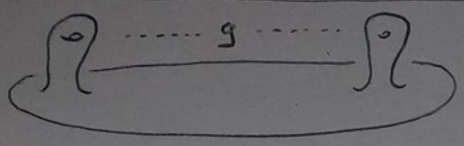
Braid groups, configuration spaces and homotopy theory — 23-27 July, 2018

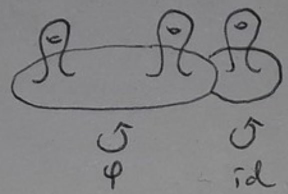


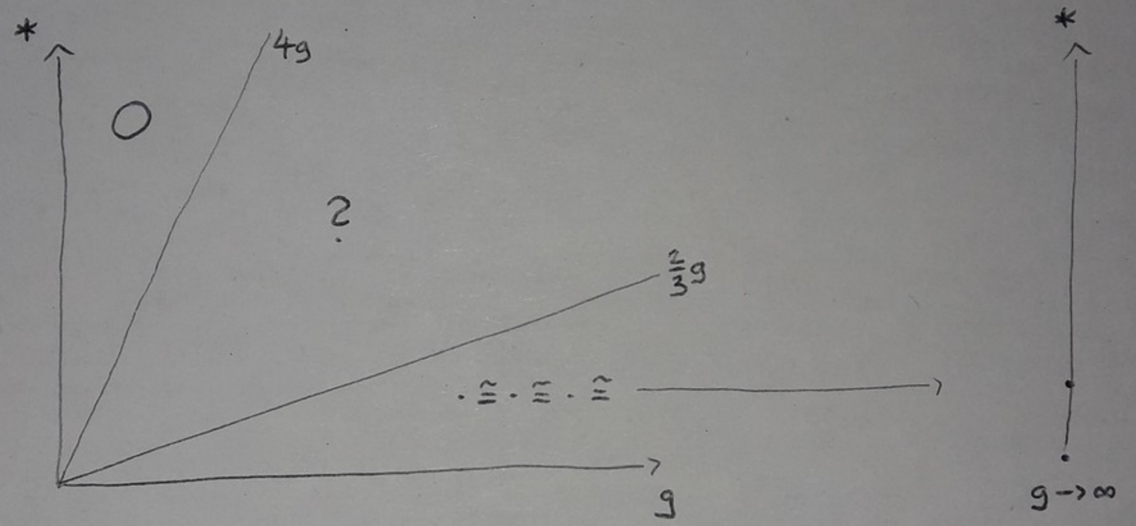
$$= S_{g,1} = D^2 \# \underbrace{(S^1\times S^1)\#(S^1\times S^1)\#\cdots}_{g}$$

Mapping class group: $\Gamma(S_{g,1}) = \pi_0 \operatorname{Diff}_\partial(S_{g,1})$

Harer '85: $H_*\Gamma(S_{g,1}) \longrightarrow H_*\Gamma(S_{g+1,1})$

isomorphism for $* \leq \frac{2}{3}g$





Madsen-Weiss '02: calculated $\operatorname{colim}_{g\to\infty} H_*\Gamma(S_{g,1})$

$\mathbb{Q}$: $H^* = \mathbb{Q}[\kappa_1, \kappa_2, \dots] \longrightarrow$ Mumford conjecture

Rmk $|\chi| \sim \zeta(1-2g) \sim g^{2g}$ (Harer-Zagier '86)

stable $H_* \sim c^{\sqrt{g}}$ (Madsen-Weiss '02)

$H_{4g-6} \sim c^{g}$ (Chan-Galatius-Payne '18)

3-dim version :

$$\left.\begin{array}{l} H_* \Gamma\left(D^3 \# (S^1\times S^2) \# (S^1\times S^2) \# \cdots\cdots\right) \\ H_* \Gamma\left(D^3 \# (\text{irreducible}) \# \cdots\cdots\right) \end{array}\right\} \begin{array}{l}\text{stabilises} \\ \text{(Hatcher-Wahl '10)}\end{array}$$

Fact (Earle-Schatz '70, Gramain '73) $\quad \underset{\displaystyle \Gamma(S_{g,1})}{\overset{\parallel}{\pi_0 \mathrm{Diff}_\partial(S_{g,1})}} \simeq \mathrm{Diff}_\partial(S_{g,1})$

Higher-dim version :

For $p \leq q < 2p-2$,

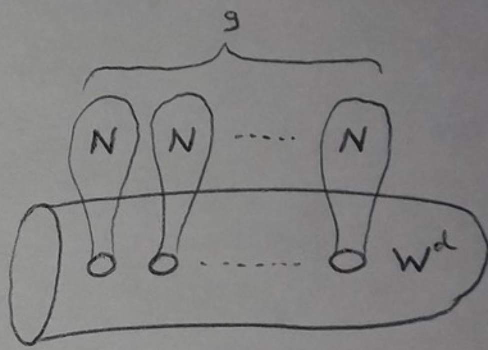
$$H_* \mathrm{Diff}_\partial\left(D^{p+q} \# (S^p\times S^q) \# (S^p \times S^q) \# \cdots\cdots\right) \quad \text{stabilises}$$

(Galatius - Randal-Williams '12 $(p=q)$, Perlmutter '14 $(p<q)$)

stable H_* : $\quad p=q$: ✓ $\quad \mathbb{Q}$: polynomial ring

$\qquad\qquad\quad p<q$: ?

Def : Symmetric diffeomorphism group



$\Sigma\mathrm{Diff}_\partial\left(W \# M \# M \# \cdots\cdots\right)$

$= \{ \varphi \in \mathrm{Diff}_\partial(-) \mid$

$\cdot\ \varphi$ preserves decomposition,

$\cdot\ \varphi$ acts on $\coprod_g S^{d-1}$ via $O(d)^g \rtimes \Sigma_g =: O(d) \wr \Sigma_g \}$

Tillmann '14 :

W, M d-manifolds, compact, connected, $\partial W \neq \phi$

$$\Rightarrow \quad \left.\begin{array}{l} H_* \Sigma\mathrm{Diff}_\partial\left(W \# N \# N \# \cdots\cdots\right) \\ H_* \pi_0 \Sigma\mathrm{Diff}_\partial\left(W \# N \# N \# \cdots\cdots\right) \end{array}\right\} \text{stabilises}$$

<u>Def</u> : Parametrised connected sum

$$L^k \overset{f}{\hookrightarrow} W^d, \quad L^k \overset{e}{\hookrightarrow} N^d \qquad \begin{array}{ccc} \nu_f & \overset{\theta}{\cong} & \nu_e \\ & \searrow \ \swarrow & \\ & L & \end{array} \quad \rightsquigarrow \quad W \underset{L}{\#} N := \left[W - \mathring{Tub}(f)\right] \underset{\theta}{\cup} \left[N - \mathring{Tub}(e)\right]$$

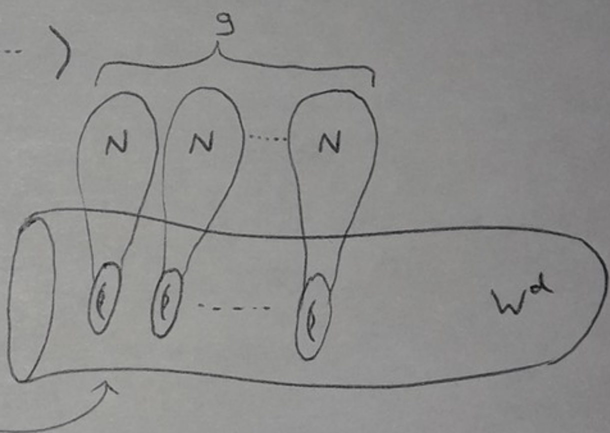
<u>E.g.</u> (1) $e : S^k \hookrightarrow S^d$, $f : S^k \hookrightarrow W$ $\rightsquigarrow$ $W \underset{L}{\#} N$ = k-surgery on W.

(2) $e : S^1 \hookrightarrow L(p,q)$ lens space, $f : S^1 \hookrightarrow W^3$ $\rightsquigarrow$ $W \underset{L}{\#} N$ = Dehn surgery of slope p/q on W.

<u>Def</u> : Symmetric diffeomorphism group, generalised

$\Sigma \mathrm{Diff}_\partial (W \underset{L}{\#} N \underset{L}{\#} N \underset{L}{\#} \cdots\cdots)$

g "parallel", disjoint copies of f



$$= \left\{ \varphi \in \mathrm{Diff}_\partial(-) \;\middle|\; \begin{array}{l} \cdot\ \varphi \text{ preserves decomposition,} \\ \cdot\ \varphi \text{ acts on } \coprod_g \partial Tub(f) \text{ via } \mathrm{Diff}_{O(d-k)}(Tub(f)) \wr \Sigma_g \end{array} \right\}$$

<u>Thm</u> (P., July-2018)

$H_* \Sigma \mathrm{Diff}_\partial (W \underset{L}{\#} N \underset{L}{\#} N \underset{L}{\#} \cdots\cdots)$ stabilises,

as long as $\cdot\ k \leq \frac{1}{2}(d-3)$

or $\cdot\ k = 1,\ d = 3$ and $f : S^1 \hookrightarrow W^3$ is an unknot

[work-in-progress : a version for $H_* \pi_0 \Sigma \mathrm{Diff}_\partial (-)$]

Proof idea:

$$\begin{array}{c} \text{(1)} \\ \text{fibre} \simeq C_{gL}(W;Z) \\ \big\downarrow \\ B(\Sigma\mathrm{Diff}_\partial(-)) = \left.\left\{ \begin{array}{ll} \overbrace{W \#_L N \#_L N \#_L \cdots\cdots}^{g} \hookrightarrow \mathbb{R}^\infty & \\ W \hookrightarrow \mathbb{R}^\infty & \underline{\text{compatible}} \end{array} \right\}\right/ \Sigma\mathrm{Diff}_\partial(-) \\ \big\downarrow \\ B\mathrm{Diff}_\partial(W) \end{array}$$

<u>Def</u> $C_n(W) = \mathrm{Emb}(\{1,\dots,n\}, W)/\Sigma_n$

$C_{nL}(W) = \mathrm{Emb}(\coprod_n L, W) / \mathrm{Diff}(\coprod_n L)$

if L is connected.
in general: $\mathrm{Diff}(L) \wr \Sigma_n$

one path-component of

$C_{nL}(W;Z)$

space of parameters: each $L \subset W$ is equipped with an element of Z

$$Z \downarrow \mathrm{Emb}(L,W)$$

In our case: $Z =$ the data needed to construct $- \#_L N$ inside $\mathbb{R}^\infty$

(1½) H_*-stability for $C_{nL}(W;Z)$ $\Longleftarrow$ H_*-stability for $C_{nL}(W)$

(2) <u>H_*-stability for $C_{nL}(W)$</u>:

$L =$ point: ✓ Arnol'd '70, Segal '73, McDuff '75

$L =$ unknot: ✓ Kupers '17

$k \leqslant \frac{1}{2}(d-3)$: ✓ P., May 2018

For the version for $H_* \underline{\underline{\pi_0}} \Sigma\mathrm{Diff}_\partial(\text{——})$, we need instead:

(2)' <u>H_*-stability for $\pi_1 C_{nL}(W)$</u>:

$L =$ point: $d=2$: $B\pi_1 C_n(W) \simeq C_n(W)$; $d \geqslant 3$: $B\pi_1 C_n(W) \simeq C_n(\mathbb{R}^\infty; B\pi_1(W))$ (*)

$L =$ unknot in $\mathbb{R}^3$: ✓ Hatcher-Wahl '10 [loop-braid groups]

$k \leqslant \frac{1}{2}(d-3)$: work-in-progress $\rightsquigarrow$ gen. of (*)