Kontsevich characteristic classes à la Watanabe

(II)

GeMAT seminar

IMAR

19 June 2020

Following T. Watanabe

"Some exotic nontrivial elements of the rational homotopy groups of $\mathrm{Diff}(S^4)$"

arXiv: 1812.02448

Plan.

Picture:

$$\pi_{k-1}\mathrm{Diff}_\partial(D^4)$$

$$\begin{tikzcd}
\pi_k B\mathrm{Diff}_\partial(D^4) \arrow[rr, "\hat{Z}_k \text{ (today)}"] \arrow[dr] & & \mathcal{A}_k \\
& \pi_k B\mathrm{Diff}_\partial(D^4)\otimes\mathbb{Q} \arrow[ur, "(*)"] & \\
& & \mathbb{Q}\left\{\begin{matrix}\text{labelled trivalent graphs without}\\ \text{loops or multiple edges}\end{matrix}\right\} \arrow[ul, "\text{talk 3}"] \arrow[uu, two heads, "\text{talk 4}"'] \\
& & \mathbb{Z}\left\{\begin{matrix}\text{labelled trivalent graphs without}\\ \text{loops or multiple edges}\end{matrix}\right\} \arrow[u, hook] \arrow[uull, bend left]
\end{tikzcd}$$

$$\pi_{k-1}\mathrm{Diff}_\partial(D^4) = \pi_k B\mathrm{Diff}_\partial(D^4)$$

$$\mathcal{A}_k = \mathbb{Q}\left\{\begin{matrix}\text{labelled}\\ \text{trivalent}\\ \text{graphs}\end{matrix}\right\}^{\text{on } 2k \text{ vertices}} \Big/ \text{IHX}, \ldots$$

$\Rightarrow$ $(*)$ is surjective.

$$\dim_{\mathbb{Q}}\left(\pi_k B\mathrm{Diff}_\partial(D^4)\otimes\mathbb{Q}\right) \geqslant \dim_{\mathbb{Q}}(\mathcal{A}_k) = \begin{cases} 1 & \text{for } k = 2,5,9 \\ \ldots\ldots & \end{cases}$$

$\Rightarrow$ "exotic" elements of $\pi_{k-1}\mathrm{Diff}(S^4)\otimes\mathbb{Q}$.

Today:

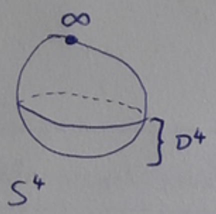
- recollection of admissible propagators —— proof of $\exists$
- "microscopic" configurations in $\mathbb{R}^d$
- construction of $\hat{Z}_k$
- sketch proof that $\tilde{Z}_k$ is well-defined (indep. of propagators)
- proof that $\hat{Z}_k$ only depends on the <u>bordism class</u> of a (D^4,∂)-bundle.

Recollections

$$\mathrm{Graphs}^{\circ}_{k} = \left\{ \text{trivalent graphs with } \begin{array}{l} \text{vertices} \leftrightarrow \{1 \ldots 2k\} \\ \text{edges} \leftrightarrow \{1 \ldots 3k\} \end{array} \right\}$$

$$= \left\{ \text{2-valued functions } \{1 \ldots 3k\} \dashrightarrow \{1 \ldots 2k\} \right\}$$

$$A_k = \mathbb{Q}\langle \mathrm{Graphs}^{\circ}_{k} \rangle \Big/ \left(\begin{array}{l} \text{IHX relation}, \ \Gamma \sim \Gamma \text{ with 2 vertices swapped}, \\ -\Gamma \sim \Gamma \text{ with 2 edges swapped} \end{array} \right)$$



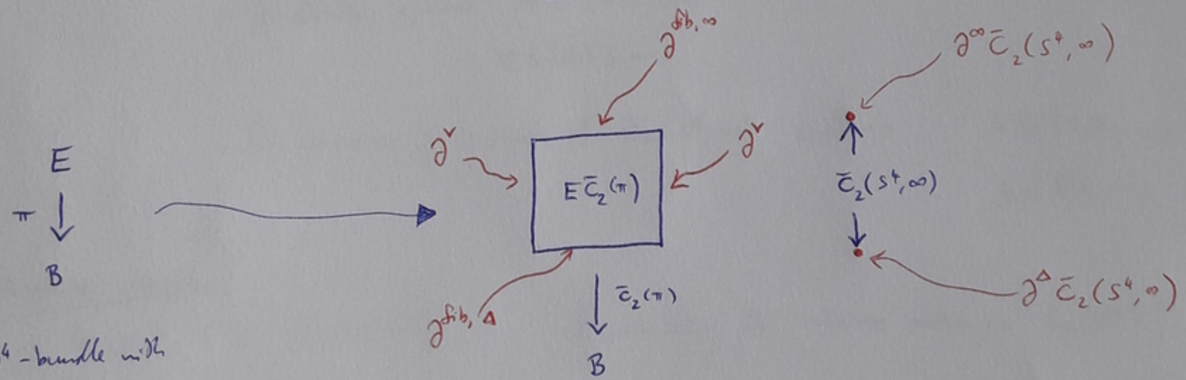
$\bar{C}_n(S^4, \infty)$ = Fulton MacPherson compactification of $C_n(S^4 - \{\infty\})$

ordered configuration space.

Facts:

- $$\underbrace{\partial \bar{C}_2(S^4,\infty)}_{\substack{\cong \ *_3 \\ S^3 \times S^4}} = \underbrace{\partial^{\infty} \bar{C}_2(S^4,\infty)}_{\substack{\cong \ *_1 \\ S^3 \times D^4}} \cup \underbrace{\partial^{\Delta} \bar{C}_2(S^4,\infty)}_{\substack{\cong \ *_2 \\ S^3 \times D^4}}$$

$$S^3 \times S^4 \xrightarrow{\phi_0 = \mathrm{proj}_1} S^3$$

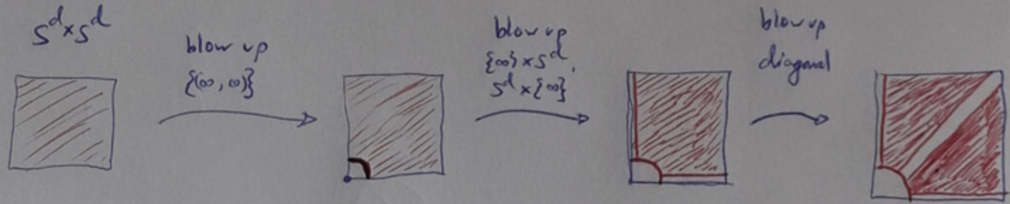


smooth D^4-bundle with $\partial E = \partial(D^4 \times B)$

- the maps $\partial^{\infty}\bar{C}_2(S^4,\infty) \xrightarrow{\phi_0} S^3$ on each fibre fit together to give a continuous map $\partial^{fib,\infty} E\bar{C}_2(\pi) \xrightarrow{\phi_0} S^3$ (the identifications $*_1$ can be made consistently)
- if we choose a vertical framing of $E \xrightarrow{\pi} B$, then the identifications $*_2$ (and hence $*_3$) can also be made canonically for each fibre, so the maps ϕ_0 fit together to give a continuous map $\partial^{fib} E\bar{C}_2(\pi) \xrightarrow{\phi_0} S^3$.

How to see $\partial \bar{C}_2(S^d, \infty)$

$\bar{C}_2 = \bar{C}_2(S^d, \infty)$:



$\partial \bar{C}_2$:

$$\emptyset \longrightarrow S^{2d-1} \xrightarrow[\substack{\text{surgery along} \\ \text{embedded} \\ S^{d-1} \sqcup S^{d-1}}]{} S^{2d-1} \xrightarrow[\substack{\text{surgery along} \\ \text{embedded} \\ S^{d-1}}]{} S^d \times S^{d-1}$$

Hopf link

unknotted

$$S^{2d-1} = (S^{d-1} \times D^d) \cup_{S^{d-1} \times S^{d-1}} (D^d \times S^{d-1})$$

surgery along the core   surgery along the core

$$(D^d \times S^{d-1}) \cup_{S^{d-1} \times S^{d-1}} (S^{d-1} \times D^d) \cong S^{2d-1}$$

Hence $\partial \bar{C}_2 \cong S^d \times S^{d-1}$

and ϕ_0 may be defined by $\mathrm{proj}_2 : \underset{\substack{\rotatebox{90}{$\cong$} \\ \partial \bar{C}_2}}{S^d \times S^{d-1}} \xrightarrow{(*)} S^{d-1}$.

$\partial^\infty \bar{C}_2$ = as above, but only half of the final surgery $\longrightarrow \boxed{D^d \times S^{d-1}}$

(cut out $S^{d-1} \times \mathrm{int}(D^d)$)

$\partial^\Delta \bar{C}_2$ = the copy of $\boxed{D^d \times S^{d-1}}$ that is glued in during the second half of the final surgery.

- The map $(*)$ depends on a choice of framing of $T(S^d - \{\infty\})$, since this is used in the final surgery / blow-up above.
- Without a choice of framing, we still have a (non-canonical) homeo. $\partial^\infty \bar{C}_2 \cong D^d \times S^{d-1}$, and the projection $\partial^\infty \bar{C}_2 \cong D^d \times S^{d-1} \longrightarrow S^{d-1}$ is still canonical
- So $(*)|_{\partial^\infty \bar{C}_2}$ does not depend on a framing.

An <u>admissible propagator</u> for $\begin{array}{c}E\\ \pi\downarrow\\ B\end{array}$ is a codim-3 chain θ on $E\bar{C}_2(\pi)$ with $\partial\theta \subseteq \partial E\bar{C}_2(\pi)$

$$\left[\text{so } \theta \in Z_{k+5}\left(E\bar{C}_2(\pi), \partial E\bar{C}_2(\pi)\right)\right]$$

such that

- θ is transverse to ∂^v
- $\partial\theta\big|_{\partial^{fib}} = \phi_0^{-1}(\{\pm a\})$

A <u>strong admissible propagator</u>

depends on choice of vertical framing of $\begin{array}{c}E\\ \pi\downarrow\\ B\end{array}$

- $\partial\theta\big|_{\partial^{fib,\infty}} = \phi_0^{-1}(\{\pm a\}) \qquad a \in S^3$
- $\partial\theta\big|_{\partial^{fib,\Delta}}$ is $\mathbb{Z}/2$-invariant

---

<u>Microscopic configuration spaces on $\mathbb{R}^d$</u>

<u>Def</u>: $$S_n(\mathbb{R}^d) = \left\{ (x_1,\dots,x_n) \in C_n(\mathbb{R}^d) \;\middle|\; \begin{array}{c} x_1 = 0 \\ \sum_i |x_i|^2 = 1 \end{array} \right\}$$

dimension $= d(n-1)-1$

<u>Compactification</u>: $\bar{S}_n(\mathbb{R}^d) :=$ closure of $S_n(\mathbb{R}^d) \subseteq C_n(\mathbb{R}^d) \subseteq \bar{C}_n(S^d,\infty)$

- compact manifold with corners
- ∂-strata $\xleftrightarrow{1:1}$ $A \subseteq \{1,2,\dots,n\}$, $\quad 2 \leq |A| \leq n-1$
- for comparison, ∂-strata of $\bar{C}_n(S^d,\infty)$ $\xleftrightarrow{1:1}$ $A \subseteq \{1,2,\dots,n,\infty\}$, $\quad 2 \leq |A|$

<u>Fibrewise version</u>:

- $O(d) \curvearrowright \bar{S}_n(\mathbb{R}^d)$ $\quad$ [extending the obvious action on $S_n(\mathbb{R}^d)$]
- $\begin{array}{c}V\\ \xi\downarrow\\ B\end{array}$ rank-d real vector bundle $\longrightarrow$
  - wlog assume structure group is $O(d)$
  - take associated principal $O(d)$-bundle
  - Borel construction $\rightsquigarrow$ change fibre to $\bar{S}_n(\mathbb{R}^d)$

$$\begin{array}{c}\bar{S}_n(V)\\ \Big\downarrow \bar{S}_n(\xi)\\ B\end{array}$$

<u>intuitively</u>: $\bar{S}_n(V) = \bigcup_{b\in B} \bar{S}_n(\xi^{-1}(b))$

Prop. · An admissible propagator θ exists.

· All such θ represent the <u>same</u> class in $H_{k+d+1}(E\bar{C}_2(\pi), \partial E\bar{C}_2(\pi))$.

("uniqueness-up-to-homology")

modification in light blue

(**Rmk**: A <u>similar proof</u> gives existence/uniqueness-up-to-homology of <u>"strong adm. prop."</u>)

proof. Note: enough to show that $\exists$ unique-up-to homology θ

satisfying $\partial\theta|_{\partial^{fib,\infty}} = \phi_0^{-1}(\{\pm a\})$ $(a \in S^{d-1})$ (*)

∂^{fib}

(the two other conditions may be ensured by (a) Thom transversality

(b) $c \longmapsto \frac{1}{2}(c + \tau . c)$).

Re-interpret (*) in terms of cohomology:

$$\begin{tikzcd}
 & & {[\partial\theta]} \arrow[dl, dashed] & {[\theta]} \arrow[l, mapsto] \\
1 \arrow[r, mapsto] & \phi_0^*(1) & H_{k+d}(\partial E\bar{C}_2(\pi)) & H_{k+d+1}(E\bar{C}_2(\pi), \partial E\bar{C}_2(\pi)) \arrow[l, "\delta"'] \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \cong & \cong \\
H^{d-1}(S^{d-1}) \arrow[r, "\phi_0^*"] & H^{d-1}(\partial^{fib,\infty} E\bar{C}_2(\pi)) & H^{d-1}(\partial E\bar{C}_2(\pi)) \arrow[l, "(incl)^*"'] & H^{d-1}(E\bar{C}_2(\pi)) \arrow[l, "(incl)^*"'] \arrow[ll, bend left, "(*)"] \\
\cong & \partial^{fib} & & \\
\mathbb{Q} & & &
\end{tikzcd}$$

<u>Need</u>: $\exists!$ lift of $\phi_0^*(1)$ along (*).

<u>In fact</u>: the inclusion $\partial^{fib,\infty}E\bar{C}_2(\pi) \hookrightarrow E\bar{C}_2(\pi)$ inducing (*)

∂^{fib}

is a <u>homotopy equivalence</u>

induces $\cong$ on H^i for all $i \leq d-1$.

proof:

$$\begin{tikzcd}
\boxed{\partial^\infty\bar{C}_2(S^d,\infty)} \arrow[r, hook] \arrow[d] & \boxed{\bar{C}_2(S^d,\infty)} \arrow[d] \\
\partial^{fib,\infty}E\bar{C}_2(\pi) \arrow[r, hook] \arrow[dr] & E\bar{C}_2(\pi) \arrow[d] \\
 & B
\end{tikzcd}$$

∂ ∂^{fib}

$$\begin{tikzcd}
 & \partial^\infty\bar{C}_2(S^d,\infty) \arrow[r] & \bar{C}_2(S^d,\infty) \\
 & D^d \times S^{d-1} \arrow[u, "\cong"'] & C_2(S^d - \{\infty\}) \arrow[u, "\cong"'] \\
S^d \times S^{d-1} \arrow[ur] \arrow[rr, bend right, "proj_2"'] & S^{d-1} \arrow[u, "\cong"'] \arrow[r, "id"] & S^{d-1} \arrow[u, "\cong"']
\end{tikzcd}$$

∂

$\square$.

$\square$.

From now on, $d=4$

Construction of Z_k:

Inputs

$$\boxed{\begin{array}{ll} E & \text{smooth } D^4\text{-bundle} \\ \downarrow \pi & \text{with } \partial E = \partial(D^4 \times B) \\ B^{k+\ell} & \end{array}} \quad \xleftrightarrow{1:1} \quad \left[(B,\partial B), (B\mathrm{Diff}_\partial(D^4), *)\right]$$

$$\boxed{\theta = (\theta^{(1)}, \dots, \theta^{(3k)}) \text{ admissible propagators for } \pi}$$

For $A \subseteq \{1, \dots, 2k\}$, $|A| = 2$ we have forgetful maps

$$\begin{tikzcd} E\bar{C}_{2k}(\pi) \arrow[rr, "\phi_A"] \arrow[dr] & & E\bar{C}_2(\pi) \arrow[dl] \\ & B & \end{tikzcd}$$

For a trivalent graph $\Gamma \in \mathrm{Graphs}^\circ_k$, define

$$\boxed{I(\pi, \theta, \Gamma) := \bigcap_{j=1}^{3k} \phi_{\Gamma(j)}^{-1}(\theta^{(j)}) \quad \in \quad C_\ell(E\bar{C}_{2k}(\pi))}$$

(assuming that the intersection is transverse)

Then define

$$\boxed{Z_k(\pi, \theta) := \lambda_k \sum_{\Gamma \in \mathrm{Graphs}^\circ_k} I(\pi, \theta, \Gamma) \otimes [\Gamma] \quad \in \quad C_\ell(E\bar{C}_{2k}(\pi)) \underset{\mathbb{Q}}{\otimes} \mathcal{A}_k}$$

$$\boxed{\lambda_k = \frac{1}{2^{3k}(2k)!(3k)!}}$$

When $\ell = 0$ (i.e. $\dim(B) = k$):

$$\begin{array}{c} Z_k(\pi,\theta) \\ \cap \\ C_0(E\bar{C}_{2k}(\pi)) \underset{\mathbb{Q}}{\otimes} \mathcal{A}_k \end{array} \xrightarrow{\text{aug} \otimes \text{id}} \mathbb{Q} \underset{\mathbb{Q}}{\otimes} \mathcal{A}_k = \mathcal{A}_k$$

By abuse of notation, we denote the image also by $\boxed{Z_k(\pi, \theta) \in \mathcal{A}_k}$.

Construction of α_k:

Inputs

$$\boxed{\begin{array}{ll} V & \text{rank-4 real} \\ \big\downarrow \xi & \text{vector bundle} \\ X^{k+5+\ell} & \end{array}} \quad \xleftrightarrow{1:1} \quad [X, BO(4)]$$

> **NB:** Reducing the structure group $GL_d(\mathbb{R}) \rightsquigarrow O(d)$ corresponds to choosing a Riem. metric for the vector bundle.
> → The space of Riem. metrics is contractible
> → After choosing this, we may speak of "unit sphere sub-bundles", etc.

$$\begin{array}{l} SV \\ \big\downarrow S\xi \\ X \end{array} = \text{unit sphere subbundle}$$

$$\boxed{\eta = (\eta^{(1)} \ldots\ldots \eta^{(3k)}) \text{ codim-3, } \mathbb{Z}/2\text{-invariant}^{*} \text{ chains on } SV}$$

For $A \subseteq \{1, \ldots, 2k\}$ we have forgetful maps

$$\begin{tikzcd} \bar{S}_{2k}(V) \arrow[rr, "\varphi_A"] \arrow[dr] & & \bar{S}_2(V) = SV \arrow[dl] \\ & X & \end{tikzcd}$$

For a trivalent graph $\Gamma \in \mathrm{Graphs}^{\circ}_k$, define

$$\boxed{J(\xi, \eta, \Gamma) := \bigcap_{j=1}^{3k} \varphi^{-1}_{\Gamma(j)}(\eta^{(j)}) \in C_\ell(\bar{S}_{2k}(V))}$$

(assuming that the intersection is transverse).

Then define

$$\boxed{\alpha_k(\xi, \eta) := \frac{\lambda_k}{N'_k} \sum_{\Gamma \in \mathrm{Graphs}^{\circ}_k} J(\xi, \eta, \Gamma) \otimes [\Gamma] \in C_\ell(\bar{S}_{2k}(V)) \underset{\mathbb{Q}}{\otimes} A_k}$$

> N'_k = integer to be defined later

When $\ell = 0$ (i.e. $\dim(X) = k+5$):

$$\begin{array}{ccc} \alpha_k(\xi, \eta) & & \\ \cap & & \\ C_0(\bar{S}_{2k}(V)) \underset{\mathbb{Q}}{\otimes} A_k & \xrightarrow{\mathrm{aug} \otimes \mathrm{id}} & \mathbb{Q} \underset{\mathbb{Q}}{\otimes} A_k = A_k \end{array}$$

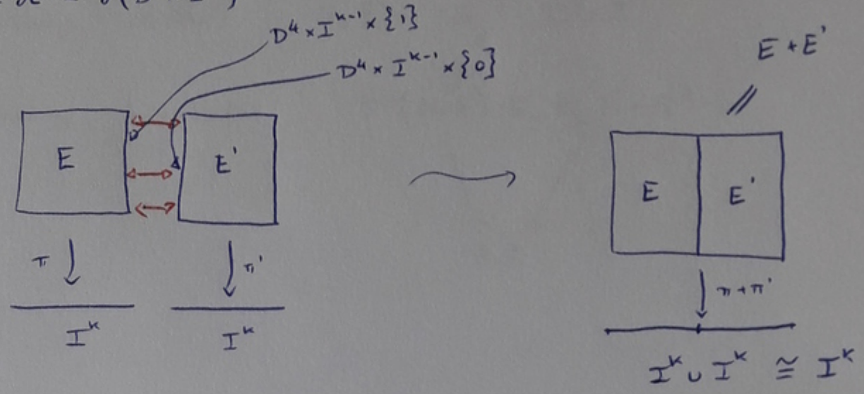
By abuse of notation, we denote the image also by $\boxed{\alpha_k(\xi, \eta) \in A_k}$.

> $*$ In general, if a bundle has str group G, then its total space has a natural action of $Z(G)$. In this case $G = O(4)$, and $-I \in Z(O(4))$, so $-I$ generates a $\mathbb{Z}/2$-action

Construction of $\hat{Z}_k$:

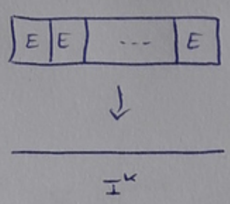
given $\begin{array}{cc} E & E' \\ \pi\downarrow & \swarrow\pi' \\ B = I^k & \end{array}$ smooth D^4- bundles with $\partial E = \partial E' = \partial(D^4 \times I^k)$

we may glue them :



This corresponds to addition in $[(I^k, \partial I^k), (B\mathrm{Diff}_\partial(D^4), *)]$
$$\| $$
$$\pi_k B\mathrm{Diff}_\partial(D^4)$$

For $n \geq 1$, let $nE = \overbrace{E + \cdots + E}^{n}$ =



"The n^{th} iteration of E"

Def : G finite abelian group $\rightsquigarrow$ $m(G) := \min\{r \in \mathbb{N} \mid g^r = id \quad \forall g \in G\}$

Thm :
- Θ_n is finite for $n \geq 5$ [Smale + Kervaire-Milnor]

  abelian group of homotopy n-spheres   h-cob. thm.

- $\Theta_n \cong \pi_0 \mathrm{Diff}_\partial(D^{n-1})$ for $n \geq 6$ [Smale + Cerf]

Lemma :
- $\pi_n O(4)$ is finite for $n \geq 4$ [proof uses fibration sequence $O(n) \hookrightarrow O(n+1) \twoheadrightarrow S^n$ and the fact [Serre] that $\pi_n S^2 \cong \pi_n S^3$ are finite for $n \geq 4$]

Def : $\boxed{N_k' := m(\Theta_{k+4}) \cdot m(\Theta_{k+5}) \cdot m(\pi_{k+3} O(4)) \cdot m(\pi_{k+4} O(4))}$

$(k \geq 1)$

Prop: For any $\begin{array}{c} E \\ \downarrow \pi \\ I^k \end{array}$ we have:

(a) $\exists$ canonical-up-to-isotopy diffeo $N_k' E \cong_\partial D^4 \times I^k$

(b) $\exists$ unique-up-to-homotopy trivialisation

$$\begin{tikzcd} T^v(N_k' E) \arrow[rr, "\cong_\partial"] \arrow[dr] & & N_k' E \times \mathbb{R}^4 \arrow[dl] \\ & N_k' E & \end{tikzcd}$$

Proof: Set $n_1 = m(\Theta_{k+4})$

$n_2 = m(\Theta_{k+4}) \cdot m(\Theta_{k+5})$

$n_3 = m(\Theta_{k+4}) \cdot m(\Theta_{k+5}) \cdot m(\pi_{k+3} O(4)) \cdot m(\pi_{k+4} O(4)) = N_k'$.

(a) • Let $S^E := E \cup_\partial (D^4 \times I^k)$ (makes sense since $\partial E = \partial(D^4 \times I^k)$)

and note that $S^E \in \Theta_{k+4}$.

Hence $S^{n_1 E} \cong \underbrace{S^E \# \cdots \# S^E}_{n_1} \cong S^{k+4}$

$S^{n_1 E} = (n_1 E) \cup_\partial (D^4 \times I^k)$

$\Rightarrow n_1 E \cong S^{k+4} - \mathrm{int}(\text{some } (k+4)\text{-disc})$

$\cong S^{k+4} - \mathrm{int}(\text{northern hemisphere})$ [Disc theorem & Palais]

$= D^{k+4}$.

• Any two such diffeo's differ (up to $\simeq$) by an element of $\pi_0 \mathrm{Diff}_\partial(D^{k+4}) \cong \Theta_{k+5}$

$\Rightarrow$ iterating such a diffeo $m(\Theta_{k+5})$ times, it becomes unique up to isotopy.

$\Rightarrow \exists$ unique-up-to-isotopy diffeo $n_2 E \cong_\partial D^4 \times I^k$ that arises as an iteration of a diffeo $n_1 E \cong_\partial D^4 \times I^k$.

(b) <u>obstruction theory</u> :

- obstructions to trivialising $\begin{array}{c} T^v(n_2 E) \\ \downarrow \\ n_2 E \end{array}$ lie in $H^*(n_2 E, \partial n_2 E \,;\, \pi_{*-1}\, O(4))$

- if $\exists$, obstructions to uniqueness (up to $\simeq$) lie in $H^*(n_2 E, \partial n_2 E \,;\, \pi_* \, O(4))$

By (a), $\boxed{n_2 E \cong_\partial D^{k+4}}$ so we just need

$$0 \overset{?}{=} \text{obs} \in H^{k+4}(n_2 E, \partial n_2 E \,;\, \pi_{k+3}\, O(4))$$

$$0 \overset{?}{=} \text{obs} \in H^{k+4}(n_2 E, \partial n_2 E \,;\, \pi_{k+4}\, O(4))$$

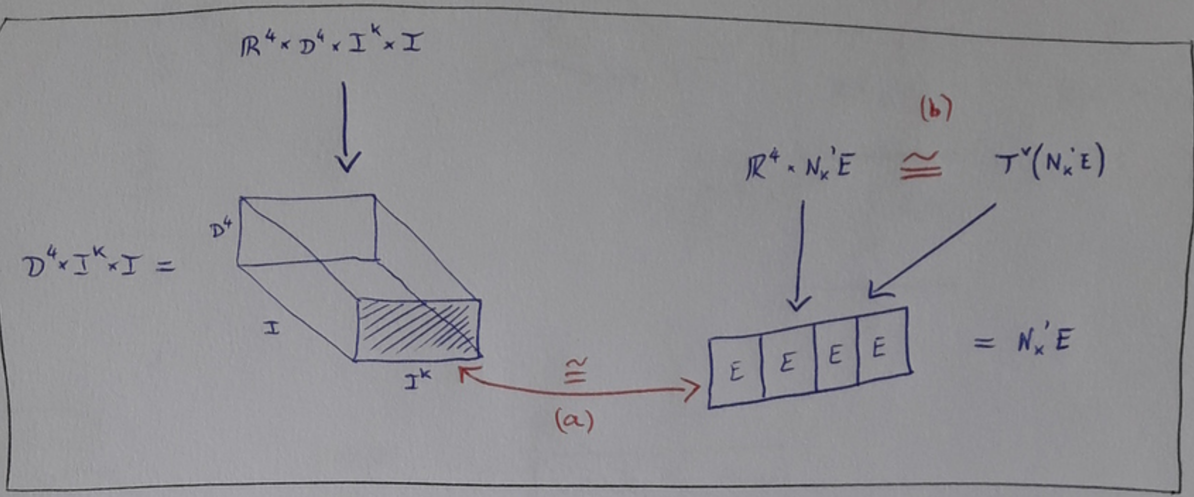
Iteration of bundles $\longrightarrow$ addition of the obstructions using the group structure of $\pi_* O(4)$

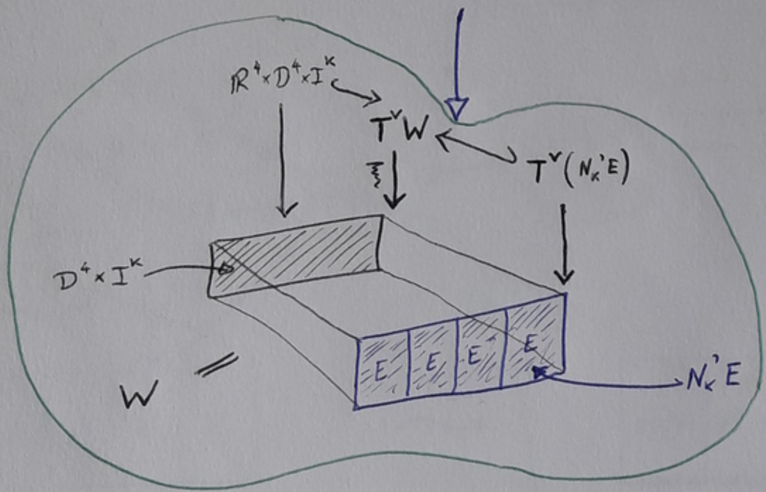
$\Rightarrow$ iterating by $m(\pi_{k+3}\, O(4)) . m(\pi_{k+4}\, O(4)) \longrightarrow$ done. $\square$.

Construction of $\hat{Z}_k$:

Step (1)

$$\begin{array}{c} E \\ \pi\downarrow \\ I^k \end{array} \quad \text{smooth } D^4\text{-bundle with } \partial E \text{ trivial} \quad \longrightarrow \quad \begin{array}{c} T^vW \\ \xi\downarrow \\ W^{k+5} \end{array} \quad \mathbb{R}^4\text{-bundle}$$





Note: W is a bordism (rel. ∂) of $(k+4)$-manifolds with $\mathbb{R}^4$-bundles

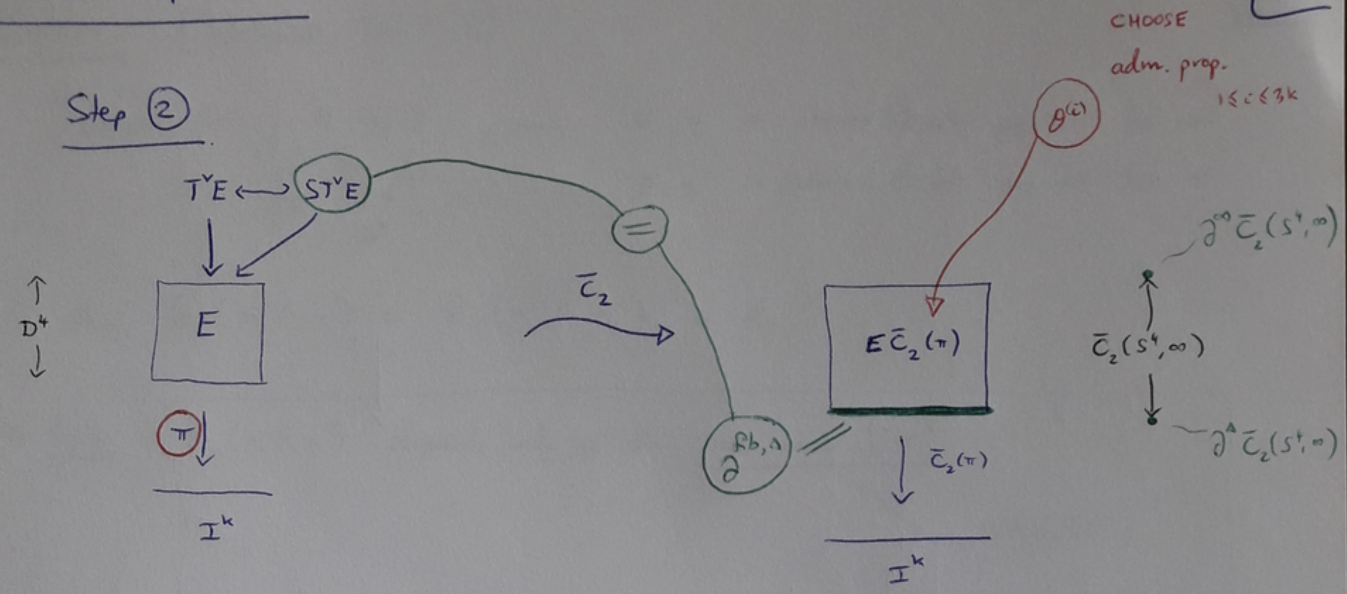
$$\begin{array}{c} \mathbb{R}^4 \times D^4 \times I^k \\ \downarrow \\ D^4 \times I^k \end{array} \rightsquigarrow \begin{array}{c} T^v(N_k' E) \\ \downarrow \\ N_k' E \end{array}$$

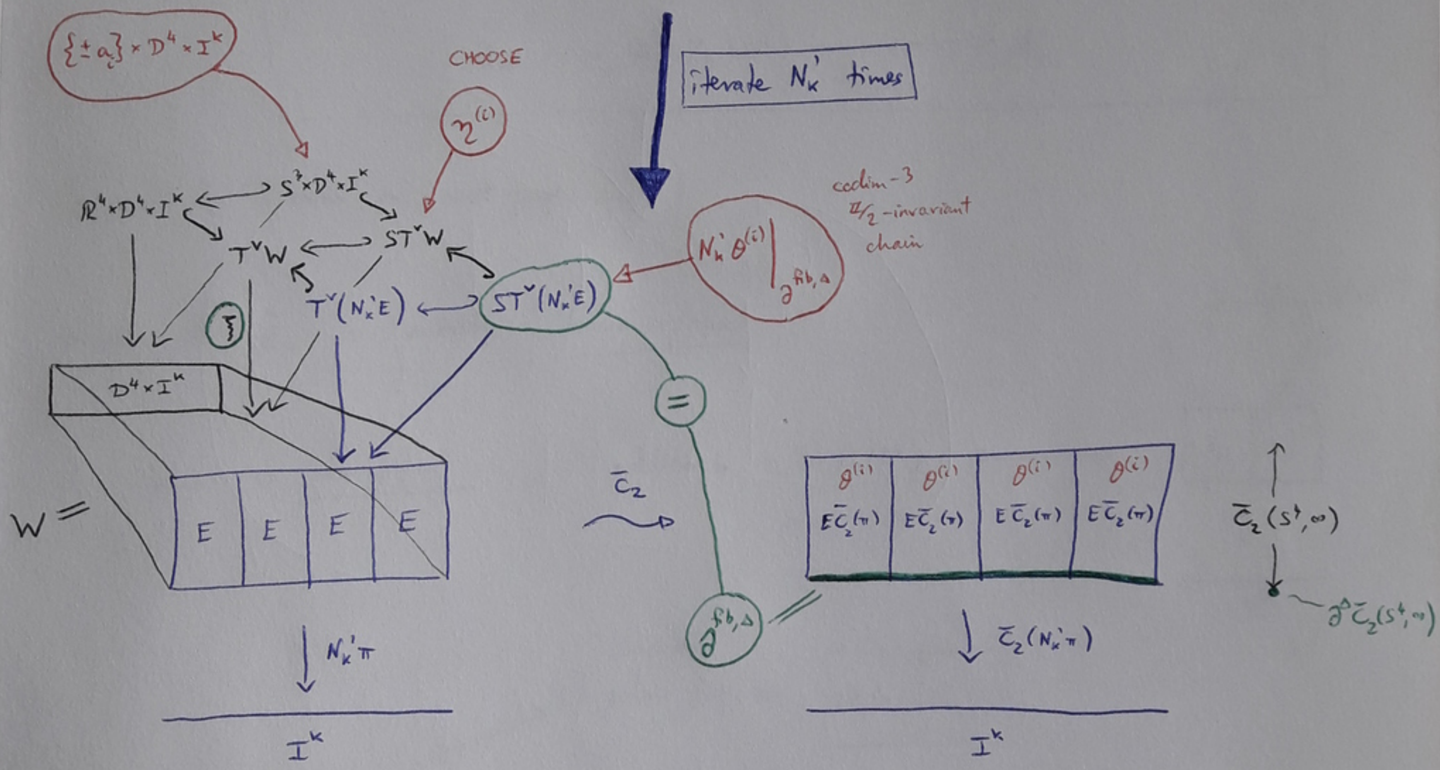
but **not** (!) a bordism of smooth D^4-bundles.

This is because (a) is a diffeo. (rel. ∂) of manifolds, not of smooth D^4-bundles

Construction of $\hat{Z}_k$:

Step ②





$\eta^{(i)}$ codim-3, $\mathbb{Z}/2$-invariant chain such that

$$\partial \eta^{(i)}\Big|_{\text{front face}} = N_k' \theta^{(i)}\Big|_{\partial^{fib,\Delta}}$$

$$\partial \eta^{(i)}\Big|_{\text{back face}} = \{\pm a_i\} \times D^4 \times I^k \qquad (a_i \in S^3)$$

Def: $\boxed{\hat{Z}_k(\pi, \theta, \eta) := Z_k(\pi, \theta) - \alpha_k(\xi, \eta) \in \mathcal{A}_k}$

Theorem [Watanabe, Thm 2.7]

suppose that
$$\begin{array}{ccc} E & \cong_\varphi & E' \\ & {}_{\pi}\searrow \;\; \swarrow_{\pi'} & \\ & I^k & \end{array}$$
and θ, η = choices of adm. prop. etc for π

θ', η' = choices of adm. prop. etc for π'

then $\hat{Z}_k(\pi, \theta, \eta) = \hat{Z}_k(\pi', \theta', \eta') \in A_k$.

$\hookrightarrow$ **Coro** $\hat{Z}_k(\pi, \theta, \eta)$ depends only on the isom. class of π $\xleftrightarrow{1:1}$ $\pi_k B\mathrm{Diff}_\partial(D^4)$

Thus we get a well-defined function

$$\hat{Z}_k : \pi_k B\mathrm{Diff}_\partial(D^4) \longrightarrow A_k$$

(proof of Theorem on next pages ......)

Lemma: $\hat{Z}_k$ is actually a homomorphism

$\longrightarrow$ idea of proof:

(i) addition in $\pi_k B\mathrm{Diff}_\partial(D^4) \ni \alpha + \beta$ $\longleftrightarrow$ $\boxed{E_\alpha \mid E_\beta} \to I^k$

glue along $D^4 \times I^{k-1}$

(ii) choice of adm. prop. on $\boxed{E\bar{C}_2(\pi_\alpha) \mid E\bar{C}_2(\pi_\beta)}$

restricts to an adm prop on $E\bar{C}_2(\pi_\alpha)$ and on $E\bar{C}_2(\pi_\beta)$.

NB: conversely, if
θ_α adm prop on $E\bar{C}_2(\pi_\alpha)$
θ_β ---------- $E\bar{C}_2(\pi_\beta)$
then $\theta_\alpha + \theta_\beta$ adm prop on $\boxed{E\bar{C}_2(\pi_\alpha) \mid E\bar{C}_2(\pi_\beta)}$

F

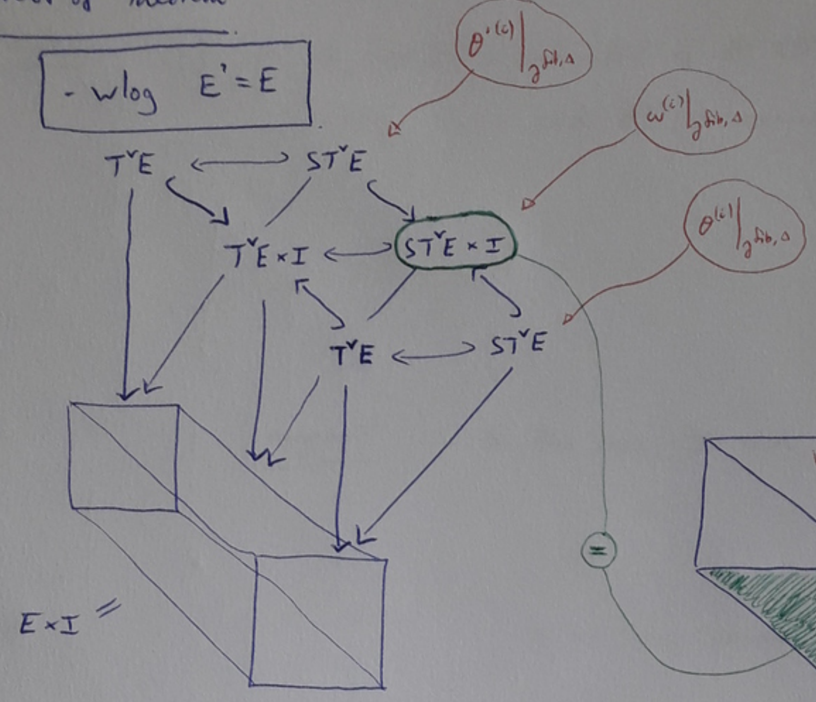
$\Leftrightarrow$ $\partial\theta_\alpha|_F = -\partial\theta_\beta|_F$

(iii) $\hat{Z}_k(\alpha+\beta) = \sum$ intersections of pullbacks of adm. prop.'s on $\boxed{\;\mid\;}$

$$= \underbrace{\sum \text{intersections} \ldots \text{on } \boxed{\;}\,\square}_{\hat{Z}_k(\alpha)} + \underbrace{\sum \text{intersection} \ldots \text{on } \square\,\boxed{\;}}_{\hat{Z}_k(\beta)}$$

proof of Theorem

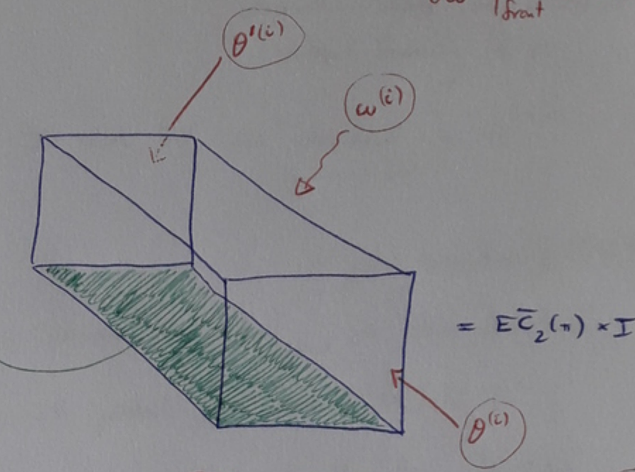
$\boxed{-\text{ wlog } E' = E}$

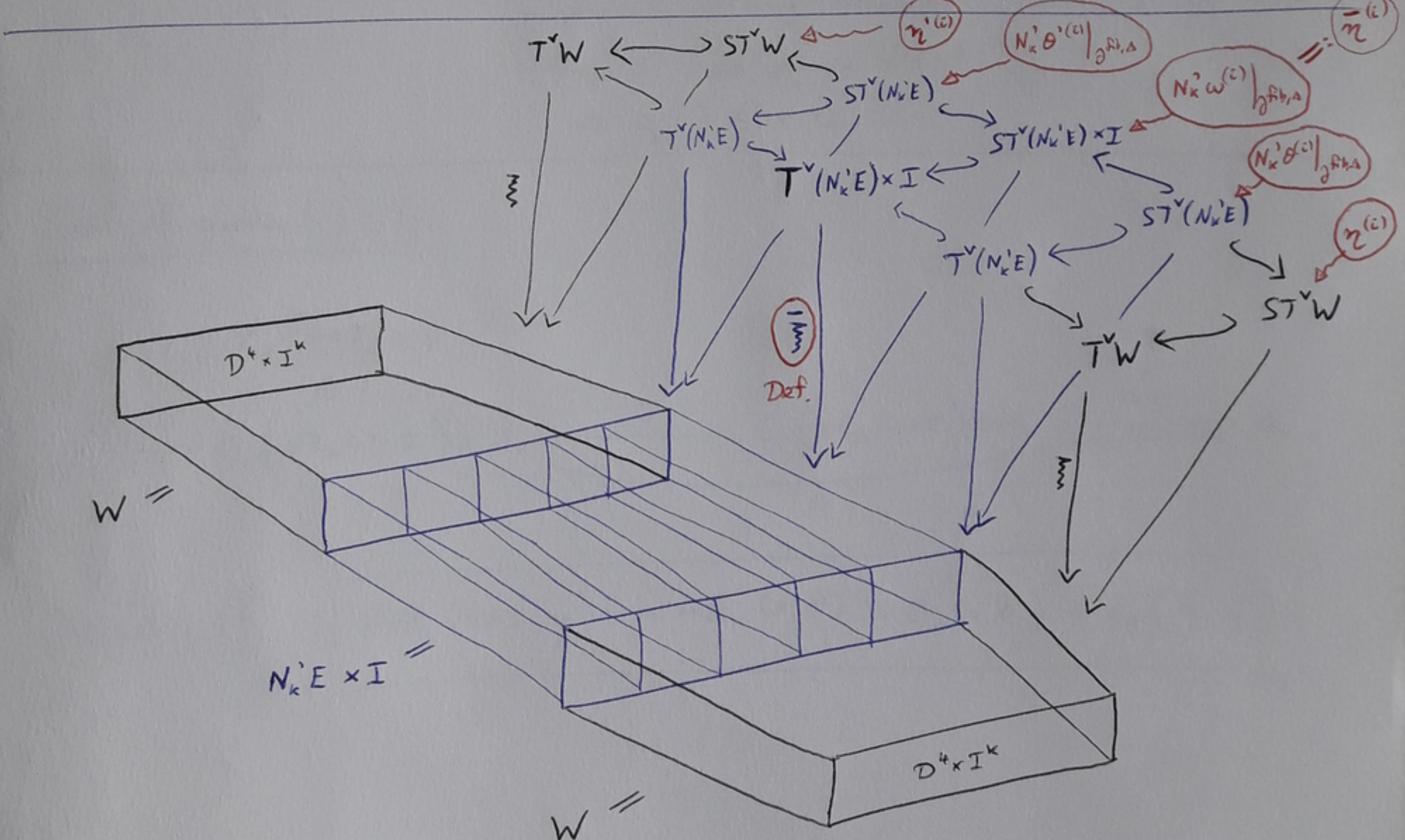


Lem: $\exists$ adm. prop. $\omega^{(i)}$ on $E\bar{C}_2(\pi) \times I$ with

$\partial \omega^{(i)}|_{\text{back}} = \theta'^{(i)}$

$\partial \omega^{(i)}|_{\text{front}} = -\theta^{(i)}$





Claim: (1) $Z_k(\pi, \theta) - Z_k(\pi, \theta') = \alpha_k(\bar{\xi}, \bar{\eta})$

(2) $\alpha_k(\xi, \eta) - \alpha_k(\xi, \eta') = \alpha_k(\bar{\xi}, \bar{\eta})$

Note that

(1) - (2)

$\Downarrow$

Theorem.

Aside: (1) $\Rightarrow$ $Z_k(\pi, \theta)$ is independent of $\theta = (\theta^{(1)} \dots \theta^{(3k)})$ <u>if</u>

each $\theta^{(i)}$ is assumed to be a <u>strong</u> adm. prop.

depends on choice (in particular, $\exists$) vertical framing of π

<u>argument</u>: - in this case, $\partial\theta$ and $\partial\theta'$ are "standard" on $\partial^{fib,\Delta}$

pulled back from $\{\pm a\} \subseteq S^3$

& we may assume that $\partial\omega\big|_{\partial^{fib,\Delta}}$ is also standard

- $\Rightarrow$ $\bar{\eta}^{(i)}$ are all pulled back from $\{\pm a_i\} \subseteq S^3$
- generically, we may assume that $a_i \neq \pm a_j$ for $i \neq j$

  $\Rightarrow$ $\bar{\eta}^{(i)}$ are pairwise disjoint

  $\Rightarrow$ $\alpha_k(\bar{\xi}, \bar{\eta}) = 0$.

<u>Proof of equation (1)</u>:

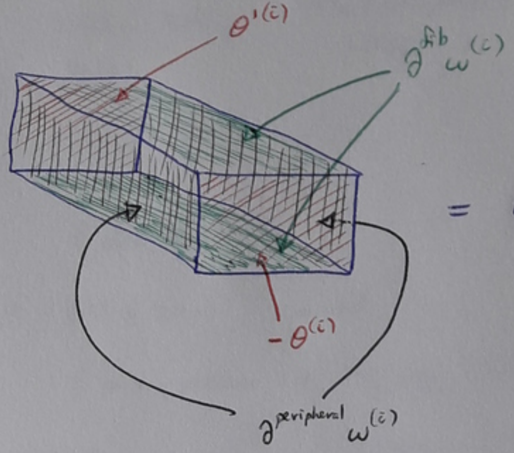
$$\begin{tikzcd}
Z_k(\pi \times I, \omega) \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & \\
\dots \longrightarrow C_1(E\bar{C}_{2k}(\pi) \times I) \otimes A_k \arrow[r, "\partial"] \arrow[rr, bend right, "0"'] & C_0(E\bar{C}_{2k}(0) \times I) \otimes A_k \arrow[r, Rightarrow, "\underset{\text{aug.}}{\overset{\#}{\shortparallel}}"] & A_k
\end{tikzcd}$$

<u>Claim (1)'</u>: $$\boxed{\#\partial Z_k(\pi \times I, \omega) = -Z_k(\pi, \theta) + Z_k(\pi, \theta') + \alpha_k(\bar{\xi}, \bar{\eta})}$$

$$\# \partial Z_k(\pi \times I, \omega) = \lambda_k \sum_{\Gamma} \underbrace{\partial I(\pi\times I, \omega, \Gamma)}. [\Gamma] \quad \in \mathcal{A}_k$$

$$\partial I(\pi \times I, \omega, \Gamma) = \sum_{\substack{X \subseteq \{1, \dots, 3k\} \\ |X| \geq 1}} I\left(\pi \times I, \left\{ \begin{matrix} \omega^{(i)} & i \notin X \\ \underbrace{\partial \omega^{(i)}} & i \in X \end{matrix} \right\}, \Gamma\right)$$

$$\partial \omega^{(i)} = \theta'^{(i)} - \theta^{(i)} + \partial^{fib} \omega^{(i)} + \partial^{peripheral} \omega^{(i)} \quad] (*)$$



$= E\bar{C}_2(\pi) \times I$

Hence $\# \partial Z_k(\pi \times I, \omega)$

$= \sum$ terms involving $\theta'^{(i)}$ $\longleftrightarrow$ $Z_k(\pi, \theta')$

$+ \sum$ terms involving $\theta^{(i)}$ $\longleftrightarrow$ $-Z_k(\pi, \theta)$

$+ \sum$ terms involving $\partial^{peripheral} \omega^{(i)}$ $\longleftrightarrow$ 0 — since $\partial^{peripheral} \omega^{(i)}$ is supported in the faces of $E\bar{C}_2(\pi) \times I$ corresponding to a <u>trivial</u> D^4-bundle over $\partial I^k \times I$

$+ \sum$ terms involving $\partial^{fib} \omega^{(i)}$ $\longleftrightarrow$ ....

+ no mixed terms — since the terms of (*) have support in different faces of $E\bar{C}_2(\pi) \times I$.

$$\partial^{fib}\omega^{(i)} = \sum_{\substack{A\subseteq\{1\dots 2k,\infty\}\\ |A|\geq 2}} \underbrace{\partial^{fib,A}\omega^{(i)}}$$

the piece supported in the fibrewise-∂-stratum of $E\bar{C}_2(\pi)\times I$ corresponding to A.

$$\sum \text{terms involving } \partial^{fib}\omega^{(i)} = \lambda_k \sum_{\Gamma}\sum_{X}\sum_{A} I\left(\pi\times I, \begin{Bmatrix} \omega^{(i)} & i\notin X \\ \partial^{fib,A}\omega^{(i)} & i\in X\end{Bmatrix}, \Gamma\right).[\Gamma]$$

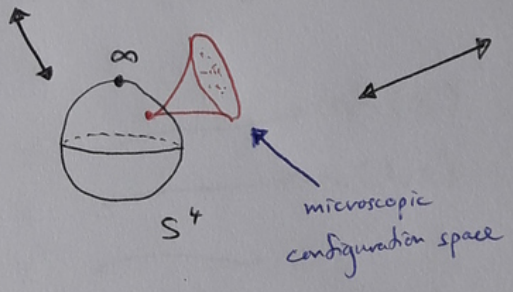
Γ: trivalent graph, vertices $=\{1\dots 2k\}$, edges $=\{1\dots 3k\}$

$X\subseteq\{1\dots 3k\}$, $|X|\geq 1$

$A\subseteq\{1,\dots,2k,\infty\}$, $|A|\geq 2$

Facts:

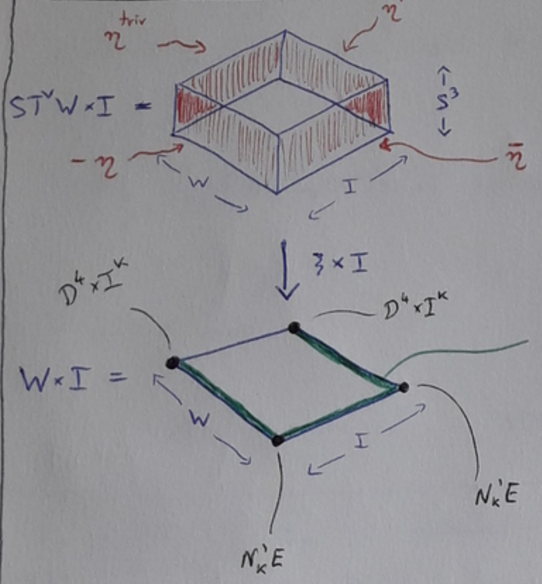
- terms with $\infty\in A$ cancel (dim. reasons)
- terms with $\infty\notin A$ and $3\leq |A|\leq 2k-1$ cancel (dim. reasons + $\mathbb{Z}/2$-symmetry)
- terms with $\infty\notin A$ and $|A|=2$ cancel (IHX relations)
- terms with $A=\{1,\dots,2k\}$



$\alpha_k(\bar{\xi},\bar{\eta})$.

Proof of equation (2): (philosophically similar)

$$\begin{tikzcd} \alpha_k(\xi\times I,\bar{\bar\eta}) \arrow[d, phantom, "\cap"] & & \\ C_1(ST^vW\times I)\otimes A_k \arrow[r,"\partial"] \arrow[rr, bend right, "0"'] & C_0(ST^vW\times I)\otimes A \arrow[r, "\overset{\#}{\overset{\|}{\text{aug.}}}"] & A_k \end{tikzcd}$$



Lem: $\exists\ \bar{\bar\eta}$ $\mathbb{Z}/2$-inv. codim.-3-chain on $ST^vW\times I$ such $\partial\bar{\bar\eta}$ restricts to η, η^{triv}, η' and $\bar\eta$

$\eta^{triv} = \{\pm a\}\times D^4\times I^{k+1}$ on $S^3\times D^4\times I^{k+1}$

= diagram at bottom of page 12 (with 3 dim. collapsed to 1 dim.)

Claim (2)': $\boxed{\#\partial\alpha_k(\xi\times I,\bar{\bar\eta}) = -\alpha_k(\xi,\eta)+\alpha_k(\xi,\eta')+\alpha_k(\bar\xi,\bar\eta)}$

Note that $\quad \partial\bar{\bar\eta} = \eta' - \eta + \eta^{triv} + \bar\eta + \partial^{fib}\bar{\bar\eta}$ $\quad$] (**)

Hence $\#\partial\alpha_k(\xi\times I,\bar{\bar\eta})$

$$\begin{aligned} &= \textstyle\sum \text{terms involving } \eta' &&\longmapsto \alpha_k(\xi,\eta')\\ &+\textstyle\sum \text{terms involving } \eta &&\longmapsto -\alpha_k(\xi,\eta)\\ &+\textstyle\sum \text{terms involving } \eta^{triv} &&\longmapsto 0\\ &+\textstyle\sum \text{terms involving } \bar\eta &&\longmapsto \alpha_k(\bar\xi,\bar\eta)\\ &+\textstyle\sum \text{terms involving } \partial^{fib}\bar{\bar\eta} && \end{aligned}$$

all strata except $A=\{1,2,\dots,2k\}$ cancel as before
(dim. reasons + $\mathbb{Z}/2$-symmetry + IHX relation)

this time $\nexists$ stratum corresponding to $A=\{1,2,\dots,2k\}$ in $\bar S_{2k}$,
since <u>not all</u> points may collide simultaneously in microscopic config spaces.

Oriented bordism-invariance of $\hat{Z}_k$

Oriented bordism:

$$\tilde{\Omega}_k(X) = \Big\{\text{closed } k\text{-manifolds with based map to } X\Big\} \Big/ \begin{matrix}\text{oriented}\\ \text{bordism}\end{matrix}$$

$$\rightsquigarrow \quad \tilde{\Omega}_k(BG) \cong \Big\{\text{closed } k\text{-manifolds with principal } G\text{-bundle} \atop \text{trivialised over the basepoint}\Big\} \Big/ \begin{matrix}\text{oriented}\\ \text{bordism}\end{matrix}$$

$$\rightsquigarrow \quad \tilde{\Omega}_k(BDiff_\partial(D^4)) \cong \Big\{\text{closed } k\text{-manifolds with smooth } D^4\text{-bundle} \atop {\text{(with trivial } \partial) \atop \text{trivialised over the basepoint}}\Big\} \Big/ \begin{matrix}\text{oriented}\\ \text{bordism}\end{matrix}$$

$\exists$ natural map

$$H = \text{"Hurewicz"} : \quad \pi_k BDiff_\partial(D^4) \longrightarrow \tilde{\Omega}_k(BDiff_\partial(D^4))$$

Proposition: $\hat{Z}_k$ is oriented-bordism-invariant, in other words:

$$\begin{tikzcd}
\pi_k BDiff_\partial(D^4) \arrow[rr, "H"] \arrow[dr, two heads] \arrow[dd, "\hat{Z}_k"'] & & \tilde{\Omega}_k(BDiff_\partial(D^4)) \\
& im(H) \arrow[ur, hook] \arrow[dl, dashed] & \\
A_k & &
\end{tikzcd}$$

More generally, if $\tilde{h}_*$ is a (reduced) generalised homology theory, choosing $x \in \tilde{h}_0(S^0) = h_0(pt)$ determines a Hurewicz map

$\tilde{h}_0(S^0) \cong \tilde{h}_k(S^k)$

$$\pi_k(X) \xrightarrow{H} \tilde{h}_k(X)$$

$$[S^k \xrightarrow{f} X] \longmapsto f_*(x)$$

Proposition': if $x \in h_0(pt)$ generates a $\mathbb{Z}$-summand then:

$$\begin{tikzcd}
\pi_k(BG) \arrow[rr, "H"] \arrow[dr, two heads] \arrow[dd, "\varphi"'] & & \tilde{h}_k(BG) \\
& im(H) \arrow[ur, hook] \arrow[dl, dashed] & \\
V & &
\end{tikzcd}$$

any homom. to a $\mathbb{Q}$-vector-space

G any topological group

Prop' ⟹ Prop : take $G = \mathrm{Diff}_\partial(D^4)$
$\varphi = \hat{Z}_K$
$\tilde{h}_* =$ oriented bordism

and note that $\Omega_0(pt^+) \cong \mathbb{Z}\langle \cdot^+ \rangle$ $\square$.

Rmk: - We could take $\tilde{h}_* =$ K-theory
- But we cannot take $\tilde{h}_* =$ unoriented bordism, since $\Omega_0^{unor}(pt) \cong \mathbb{Z}/2\langle \cdot \rangle$

---

sketch proof of Prop':

$$\begin{tikzcd}
\pi_K BG \arrow[rr, "H"] \arrow[dr] \arrow[dd, "\varphi"'] & & \tilde{h}_K(BG) \arrow[dr] & \\
& \pi_K BG \otimes \mathbb{Q} \arrow[rr, "H \otimes \mathbb{Q}"] \arrow[dl] & & \tilde{h}_K(BG) \otimes \mathbb{Q} \\
V & & &
\end{tikzcd}$$

- By a diagram chase, it suffices to prove that <u>$H \otimes \mathbb{Q}$ is injective</u>.
- To see this:

class.^{n} of rational gen. hom. theories

$$\begin{tikzcd}
\pi_* BG \otimes \mathbb{Q} \arrow[r, "H \otimes \mathbb{Q}"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{h}_*(BG) \otimes \mathbb{Q} \cong \tilde{H}_*(BG;\mathbb{Q}) \otimes h_*(pt) \\
& H_*(BG;\mathbb{Q}) \otimes h_*(pt) \arrow[u, "(\cong \text{ in deg} > 0) \otimes \mathrm{id}"'] \\
\mathcal{U}(\pi_* BG \otimes \mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r, "\mathrm{id} \otimes (1 \mapsto x)"] & \mathcal{U}(\pi_* BG \otimes \mathbb{Q}) \otimes h_*(pt) \arrow[u, phantom, "\cong \text{ Milnor-Moore}"]
\end{tikzcd}$$

split-injective iff $x \in h_0(pt)$ generates a $\mathbb{Z}$-summand.

universal enveloping Hopf algebra

$\square$.