Towards stable homology of moduli spaces of disconnected submanifolds

Martin Palmer-Anghel // Workshop for Young Researchers in Mathematics, 3–4 June 2019

Configuration spaces — stability — stable homology.

Configuration spaces of unordered points in a manifold M appear ubiquitously in mathematics, in knot theory (via the braid groups), homotopy theory and algebraic geometry. Their homology $H_i(C_n(M);\mathbb{Z})$ is known to *stabilise*, in each degree i, as the number n of points in the configuration goes to infinity [S1,McD,S2], as long as M is connected and non-compact.

More precisely, there are canonical isomorphisms

$$H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+1}(M);\mathbb{Z})$$

whenever $n \geqslant 2i$. In this range of degrees, their homology is therefore isomorphic to the limiting homology as $n \to \infty$ (which is called the *stable homology* due to this stabilisation phenomenon).

In fact, McDuff and Segal [S1,McD] also identified this stable homology, by describing a specific space $X(M)$ whose homology is the stable homology of $C_n(M)$ as $n \to \infty$, and which is moreover easily accessible to the tools of algebraic topology. For example, $X(\mathbb{R}^2) = \Omega^2 S^3 = \mathrm{Map}_*(S^2, S^3)$.

Moduli spaces of disconnected submanifolds — stability — stable homology?

Fixing any closed manifold L, we may more generally consider the moduli spaces

$$C_{nL}(M)$$

of all submanifolds of M that are diffeomorphic to n disjoint copies of L and isotopic to a chosen "basepoint configuration". These moduli spaces also stabilise as $n \to \infty$ under certain conditions on the dimensions of L and M [K,P] (assuming as before that M is connected and non-compact). However, their stable homology is unknown, unless L is a point.

The goal of this talk will be to outline current work in progress on identifying this stable homology, in the above sense of finding appropriate spaces $X_L(M)$ modelling the stable homology.

References.

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