Problems related to lecture 3 of the GSS lecture course by Søren Galatius.

Problem 1 Let C be a small category and let $\gamma : C \to C[C^{-1}]$ the universal functor to a groupoid. In particular we have, for each object $x \in C$,

$$\gamma : \mathrm{End}_C(x) \to \mathrm{End}_{C[C^{-1}]}(x), \tag{1}$$

a monoid homomorphism into a group. The purpose of this problem is to work out some useful rules for determining the group $\mathrm{End}_{C[C^{-1}]}(x)$, under the additional assumption on C that for any other object y, both $C(x, y)$ and $C(y, x)$ are non-empty. Unless stated otherwise, we shall in the rest of this problem make this assumption on C and x.

(a) Prove that the image of (1) generates.

(b) Let $y \in C$ be any object, let $w_1, w_2 \in C(y, x)$ and $w_3, w_4 \in C(x, y)$ be any morphisms, and define $a, b, c, d \in \mathrm{End}_C(x)$ by

$$a = w_1 \circ w_3, \quad b = w_2 \circ w_3 \quad c = w_1 \circ w_4, \quad d = w_2 \circ w_4.$$

Then $\gamma(a)$, ..., $\gamma(d)$ are elements of the group $\mathrm{End}_{C[C^{-1}]}(x)$. Prove that

$$\gamma(a) \circ (\gamma(b))^{-1} = \gamma(c) \circ (\gamma(d))^{-1}. \tag{2}$$

Let us now specialize to $C \subset h\mathcal{C}_d^V$ the full subcategory on those objects admitting a morphism from $\emptyset$, and set $x = \emptyset$.

(c) Prove that this C and x satisfy the assumption above.

(d) Convince yourself that (for $\dim(V) > 0$) the domain of (1) is a commutative monoid, and deduce that the codomain is an abelian group.

(e) Convince yourself that, moreover, the domain of (1) is a *free* commutative monoid.

(f) Use the tools developed above to show that, in $\mathrm{Cob}_2[(\mathrm{Cob}_2)^{-1}]$, the endomorphisms of $\emptyset$ given by the torus, by the Klein bottle and by the empty surface are equal.

(g) Return to Problem 2(c) of problem set 1 (from Monday) and Problem 2(b) of problem set 2 (from Tuesday).

See also Bökstedt–Dupont–Svane: *A geometric interpretation of the homotopy groups of the cobordism category*, section 6.

Problem 2 Let D be a rigid symmetric monoidal groupoid, let 1 denote the monoidal unit, and let $x \in D$ be any object. Prove that

$$\begin{aligned} \mathrm{End}_D(1) &\to \mathrm{End}_D(x \otimes 1) \\ f &\mapsto f \otimes \mathrm{id}_x \end{aligned}$$

is an isomorphism of groups. Using the unitor and its inverse, the codomain may be identified with $\mathrm{End}_D(x)$.

Does this say anything useful when $D = \mathrm{Cob}_d[\mathrm{Cob}_d^{-1}]$? (Hint: first show that D is rigid.)

Problem 3 Recall that $\mathcal{C}_d$ is the topologically enriched cobordism category, where cobordisms are embedded in $[0, \infty) \times \mathbb{R}^\infty$ (or alternatively in $[0, \infty) \times V$ for $\dim(V) \gg d$, if you prefer not to take a colimit). Let $\mathcal{C}_d^k$, for an integer $k \geqslant 0$, be the subcategory with the same objects and whose morphisms are those cobordisms $W \subset [0, t] \times V$ with the property that the inclusion of the outgoing boundary

$$W \cap (\{t\} \times V) \longrightarrow W$$

is k-connected.

(a) Verify that this is indeed a subcategory of $\mathcal{C}_d$.

In the lectures it was stated that the inclusion $\mathcal{C}_d^k \hookrightarrow \mathcal{C}_d$ induces a weak homotopy equivalence of classifying spaces

$$B\mathcal{C}_d^k \longrightarrow B\mathcal{C}_d \tag{3}$$

as long as $k \leqslant \frac{d-2}{2}$. The purpose of this problem is to investigate, by more elementary means, when the map (3) induces a bijection on π_0.

(b) Prove "by hand" that, when $d \geqslant 2$ and $k = 0$, the map (3) induces a bijection on π_0.
(c) Rephrase bijectivity of (3) on π_0 as the statement that, given any cobordism $W\colon M_0 \rightsquigarrow M_1$ in $\mathcal{C}_d$, there is a zig-zag of cobordisms between M_0 and M_1 that each satisfy the k-connectivity condition on the outgoing boundary.
(d) Prove bijectivity of (3) on π_0 more generally whenever $k < d/2$. (*Hint*: if you haven't already, learn about *elementary cobordisms* e.g. from Milnor's book on the h-cobordism theorem.)

Problem 4 Let D be the groupoid with one object $*$ and $\mathrm{End}_D(*) = \mathbb{Z}$. Define

$$E : \mathrm{Cob}_d \to D$$

by sending any object to $*$ and a morphism $W : M_0 \rightsquigarrow M_1$ to $\chi(W) - \chi(M_0) \in \mathbb{Z}$.

(a) Briefly explain why this is a functor.
(You may have already done this in Problem 2(b) on problem set 1 on Monday.)
(b) Can you promote E to a symmetric monoidal functor? (Hint: use addition in $\mathbb{Z}$ as $\otimes$. In this case the associator, symmetry, and unitor can all be taken to be the identity.)

This is sometimes called the "Euler TQFT".

(c) Let $V = \mathbb{R}^m$, let $e\colon T_{d,\mathbb{R}^{m+1}} \to K(\mathbb{Z}, m+1)$ be a map, and let $\Omega^m T_{d,\mathbb{R}^{m+1}} \to \Omega^m K(\mathbb{Z}, m+1)$ be the m-fold loop of e. Prove that the fundamental groupoid of $\Omega^m K(\mathbb{Z}, m+1)$ is equivalent to D, and explain how any such map e gives rise to a symmetric monoidal invertible field theory $\mathrm{Cob}_d \to D$ if $m \geqslant 3$.

Problem 5 If X is a rigid symmetric monoidal groupoid, it is determined up to equivalence by three pieces of data: $\pi_0 X$ (the abelian group of isomorphism classes of objects), $\pi_1 X = \mathrm{Aut}(1_X)$, and something called the *k-invariant*, which we proceed to define. Given any $x \in X$, there is a canonical isomorphism $-\otimes \mathrm{id}_x\colon \mathrm{Aut}(1_X) \to \mathrm{Aut}(x)$. The k-invariant of X is the map $\pi_0 X \otimes \mathbb{Z}/2 \to \pi_1 X$ which to $x \in \pi_0 X$ assigns the image of the symmetry $\sigma\colon x \otimes x \to x \otimes x$ in $\mathrm{Aut}(x \otimes x) \cong \mathrm{Aut}(1_X) = \pi_1 X$.

Compute π_0, π_1, and k for the following rigid symmetric monoidal groupoids.

(a) The category $\mathrm{Vect}_k^{\sim}$ of invertible vector spaces over a field k.
(b) Assuming $\mathrm{char}(k) \neq 2$, the category $\mathrm{sVect}_k^{\sim}$ of invertible *super vector spaces*, i.e. the category of invertible $\mathbb{Z}/2$-graded vector spaces with the symmetry $a \otimes b \mapsto (-1)^{\deg a \deg b} b \otimes a$.
(c) $\mathrm{Cob}_1[\mathrm{Cob}_1^{-1}]$.
(d) The same as in (c), but with the oriented 1-dimensional cobordism category.

Problem 6 Let us denote by $h\mathring{\mathcal{C}}_d^V$ the oriented version of the category $h\mathcal{C}_d^V$, where both $(d-1)$-manifolds and cobordisms are equipped with compatible orientations.[1] There is a functor

$$F_{d,V}\colon h\mathring{\mathcal{C}}_d^V \longrightarrow h\mathcal{C}_d^V$$

that forgets all orientations.

(a) When $d = \dim(V)$ or $d = 0$, construct a section of $F_{d,V}$.
(b) When $0 < d < \dim(V)$, prove that the functor $F_{d,V}$ does not admit a section.

(*Suggestion*: first consider the cases $(d, V) = (1, \mathbb{R})$ and $(d, V) = (1, \mathbb{R}^2)$, and look at the picture in Problem 2 of problem set 2.)

[1] This was called $h\mathcal{C}_{\{\pm 1\}}^V$ in Problem 3 of problem set 2.