Stability and stable homology for moduli spaces of disconnected submanifolds

Martin Palmer-Anghel
Universität Bonn

PCMI, Utah, 12 July 2019

Definition

Fix a choice of

$$e \colon L \hookrightarrow \partial \bar{M}$$

where

- L is a closed, connected, smooth manifold,
- $\bar{M}$ is a connected, smooth manifold.
  Write $M = \mathrm{int}(\bar{M})$.

Then

$$C_{nL}(M) = \text{path-component of } \frac{\mathrm{Emb}\left(\sqcup_n L, M\right)}{\mathrm{Diff}\left(\sqcup_n L\right)}$$

containing $[ne]$

where

- the embedding ne is n parallel copies of e in a collar neighbourhood of $\partial \bar{M}$.

Moduli space of n unlinked copies of L in M.

Examples

- Configuration spaces $(L = \text{point})$
- Space of n-component unlinks in $\mathbb{R}^3$
  $(e \colon L = S^1 \hookrightarrow \partial \mathbb{B}^3)$

Stability and stable homology for moduli spaces of disconnected submanifolds

Martin Palmer-Anghel

Universität Bonn

PCMI, Utah, 12 July 2019

Remark

- $\pi_1 C_{nL}(M) =$ *motion groups* of $(L \hookrightarrow \partial\bar{M})$

(a) • (point $\hookrightarrow \partial S$) $\rightsquigarrow$ surface braid groups
(b) • $(S^1 \hookrightarrow \mathbb{B}^3)$ $\rightsquigarrow$ extended loop-braid groups

Aim Understand $H_*(C_{nL}(M))$ ... *in a stable range.*

Remark

This is typically not the same as $H_*(B\pi_1 C_{nL}(M))$.

- In example (a), it is the same if $\mathbb{S}^2 \neq S \neq \mathbb{RP}^2$.
  [Fadell-Neuwirth]
- In example (b), it is not the same:

  H_i(loop braid groups) $\neq 0$ for infinitely many i
  since the loop braid groups contain torsion

  $H_i(C_{nS^1}(\mathbb{R}^3)) = 0$ for all $i > 6n$
  since $C_{nS^1}(\mathbb{R}^3) \simeq 6n$-dimensional manifold
  [Brendle-Hatcher]

Moduli spaces $C_{nL}(M)$ $n \to \infty$ Singularities $n = \infty$

Definition

Fix a choice of

$$e \colon L \longrightarrow \partial\bar{M}$$

where

- L is a closed, connected, smooth manifold,
- $\bar{M}$ is a connected, smooth manifold.
  Write $M = \mathrm{int}(\bar{M})$.

Then

$$C_{nL}(M) = \text{path-component of } \frac{\mathrm{Emb}\left(\bigsqcup_n L, M\right)}{\mathrm{Diff}\left(\bigsqcup_n L\right)}$$

containing $[ne]$

where

- the embedding ne is n parallel copies of e in a collar neighbourhood of $\partial\bar{M}$.

Moduli space of n unlinked copies of L in M.

Examples

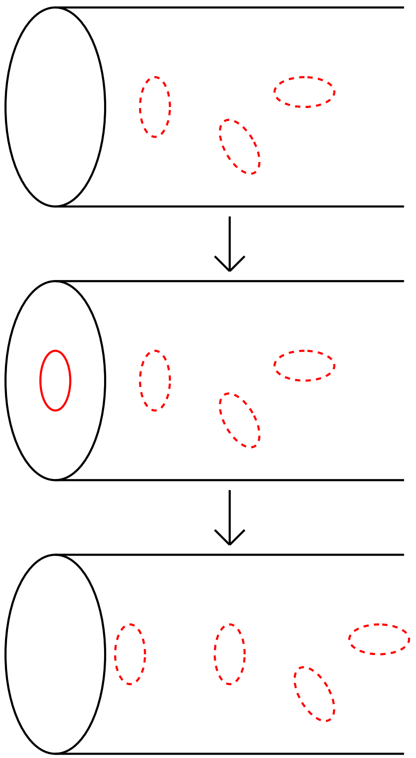
- Configuration spaces $(L = \text{point})$
- Space of n-component unlinks in $\mathbb{R}^3$
  $(e \colon L = S^1 \longrightarrow \partial\mathbb{B}^3)$

Consider $n \longrightarrow \infty$

Definition (*Stabilisation maps*)

$$s\colon C_{nL}(M) \longrightarrow C_{(n+1)L}(M)$$

- Adjoin $[e]$ to the configuration
  $\rightsquigarrow$ $n+1$ copies of L in $\bar{M}$
- Push new configuration inwards along collar nbhd
  $\rightsquigarrow$ $n+1$ copies of L in M



Moduli spaces $C_{nL}(M)$ $n \to \infty$ Singularities $n = \infty$

Remark

- $\pi_1 C_{nL}(M) =$ *motion groups* of $(L \hookrightarrow \partial\bar{M})$

(a) • (point $\hookrightarrow \partial S$) $\rightsquigarrow$ surface braid groups
(b) • $(S^1 \hookrightarrow \mathbb{B}^3)$ $\rightsquigarrow$ extended loop-braid groups

Aim Understand $H_*(C_{nL}(M))$ … *in a stable range.*

Remark

This is typically not the same as $H_*(B\pi_1 C_{nL}(M))$.

- In example (a), it is the same if $\mathbb{S}^2 \neq S \neq \mathbb{RP}^2$.
  [Fadell-Neuwirth]
- In example (b), it is not the same:

  H_i(loop braid groups) $\neq 0$ for infinitely many i
  since the loop braid groups contain torsion

  $H_i(C_{nS^1}(\mathbb{R}^3)) = 0$ for all $i > 6n$
  since $C_{nS^1}(\mathbb{R}^3) \simeq 6n$-dimensional manifold
  [Brendle-Hatcher]

When $L =$ point:

Theorem (McDuff, Segal)

(a) *The map s induces isomorphisms on homology up to degree $n/2$.*

(b) *Construct "computable" spaces $X(M)$ such that*

$$\lim_{n\to\infty} H_*(C_n(M)) \cong H_*(X(M)).$$

For example: $X(\mathbb{R}^d) = \Omega^d_\bullet S^d$

Where $\Omega^d_\bullet(-) =$ one path-component of $\mathrm{Map}_*(S^d, -)$.

When $\dim(L) > 0$:

Theorems

- *In the case of $C_{nS^1}(\mathbb{R}^3)$, the map s induces isomorphisms on homology up to degree $n/2$.* [Kupers, 2013]
- *If $\dim(L) \leqslant \frac{1}{2}(\dim(M) - 3)$, the map s induces isomorphisms on homology up to degree $n/2$.* [P., 2018]

Remark For the extended loop-braid groups LB_n, we also have:

- Homological stability for LB_n [Hatcher-Wahl, 2010]
- Calculation of integral homology of LB_n [Griffin, 2013]

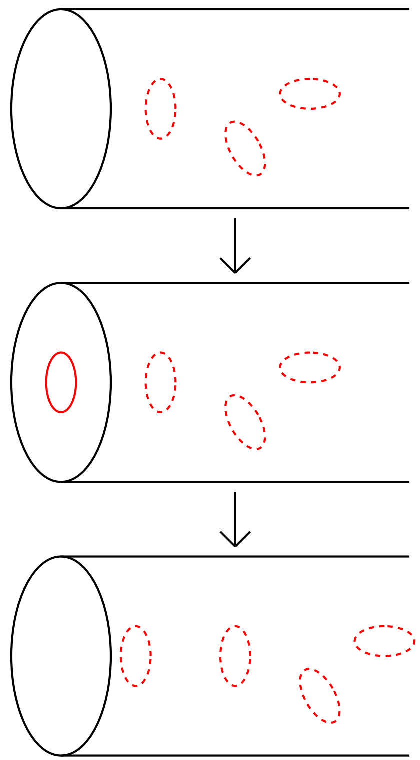
Moduli spaces $C_{nL}(M)$ $n \to \infty$ Singularities $n = \infty$

Consider $n \longrightarrow \infty$

Definition (*Stabilisation maps*)

$$s\colon C_{nL}(M) \longrightarrow C_{(n+1)L}(M)$$

- Adjoin $[e]$ to the configuration
  $\rightsquigarrow$ $n+1$ copies of L in $\bar{M}$
- Push new configuration inwards along collar nbhd
  $\rightsquigarrow$ $n+1$ copies of L in M



Sketch of proof

General strategy:

- Build a simplicial complex X_n of *"ways to undo the map $s\colon C_{nL}(M) \to C_{(n+1)L}(M)$"*
- Prove that X_n is *highly-connected* $(\pi_{\leqslant n/2} = 0)$

$$\boxed{\begin{matrix}\text{Homological stability machine}\\ \text{[Quillen, ...]}\end{matrix}} \rightsquigarrow \square$$

Here is a "toy model" of the complex X_n in our case:

- Fix $f \in \mathrm{Emb}(\sqcup_n L, M)$
- Vertices:
$$\left\{ e\colon L \times [0,1] \hookrightarrow \bar{M} \;\middle|\; \begin{matrix} e(L \times \{0\}) \subseteq \partial \bar{M} \\ e(L \times \{1\}) \subseteq f\left(\sqcup_n L\right) \end{matrix} \right\}$$
- A set $\{e_0, \ldots, e_p\}$ spans a p-simplex if and only if the images $e_i(L \times [0,1))$ are pairwise disjoint.

This is *contractible*:

- Any map $S^i \to X_n$ lands in $\mathrm{span}_{X_n}(e_1, \ldots, e_k)$
- Transversality & $2.\dim(L \times [0,1]) < \dim(M) \Rightarrow$

$$\begin{matrix} D^{i+1} & \longrightarrow & \mathrm{Cone}(\mathrm{span}_{X_n}(e_1, \ldots, e_k)) & & \\ \cup & & \wr\Vert & & \\ S^i & & \mathrm{span}_{X_n}(e_1, \ldots, e_k, \bar{e}) & \subset & X_n \end{matrix}$$

Moduli spaces $C_{nL}(M)$ $n \to \infty$ Singularities $n = \infty$

When $L =$ point:

Theorem (McDuff, Segal)

(a) *The map s induces isomorphisms on homology up to degree $n/2$.*

(b) *Construct "computable" spaces $X(M)$ such that*
$$\lim_{n\to\infty} H_*(C_n(M)) \;\cong\; H_*(X(M)).$$

For example: $X(\mathbb{R}^d) = \Omega^d_\bullet S^d$

Where $\Omega^d_\bullet(-) =$ one path-component of $\mathrm{Map}_*(S^d, -)$.

When $\dim(L) > 0$:

Theorems

- *In the case of $C_{nS^1}(\mathbb{R}^3)$, the map s induces isomorphisms on homology up to degree $n/2$.* [Kupers, 2013]
- *If $\dim(L) \leqslant \frac{1}{2}(\dim(M) - 3)$, the map s induces isomorphisms on homology up to degree $n/2$.* [P., 2018]

Remark For the extended loop-braid groups LB_n, we also have:

- Homological stability for LB_n [Hatcher-Wahl, 2010]
- Calculation of integral homology of LB_n [Griffin, 2013]

Manifolds with conical singularities

Fix a closed, smooth $(d-1)$-manifold P and let

$$\mathrm{Cone}(P) = (P \times [0,\infty)) \,/\, (P \times \{0\})$$

Definition (*Manifold with conical P-singularities*)

- space M
- discrete subset $A \subset M$ (set of singularities)
- smooth d-dimensional atlas on $M \smallsetminus A$
- $\forall a \in A$: *germ* of (U_a, u_a), where
  - U_a is an open neighbourhood of a in M
  - $u_a\colon U_a \to \mathrm{Cone}(P)$ is a homeomorphism
    - taking a to the tip of the cone,
    - restricting to a diffeomorphism

$$U_a \smallsetminus \{a\} \cong P \times (0,\infty)$$

Definition ($\mathrm{Diff}^P(M)$)

Homeomorphisms $\varphi\colon M \to M$ that

- fix A setwise
- act on $M \smallsetminus A$ by a diffeomorphism
- act on $\partial(M \smallsetminus A)$ by the identity
- act on $\bigsqcup\{U_a \mid a \in A\}$ by $\mathrm{Diff}(P)^A \rtimes \mathfrak{S}_A$

  (φ acts "cylindrically" near each singularity)

Example

- Graph of uniform valency v $P = \{1,2,\ldots,v\}$

Sketch of proof

General strategy:

- Build a simplicial complex X_n of *"ways to undo the map $s\colon C_{nL}(M) \to C_{(n+1)L}(M)$"*
- Prove that X_n is *highly-connected* $(\pi_{\leqslant n/2} = 0)$

$$\boxed{\begin{matrix}\text{Homological stability machine}\\ \text{[Quillen, ...]}\end{matrix}} \rightsquigarrow \square$$

Here is a "toy model" of the complex X_n in our case:

- Fix $f \in \mathrm{Emb}(\bigsqcup_n L, M)$
- Vertices:

$$\left\{ e\colon L \times [0,1] \hookrightarrow \bar{M} \;\middle|\; \begin{matrix} e(L \times \{0\}) \subseteq \partial \bar{M} \\ e(L \times \{1\}) \subseteq f\left(\bigsqcup_n L\right) \end{matrix} \right\}$$

- A set $\{e_0,\ldots,e_p\}$ spans a p-simplex if and only if the images $e_i(L \times [0,1))$ are pairwise disjoint.

This is *contractible*:

- Any map $S^i \to X_n$ lands in $\mathrm{span}_{X_n}(e_1,\ldots,e_k)$
- Transversality & $2.\dim(L \times [0,1]) < \dim(M) \Rightarrow$

$$\begin{matrix} D^{i+1} & \longrightarrow & \mathrm{Cone}(\mathrm{span}_{X_n}(e_1,\ldots,e_k)) & & \\ \cup & & \wr\| & & \\ S^i & & \mathrm{span}_{X_n}(e_1,\ldots,e_k,\bar{e}) & \subset & X_n \end{matrix}$$

More examples

For a submanifold $N \subset M$, let

$M /\!\!/ N$ = result of collapsing each component of N to a point

- $\mathbb{R}^3 /\!\!/ \mathcal{L}$ for a link $\mathcal{L}$ $\qquad P = S^1 \times S^1$
- $M /\!\!/ c_n$ for a point $c_n \in C_{nL}(M)$ $\qquad P = \partial T$

where $T = \mathrm{Tub}(L \hookrightarrow M)$

Theorem (P., 2018)

$$H_i(B\mathrm{Diff}^{\partial T}(M /\!\!/ c_n)) \text{ stabilises as } n \to \infty$$

as long as $\dim(L) \leqslant \frac{1}{2}(\dim(M) - 3)$.

Sketch of proof ... that $B\mathrm{Diff}_\partial(M, c_n)$ stabilises

$$\begin{aligned} C_{nL}(M) &= \text{path-comp}^{\text{t}} \text{ of } \mathrm{Emb}(c_n, M)/\mathrm{Diff}(c_n) \\ &= \text{orbit of } \mathrm{Emb}(c_n, M)/\mathrm{Diff}(c_n) \curvearrowleft \mathrm{Diff}_\partial(M) \end{aligned}$$

isotopy extension theorem

$$\cong \mathrm{Diff}_\partial(M)/\mathrm{Diff}_\partial(M, c_n)$$

topological orbit-stabiliser theorem

$$\cong \text{ fibre of } \Phi$$

$$\begin{array}{ccc} \dfrac{\mathrm{Emb}(M, \mathbb{R}^\infty)}{\mathrm{Diff}_\partial(M, c_n)} & \xrightarrow{\quad\Phi\quad} & \dfrac{\mathrm{Emb}(M, \mathbb{R}^\infty)}{\mathrm{Diff}_\partial(M)} \\ \| & & \| \\ B\mathrm{Diff}_\partial(M, c_n) & & B\mathrm{Diff}_\partial(M) \end{array}$$

Manifolds with conical singularities

Fix a closed, smooth $(d-1)$-manifold P and let

$$\mathrm{Cone}(P) = (P \times [0, \infty)) \;/\; (P \times \{0\})$$

Definition (*Manifold with conical P-singularities*)

- space M
- discrete subset $A \subset M$ (set of singularities)
- smooth d-dimensional atlas on $M \setminus A$
- $\forall a \in A$: *germ* of (U_a, u_a), where
  - U_a is an open neighbourhood of a in M
  - $u_a \colon U_a \to \mathrm{Cone}(P)$ is a homeomorphism
    - taking a to the tip of the cone,
    - restricting to a diffeomorphism $U_a \setminus \{a\} \cong P \times (0, \infty)$

Definition ($\mathrm{Diff}^P(M)$)

Homeomorphisms $\varphi \colon M \to M$ that

- fix A setwise
- act on $M \setminus A$ by a diffeomorphism
- act on $\partial(M \setminus A)$ by the identity
- act on $\bigsqcup\{U_a \mid a \in A\}$ by $\mathrm{Diff}(P)^A \rtimes \mathfrak{S}_A$

(φ acts "cylindrically" near each singularity)

Example

- Graph of uniform valency v $\qquad P = \{1, 2, \ldots, v\}$

Consider $n = \infty$

Restrict to the case of $M = \mathbb{R}^d$ and write

$$C_{\infty L}(\mathbb{R}^d) = \underset{\substack{n \to \infty \\ s}}{\operatorname{colim}}\ C_{nL}(\mathbb{R}^d)$$

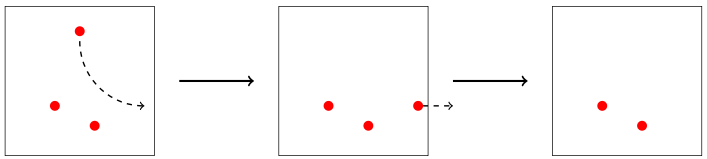
When $L =$ point: McDuff-Segal prove that

$$H_*(C_\infty(\mathbb{R}^d)) \cong H_*(\Omega^d_\bullet S^d).$$

(1) They construct a homology-equivalence:

$$\begin{array}{ccc} C_\infty(\mathbb{R}^d) & \xrightarrow{H_* \cong} & \Omega^d_\bullet Z(\mathbb{R}^d) \\ & & \| \\ & & \coprod_n C_n(\mathbb{R}^d) \big/ \sim \end{array}$$

$$c \sim d \;\Leftrightarrow\; c \cap (0,1)^d = d \cap (0,1)^d$$



(2) Geometric argument:

$$Z(\mathbb{R}^d) \simeq (\mathbb{R}^d)^+ = S^d$$

More examples

For a submanifold $N \subset M$, let

$M /\!\!/ N =$ result of collapsing each component of N to a point

- $\mathbb{R}^3 /\!\!/ \mathcal{L}$ for a link $\mathcal{L}$ $\qquad P = S^1 \times S^1$
- $M /\!\!/ c_n$ for a point $c_n \in C_{nL}(M)$ $\qquad P = \partial T$

where $T = \mathrm{Tub}(L \hookrightarrow M)$

Theorem (P., 2018)

$H_i(B\mathrm{Diff}^{\partial T}(M /\!\!/ c_n))$ stabilises as $n \to \infty$

as long as $\dim(L) \leqslant \frac{1}{2}(\dim(M) - 3)$.

Sketch of proof … that $B\mathrm{Diff}_\partial(M, c_n)$ stabilises

$$\begin{aligned} C_{nL}(M) &= \text{path-comp}^{\text{t}} \text{ of } \mathrm{Emb}(c_n, M)/\mathrm{Diff}(c_n) \\ &= \text{orbit of } \mathrm{Emb}(c_n, M)/\mathrm{Diff}(c_n) \curvearrowleft \mathrm{Diff}_\partial(M) \end{aligned}$$

isotopy extension theorem

$$\cong \mathrm{Diff}_\partial(M)/\mathrm{Diff}_\partial(M, c_n)$$

topological orbit-stabiliser theorem

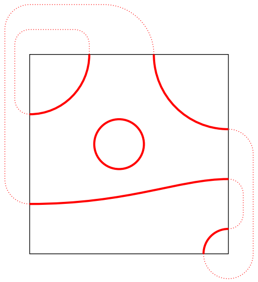
$$\cong \text{ fibre of } \Phi$$

$$\begin{array}{ccc} \dfrac{\mathrm{Emb}(M, \mathbb{R}^\infty)}{\mathrm{Diff}_\partial(M, c_n)} & \xrightarrow{\Phi} & \dfrac{\mathrm{Emb}(M, \mathbb{R}^\infty)}{\mathrm{Diff}_\partial(M)} \\ \| & & \| \\ B\mathrm{Diff}_\partial(M, c_n) & & B\mathrm{Diff}_\partial(M) \end{array}$$

Guess for $\dim(L) > 0$:

$$C_{\infty L}(\mathbb{R}^d) \xrightarrow{??} \Omega^d_\bullet Z_L(\mathbb{R}^d) = \coprod_n C_{nL}(\mathbb{R}^d) \big/ \sim$$

$$c \sim d \iff c \cap (0,1)^d = d \cap (0,1)^d$$

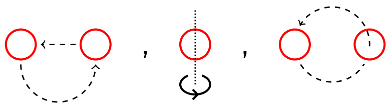
e.g.  $\in Z_{S^1}(\mathbb{R}^2)$

Geometric argument:

$$\begin{aligned} Z_L(\mathbb{R}^d) &\simeq T_{\dim(L),\mathbb{R}^d} \\ &= \{\text{affine } \dim(L)\text{-planes in } \mathbb{R}^d\}^+ \end{aligned}$$

Counterexample: in the case $L = S^1 \hookrightarrow \partial\mathbb{B}^3$

- $H_1(\Omega^3_\bullet T_{1,\mathbb{R}^3}) \otimes \mathbb{Q} \cong \mathbb{Q}$
- $H_1(C_{nS^1}(\mathbb{R}^3)) \cong (\mathbb{Z}/2\mathbb{Z})^3$ (for $n \geqslant 2$)

$= (\mathbb{Z}/2\mathbb{Z})\{$  $\}$

[Brendle-Hatcher, 2010]

Moduli spaces $C_{nL}(M)$ $n \to \infty$ Singularities $n = \infty$

Consider $n = \infty$

Restrict to the case of $M = \mathbb{R}^d$ and write

$$C_{\infty L}(\mathbb{R}^d) = \underset{\substack{n\to\infty \\ s}}{\operatorname{colim}}\, C_{nL}(\mathbb{R}^d)$$

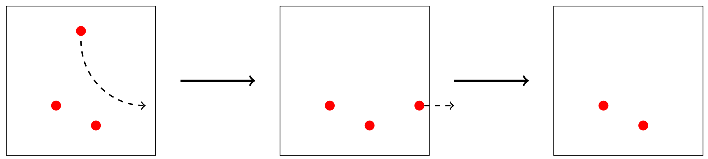
When $L =$ point: McDuff-Segal prove that

$$H_*(C_\infty(\mathbb{R}^d)) \cong H_*(\Omega^d_\bullet S^d).$$

(1) They construct a homology-equivalence:

$$C_\infty(\mathbb{R}^d) \xrightarrow{H_* \cong} \Omega^d_\bullet Z(\mathbb{R}^d) = \coprod_n C_n(\mathbb{R}^d) \big/ \sim$$

$$c \sim d \iff c \cap (0,1)^d = d \cap (0,1)^d$$



(2) Geometric argument:

$$Z(\mathbb{R}^d) \simeq (\mathbb{R}^d)^+ = S^d$$

New idea:

Definition

$$Z_L(\mathbb{R}^d) \supseteq \hat{Z}_L(\mathbb{R}^d)$$

Those submanifolds of $\mathbf{I}^d$ that are disjoint from the *union of orthogonal hyperplanes*

$$\mathbf{I}^{i-1} \times \{t_i\} \times \mathbf{I}^{d-i}$$

for some $t_1, \dots, t_d \in \mathbf{I} = (0,1)$.

Theorem (P., 2019 (*in progress*))

There is a (twisted-)homology-equivalence

$$C_{\infty L}(\mathbb{R}^d) \longrightarrow \Omega^d_\bullet \hat{Z}_L(\mathbb{R}^d).$$

Remark

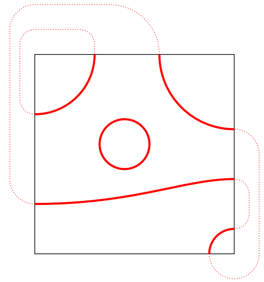
In general, $\hat{Z}_L(\mathbb{R}^d) \not\simeq T_{\dim(L),\mathbb{R}^d}$

$$\hat{Z}_{\text{point}}(\mathbb{R}^d) = Z_{\text{point}}(\mathbb{R}^d) \simeq T_{0,\mathbb{R}^d} = S^d$$
$$\hat{Z}_{S^1}(\mathbb{R}^3) \not\simeq T_{1,\mathbb{R}^3}$$

Guess for $\dim(L) > 0$:

$$\begin{array}{ccc} C_{\infty L}(\mathbb{R}^d) & \xrightarrow{??} & \Omega^d_\bullet Z_L(\mathbb{R}^d) \\ & & \| \\ & & \coprod_n C_{nL}(\mathbb{R}^d)/\sim \end{array}$$

$$c \sim d \iff c \cap (0,1)^d = d \cap (0,1)^d$$

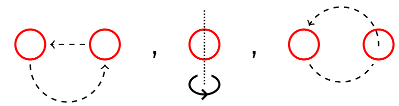
e.g.  $\in Z_{S^1}(\mathbb{R}^2)$

Geometric argument:

$$Z_L(\mathbb{R}^d) \simeq T_{\dim(L),\mathbb{R}^d}$$
$$= \{\text{affine } \dim(L)\text{-planes in } \mathbb{R}^d\}^+$$

Counterexample: in the case $L = S^1 \hookrightarrow \partial\mathbb{B}^3$

- $H_1(\Omega^3_\bullet T_{1,\mathbb{R}^3}) \otimes \mathbb{Q} \cong \mathbb{Q}$
- $H_1(C_{nS^1}(\mathbb{R}^3)) \cong (\mathbb{Z}/2\mathbb{Z})^3$ (for $n \geqslant 2$)

$= (\mathbb{Z}/2\mathbb{Z})\{$  $\}$

[Brendle-Hatcher, 2010]

Thank you for your attention!

New idea:

Definition

$$Z_L(\mathbb{R}^d) \supseteq \hat{Z}_L(\mathbb{R}^d)$$

Those submanifolds of $\mathbf{I}^d$ that are disjoint from the *union of orthogonal hyperplanes*

$$\mathbf{I}^{i-1} \times \{t_i\} \times \mathbf{I}^{d-i}$$

for some $t_1, \ldots, t_d \in \mathbf{I} = (0, 1)$.

Theorem (P., 2019 (*in progress*))

There is a (twisted-)homology-equivalence

$$C_{\infty L}(\mathbb{R}^d) \longrightarrow \Omega^d_\bullet \hat{Z}_L(\mathbb{R}^d).$$

Remark

In general, $\hat{Z}_L(\mathbb{R}^d) \not\simeq T_{\dim(L),\mathbb{R}^d}$

$$\hat{Z}_{\text{point}}(\mathbb{R}^d) = Z_{\text{point}}(\mathbb{R}^d) \simeq T_{0,\mathbb{R}^d} = S^d$$
$$\hat{Z}_{S^1}(\mathbb{R}^3) \not\simeq T_{1,\mathbb{R}^3}$$