**Problems related to lecture 1 of the GSS lecture course by Søren Galatius.**[1]

*Themes*: Cobordism categories, elementary invertible field theories, classifying spaces of categories and fundamental groupoids.

*There are many more problems here than can be attempted in a single problem session!* If you do just one problem, the most useful for understanding the course would be Problem 2 on constructing elementary invertible field theories (Problem 1 is a good warm-up for this). The problems from this set may also be discussed in the problem sessions later in the week, alongside the later problem sets.

**Problem 1** Recall that $\mathfrak{N}_d$ denotes the abelian group of smooth, closed $d$-manifolds up to cobordism, with respect to the operation of disjoint union.

(a) Show that the Klein bottle is nullbordant.
(b) Prove that, up to cobordism, disjoint union is the same as connected sum, and that every cobordism class contains a connected manifold.
(c) Prove, without invoking the theorem of Thom (but you may use the classification of surfaces), that the abelian groups $\mathfrak{N}_0$, $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, 0 and $\mathbb{Z}/2\mathbb{Z}$, respectively.

Write $\Omega_d$ for the analogous oriented cobordism group: the abelian group of smooth, closed, *oriented* $d$-manifolds up to *oriented* cobordism, with respect to the operation of disjoint union.

(d) Prove that the abelian groups $\Omega_0$, $\Omega_1$ and $\Omega_2$ are isomorphic to $\mathbb{Z}$, 0 and 0 respectively.

**Problem 2** In this problem we construct two elementary "invertible field theories" (in the naive sense, ignoring symmetric monoidal structures). Recall that a (naive) invertible field theory is a functor from the abstract cobordism category $\mathrm{Cob}_d$ to a groupoid; we will construct two examples where the target is a group.

(a) Construct a functor

$$F_d\colon \mathrm{Cob}_d \longrightarrow \mathfrak{N}_d$$

(where the group $\mathfrak{N}_d$ is considered as a category with one object) such that the restriction to $\mathrm{End}_{\mathrm{Cob}_d}(\varnothing)$ sends a diffeomorphism class of $d$-manifolds to its cobordism class.
(*Hint*: note that $\mathrm{Cob}_d$ is a disjoint union of subcategories indexed by the elements of $\mathfrak{N}_{d-1}$, so it will suffice to define $F$ on the full subcategory of $\mathrm{Cob}_d$ on nullbordant $(d-1)$-manifolds.)

(b) Construct a functor

$$E_d\colon \mathrm{Cob}_d \longrightarrow \mathbb{Z}$$

(where the group $\mathbb{Z}$ is considered as a category with one object) such that the restriction to $\mathrm{End}_{\mathrm{Cob}_d}(\varnothing)$ sends a diffeomorphism class of $d$-manifolds to its Euler characteristic.

(c) $(*)$ In the case $d = 2$, using the classification of surfaces, show that the functor

$$\mathrm{Cob}_2[\mathrm{Cob}_2^{-1}] \longrightarrow \mathbb{Z},$$

induced by $E_2$, is an equivalence.[2]

**Problem 3** Give an explicit combinatorial description of the categories $\mathrm{Cob}_0$ and $\mathrm{Cob}_1$, and of the subcategory $\mathrm{Cob}_2^{\mathrm{conn}} \subset \mathrm{Cob}_2$ consisting of connected cobordisms between non-empty 1-manifolds (i.e. its morphisms are *connected* 2-manifolds, but its objects may be *disconnected* (but non-empty) 1-manifolds). Note that $\mathrm{Cob}_2^{\mathrm{conn}}$ is a *non-unital* category.

Using this combinatorial description of $\mathrm{Cob}_2^{\mathrm{conn}}$, we may investigate its localisation, and compare it to the localisation of $\mathrm{Cob}_2$ (which was shown above to be equivalent to $\mathbb{Z}$). Since $B\mathrm{Cob}_2^{\mathrm{conn}}$ is path-connected, its fundamental groupoid is equivalent to its fundamental group (based at any point), which depends only on the 2-skeleton of the classifying space.

[1] Updated: July 10, 2019.
[2] See also Theorem 3.7 of [R. Juer, U. Tillmann, *Localisations of cobordism categories and invertible TFTs in dimension two*, Homology, Homotopy and Applications vol. 15, no. 2, pp. 1–31 (2013).].

(a) Using your combinatorial description of $\mathrm{Cob}_2^{\mathrm{conn}}$, find a presentation (with infinitely many generators and relations) of the fundamental group of its classifying space.
(b) Investigate how this may be simplified, by cancelling relations against generators. Does the group $\pi_1(B\mathrm{Cob}_2^{\mathrm{conn}}, x)$ contain torsion?

**Problem 4**

(a) Let $C$ be the poset of all non-empty, proper subsets of $\{0, 1, 2, 3\}$, considered as a category. Prove that $BC$ is homeomorphic to $S^2$ and hence that $\pi_1(BC)$ is equivalent to a trivial category.
(b) Let $C$ be the category with exactly two objects $a$, $b$ and exactly two non-identity morphisms $f$, $g$, which both have source $a$ and target $b$. Prove that $BC$ is homeomorphic to $S^1$ and describe explicitly an equivalence of groupoids $\pi_1(BC) \to \mathbb{Z}$.
(c) Combining the above ideas, find a finite category $C$ such that $\pi_2(BC, x)$ is infinitely-generated for any object $x$.
(d) Find other examples of interesting behaviour of the functor $C \to \pi_1(BC)$.

**Problem 5** $(*)$ *This problem is more difficult and less directly relevant to the lecture course, so it is recommended to do this problem after the other problems above (and for fun!).*

Another variant of the cobordism group is the *group of homotopy $d$-spheres* $\Theta_d$. To define this, let us first denote by $\mathcal{M}_d$ the abelian monoid of smooth, closed, connected and oriented $d$-manifolds under the operation of connected sum. Let $\mathcal{S}_d$ be the subset of those $d$-manifolds that are homotopy equivalent to the sphere $S^d$, which we call *homotopy spheres*.

(a) Prove that, if $M$ and $N$ are homotopy spheres, so is their connected sum.

Hence $\mathcal{S}_d$ is a submonoid. Two manifolds $M, N \in \mathcal{M}_d$ are called *$h$-cobordant* if there is a cobordism $W$ between them such that the inclusions $c_{\mathrm{in}}$ and $c_{\mathrm{out}}$ are both homotopy equivalences.

(b) Prove that $h$-cobordism $\sim_h$ induces an equivalence relation on $\mathcal{S}_d$.
(c) Prove that this equivalence relation is compatible with the connected sum operation.

Hence we have a well-defined quotient monoid $\Theta_d \coloneqq \mathcal{S}_d/{\sim_h}$ with respect to the operation of connected sum.

(d) Prove that $\Theta_d$ is a group.

In fact, except possibly for $d = 4$, the quotient $\mathcal{S}_d \to \Theta_d$ is an isomorphism, so $\mathcal{S}_d$ is also a group. The group $\Theta_d$ is known to be finite for all $d$ (including 4).