Functorial homological representations of motion groups and mapping class groups, polynomiality and twisted homological stability

Martin Palmer-Anghel // Talk at the topology seminar, IMAR, 20 December 2019

Abstract.

The classical braid groups (on at least 3 strands) are known to have "wild" representation theory, so there is no easy classification system for their representations.[1] It is therefore useful to be able to construct representations of the braid groups geometrically or topologically, so that they may be studied using geometry and topology.

Important examples of topologically-defined representations are the *Lawrence-Bigelow representations*, introduced by Ruth Lawrence in 1990 and later generalised by Stephen Bigelow to a construction that inputs a B_m-representation V over k and outputs a B_n-representation over $k[t^{\pm 1}]$ for all n.[2] Other examples are given by the *Long-Moody construction*, which inputs a B_m-representation V over k and outputs a B_{m-1}-representation over k. Both of these constructions are topological, induced by the action (up to homotopy) of B_n on a space equipped with a certain local system.

Braid groups are simultaneously examples of *motion groups* and of *mapping class groups*; other examples are surface braid groups, loop braid groups, mapping class groups of surfaces and automorphism groups of free groups. I will describe a general construction of "*homological representations*" for motion groups and mapping class groups that recovers the constructions of Lawrence-Bigelow and Long-Moody for the classical braid groups (so in a sense it "unifies" these constructions).

Moreover, the construction produces "coherent" families of representations, in the sense that they extend to a functor on a category with object set $\mathbb{N}$ and whose automorphism groups are the family of groups under consideration (e.g. braid groups on any number of strands). The richer structure of this category may then be used

(i) to organise the representation theory of the family of groups, and
(ii) to prove twisted homological stability results, via a certain notion of polynomiality.

I will explain why, via our general homological construction, the Lawrence-Bigelow representations $\mathcal{LB}_{m,n}$ may be extended to such a polynomial functor, and deduce that the twisted homology groups $H_i(B_n; \mathcal{LB}_{m,n})$ are stable (independent of n for $n \gg i$) for any fixed m.

This represents joint work with Arthur Soulié.
Preliminary version of our preprint: `arXiv:1910.13423v1`.

[1] More precisely, their representation theory is "wild" in the sense that the representation theory of the free group F_2 may be embedded into the representation theory of B_n for any $n \geqslant 3$. This also implies that the representation theory of B_n (for any fixed $n \geqslant 3$) contains the representation theory of all finite groups, and that there are k-parameter families of irreducible representations of B_n for arbitrarily large k.

[2] Lawrence's original construction $\mathcal{LB}_{m,n}$ corresponds to the case $V = k = \mathbb{Z}[q^{\pm 1}] = \mathbb{Z}[\mathbb{Z}]$, where B_m acts through its abelianisation, for $m \geqslant 2$, and $V = k = \mathbb{Z}$, for $m = 1$.