especially in the two special cases where (a) $\dim(L) = 0$ or (b) $M = \mathbb{R}^\infty$. In the first case, $\mathcal{M}(L,M)$ is the configuration space $C_n(M)$ of $n = |L|$ non-colliding, indistinguishable particles in M. An important special case of this is the configuration spaces $C_n(\mathbb{R}^2)$, which are classifying spaces of the braid groups B_n. In the second case, $\mathcal{M}(L,\mathbb{R}^\infty)$ is a classifying space of the diffeomorphism group of L. An important special case of this is when L is a closed, connected, oriented surface (this is the subject of the *Madsen-Weiss theorem*, and is related to moduli spaces of Riemann surfaces). Much less is known in the case where (c) $\dim(L) > 0$ and $\dim(M) < \infty$.

A common and very effective technique in studying the homology of these moduli spaces is to consider a sequence L_n and try to prove that the homology of $\mathcal{M}(L_n,M)$ *stabilises* as $n \to \infty$. The next step is then to calculate the homology of the *stable moduli space* $\mathcal{M}(L_\infty,M)$. This is typically easier to deal with than $\mathcal{M}(L_n,M)$, since it can be given some additional structure coming from the fact that it contains information about submanifolds of M of diffeomorphism type L_n for all n.

I will briefly review what is known (via this technique) about the homology of $\mathcal{M}(L,M)$ in the three different cases (a)–(c), and I will present a stabilisation result for $\mathcal{M}(L_n,M)$ in the case where M is non-compact and L_n is the disjoint union of n copies of a fixed manifold P, under a certain condition on their relative dimensions. If time permits, I will also discuss the (still open) related question of identifying the homology of the corresponding stable moduli space $\mathcal{M}(L_\infty,M)$.

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