High-connectivity of complexes of arcs on surfaces
and
homological stability for mapping class groups

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Abstract:

This will be an expository talk on the proof of the fact, due originally to John Harer in 1985, that the mapping class groups of compact, connected, orientable surfaces are *homologically stable*.

Write $\Sigma_{g,b}$ for the compact, connected, orientable surface of genus g with b boundary components. Its mapping class group is

$$\Gamma_{g,b} \coloneqq \pi_0(\mathrm{Diff}_\partial(\Sigma_{g,b})),$$

where $\mathrm{Diff}_\partial(-)$ denotes the topological group of self-diffeomorphisms of a manifold that act by the identity on an open neighbourhood of its boundary.[1] Homological stability says that the homology of these groups in degree i is independent of g and b when g is sufficiently large compared to i. This was first proved by Harer, and has been reproved many times since then, with various improvements and generalisations. In the first part of my talk, I will give an overview of the best-known results to date.

Proofs of homological stability usually split into two parts: a more algebraic part, involving inductive arguments with spectral sequences, and a more geometric part: I will focus on the geometric part in this talk. The idea is to construct certain simplicial complexes (whose vertices correspond to arcs, curves, or other geometric objects in the surface) on which the mapping class groups act in a "nice" way, and then to prove that these simplicial complexes are *highly-connected*, i.e. that their homotopy groups vanish up to a certain degree. There is a different simplicial complex for each g and b, and we want this vanishing degree (the *connectivity* of the simplicial complex) to go to infinity as $g \to \infty$.

I will describe various different simplicial complexes that have been used in such arguments, and then sketch a proof of high-connectivity for one of them. There are many steps to this proof, but there are essentially three different techniques that are used for all steps, and – if time permits – I will go through the details of one instance of each technique.

Mathematisches Institut der Universität Bonn
Endenicher Allee 60
53115 Bonn
Germany

palmer@math.uni-bonn.de

[1] We don't really lose anything by studying just the (abstract) group of path-components of $\mathrm{Diff}_\partial(\Sigma_{g,b})$, since its path-components are contractible (except for the sphere and the torus), by a result of Earle-Eells and Gramain. This fact is very special to the case of surfaces – it is certainly false for diffeomorphism groups of higher-dimensional manifolds!