Problems related to lecture 2 of the GSS lecture course by Søren Galatius.[1]

Problem 1 Discuss the difference between cobordisms being diffeomorphic *as cobordisms* (which depends on the collars c_{in} and c_{out}) and merely having diffeomorphic underlying manifolds (which does not).

Problem 2 Let D be the groupoid defined abstractly as $\mathrm{Ob}(D) = \mathbb{Z}/2\mathbb{Z}$, morphism sets

$$D(a,b) = \begin{cases} \mathbb{Z} & a = b \\ \varnothing & a \neq b, \end{cases}$$

and composition given by addition in $\mathbb{Z}$. As explained in Example 2.19 in the notes, this groupoid is equivalent to the fundamental groupoid of $\Omega T_{1,\mathbb{R}^2} \cong \Omega\mathbb{RP}^2$, so it should also be the target of a universal functor from $h\mathcal{C}_1^{\mathbb{R}}$ to a discrete groupoid. The goal of this exercise is to verify this directly, by geometric constructions.

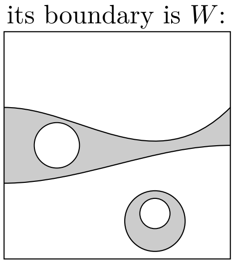
(a) Construct a functor

$$f\colon h\mathcal{C}_1^{\mathbb{R}} \longrightarrow D,$$

defined by sending an object (finite subset of $\mathbb{R}$) to its cardinality modulo 2, and a morphism $W \subset [0,t] \times \mathbb{R}$ from $M_0 \subset \mathbb{R}$ to $M_1 \subset \mathbb{R}$ to the integer

$$\chi(X) - \chi(X \cap (\{0\} \times \mathbb{R})),$$

where $X \subset [0,t] \times \mathbb{R}$ is the union of components of the complement $([0,t] \times \mathbb{R}) \smallsetminus W$ obtained by colouring them green or red in an alternating way, starting with green for the unbounded component in the $[0,t] \times \{-\infty\}$ direction, and setting X to be the union of the red components. More rigorously: a component C of $([0,t] \times \mathbb{R}) \smallsetminus W$ is included in the union X if and only if a ray starting from the interior of C, intersecting W transversely and asymptotically equal to $s \mapsto (t/2, -s)$, has an odd number of intersections with W. For example, in the following picture, the shaded region is X and its boundary is W:



Check that this indeed defines a functor as claimed.

(b) Verify by pictures that f factors over an equivalence $h\mathcal{C}_1^{\mathbb{R}}[(h\mathcal{C}_1^{\mathbb{R}})^{-1}] \simeq D$.

Problem 3 (Cobordism categories with tangential structures.)

As mentioned in the lectures, there are versions of the cobordism category for manifolds equipped with tangential structures, where a *tangential structure* is a $\mathrm{GL}_d(\mathbb{R})$-space Θ. A Θ-*structure* on a real vector bundle $E \to X$ is a $\mathrm{GL}_d(\mathbb{R})$-equivariant map $\mathrm{Fr}(E) \to \Theta$, where $\mathrm{Fr}(E)$ is the total space of the frame bundle of the vector bundle. As explained in the lectures, the topological cobordism category $\mathcal{C}_\Theta^V$ is defined similarly to $\mathcal{C}_d^V$, except that each object M is equipped with a Θ-structure on $\epsilon \oplus TM$ and each morphism W is equipped with a Θ-structure on TW.

(a) Unwind the definition of Θ-structure in the cases:
   (i) $\Theta = \{*\}$,
   (ii) $\Theta = \{\pm 1\}$, where an element A of $\mathrm{GL}_d(\mathbb{R})$ acts by multiplication by $\det(A)/|\det(A)|$,
   (iii) $\Theta = \mathrm{GL}_d(\mathbb{R})$, where $\mathrm{GL}_d(\mathbb{R})$ acts on itself by left-multiplication,
   (iv) $\Theta = \mathrm{GL}_{2n}(\mathbb{R})/\mathrm{GL}_n(\mathbb{C})$, when $d = 2n$ is even,
   (v) $\Theta = Z$, where $\mathrm{GL}_d(\mathbb{R})$ acts trivially on Z.

[1] Updated: July 10, 2019.

(b) Fill in the details of the definition of $\mathcal{C}_\Theta^V$ above. As before, $\mathcal{C}_\Theta$ is then defined as the colimit of $\mathcal{C}_\Theta^V$ over all finite-dimensional linear subspaces $V \subset \mathbb{R}^\infty$.
(c) Define the abstract cobordism category Cob_Θ and show that it is equivalent to $h\mathcal{C}_\Theta$.
(d) Describe explicitly the categories $\mathrm{Cob}_{\{\pm 1\}}$ and Cob_Z for $d = 1$ ($\mathrm{GL}_1(\mathbb{R})$ acts trivially on Z).

Problem 4 Let $\widetilde{\mathcal{C}}_d^V$ be the (ordinary) category obtained by giving the morphism spaces of $\mathcal{C}_d^V$ the discrete topology. What can you say about the connectivity of the map

$$B\widetilde{\mathcal{C}}_d^V \longrightarrow B\mathcal{C}_d^V?$$

Problem 5 For a space X we have the Postnikov truncation

$$X \longrightarrow \tau_{\leqslant n} X,$$

i.e. a map inducing isomorphisms on π_i for $i \leqslant n$, whose codomain has vanishing π_i for $i > n$. Explain how to apply this functor to all morphism spaces in a topologically enriched category C: try to construct a topologically enriched functor $C \to D$ which is a bijection on object sets, while $C(x, y) \to D(x, y)$ is a model for $C(x, y) \to \tau_{\leqslant n} C(x, y)$. Failing that, construct a zig-zag $C \leftarrow C' \to D$ of topologically enriched functors which are bijections on object sets, $C' \to D$ has the above property and $C' \to C$ is a weak equivalence on each morphism space. What does the case $n = 0$ have to do with hC? (And how should the case $n = -1$ be interpreted?)

Problem 6

(a) Prove that the "constant simplicial set" functor Sets $\to$ sSets is right adjoint to the functor $\pi_0 \colon$ sSets $\to$ Sets.
(b) Prove that the "discrete topology" functor Sets $\to$ Top does not have a left adjoint. (*Hint*: what is the space $\{0, 1\}^{\mathbb{N}}$?)