Kontsevich characteristic classes à la Watanabe

(III)

GeMAT seminar
IMAR
3 July 2020

Following T. Watanabe
"Some exotic nontrivial elements of the rational homotopy groups of $\mathrm{Diff}(S^4)$"

arXiv: 1812.02448

Recollections.

[Antonelli - Burghelea - Kahn '72]

Aim: Study $\pi_{k-1}\mathrm{Diff}(S^4) \cong \underbrace{\pi_{k-1}\mathrm{Diff}_\partial(D^4)}_{\text{"exotic"}} \times \pi_{k-1} O(5)$

$$\pi_{k-1}\mathrm{Diff}_\partial(D^4) \cong \pi_k B\mathrm{Diff}_\partial(D^4) \cong \{\text{smooth } (D^4,\partial)\text{-bundles } E \text{ over } I^k\}/\cong$$

$\partial E = \partial(I^k \times D^4)$

$\rightsquigarrow$ need invariants of (D^4,∂)-bundles.

(D^4,∂)-bundle $E \xrightarrow{\pi} B^k$ $\xmapsto{\text{Borel construction}}$ $E\bar{C}_n(\pi) \xrightarrow{\bar{C}_n(\pi)} B^k$ bundle with fibre $= \bar{C}_n(S^4,\infty)$

Fulton-MacPherson compactification of $C_n(\mathbb{R}^4)$

$n=2$

notion of admissible propagator on $E\bar{C}_2(\pi)$

$$\boxed{\begin{array}{ccccc} E\bar{C}_2(\pi) & \supseteq & \partial^{fib,\infty} E\bar{C}_2(\pi) & \xrightarrow{\phi_0} & S^3 \\ \theta & \longmapsto & \partial\theta|_{\partial^{fib,\infty}} & \longleftarrow & \{\pm a\} \end{array}}$$

codim-3 chain

Last time:

- Admissible propagators <u>exist</u>

- (D^4, ∂) - bundle over B^k $\quad \pi$
  3k - tuple of adm. prop. $\quad \theta = (\theta^{(1)}, \ldots, \theta^{(3k)})$ $\quad \longrightarrow Z_k(\pi, \theta) \in \mathcal{A}_k$

$$\mathcal{A}_k = \{\text{labelled trivalent graphs with } 2k \text{ vertices}\} / \text{IHX} \ldots$$

("infinitesimal version")

- rank-4 real vector bundle $V \longrightarrow X^{k+5}$ $\quad \xi$
  3k - tuple of codim-3 chains on SV $\quad \eta = (\eta^{(1)}, \ldots, \eta^{(3k)})$ $\quad \longrightarrow \alpha_k(\xi, \eta) \in \mathcal{A}_k$

Today:

- (D^4, ∂) - bundle over $I^k = [0,1]^k$ $\quad \pi$
  ~~3k - tuple of adm. prop.~~ $\quad \cancel{\theta = (\theta^{(1)}, \ldots, \theta^{(3k)})}$

  $\longrightarrow$ associated ξ and η $\longrightarrow \hat{Z}_k(\pi, \cancel{\theta}) \in \mathcal{A}_k$

$$\hat{Z}_k(\pi, \theta) = Z_k(\pi, \theta) - \alpha_k(\xi, \eta)$$

Preliminaries

Thm (Smale + Kervaire-Milnor)

$(n \geq 5)$: $\Theta_n = \left(\{\text{homotopy } n\text{-spheres}\}/_{\text{diffeo}} , \# \right)$ is a finite abelian group.

Lemma (corollary of [Serre])

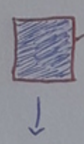
$(n \geq 4)$: $\pi_n O(4)$ is a finite abelian group.

G finite abelian group $\longrightarrow$ $m(G) := \mathrm{lcm}\{\mathrm{ord}(g) \mid g \in G\}$

For $k \geq 1$: $\boxed{N_k' = m(\Theta_{k+4}) \cdot m(\Theta_{k+5}) \cdot m(\pi_{k+3} O(4)) \cdot m(\pi_{k+4} O(4))}$

Proposition:

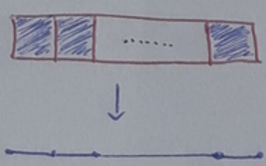
Let $E \longrightarrow [0,1]^k$ be a (D^4, ∂)-bundle



$= \partial(I^k \times D^4)$

I^k

For $n \in \mathbb{N}$ let $nE :=$



$nI^k \cong I^k$

(another (D^4,∂)-bundle over I^k)

Then (a) $N_k' E$ is canonically $\cong$ $I^k \times D^4$ (rel. ∂)

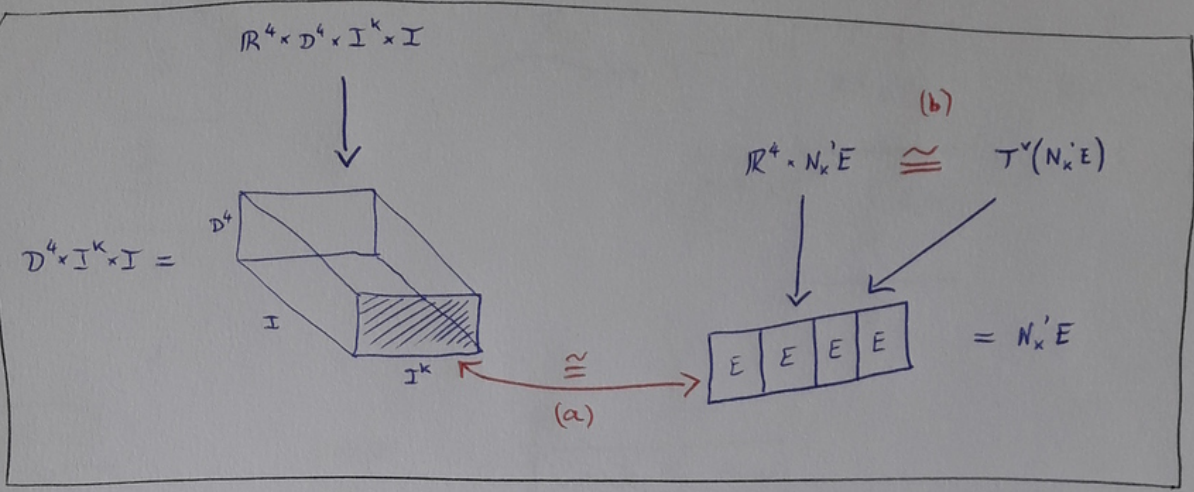
diffeo of $(k+4)$-manifolds
(not isom. of (D^4,∂)-bundles)

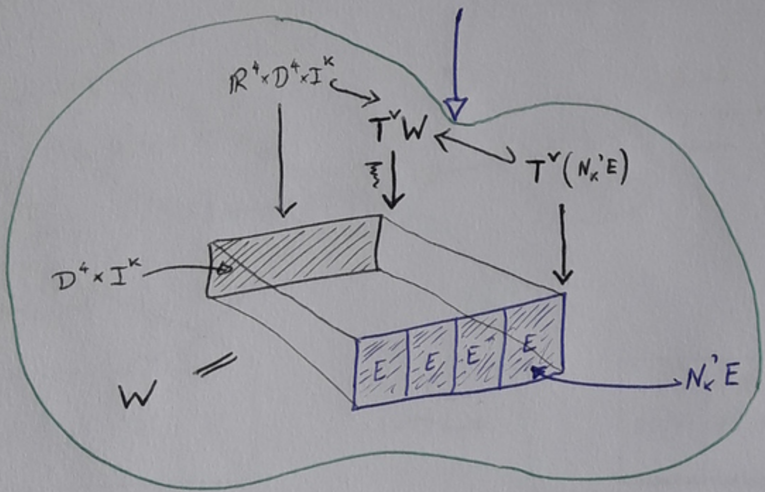
(b) the vertical tangent bundle $\begin{array}{c} T^v(N_k'E) \\ \downarrow \\ N_k'E \end{array}$ is uniquely (up to $\cong$) $\cong$ $\begin{array}{c} N_k'E \times \mathbb{R}^4 \\ \downarrow \\ N_k'E \end{array}$

proof (last time) uses
- Palais' "Disc Theorem"
- $\pi_0 \mathrm{Diff}_\partial(D^k) \cong \Theta_{k+1}$ (Cerf)
- obstruction theory

Construction of $\hat{Z}_k$:

Step ① $\begin{array}{c} E \\ \pi \downarrow \\ I^k \end{array}$ smooth D^4-bundle with ∂E trivial $\leadsto$ $\begin{array}{c} T^v W \\ \xi \downarrow \\ W^{k+5} \end{array}$ $\mathbb{R}^4$-bundle





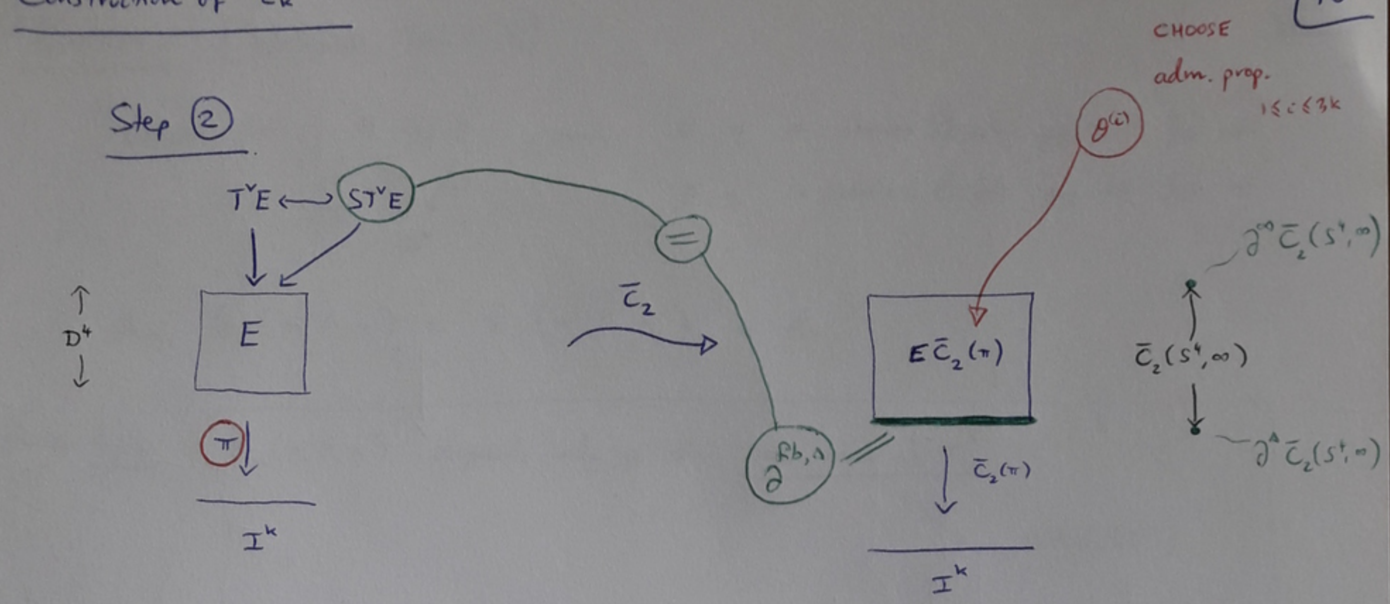
Note: W is a bordism (rel. ∂) of $(k+4)$-manifolds with $\mathbb{R}^4$-bundles $\begin{array}{c} \mathbb{R}^4 \times D^4 \times I^k \\ \downarrow \\ D^4 \times I^k \end{array} \leadsto \begin{array}{c} T^v(N_k' E) \\ \downarrow \\ N_k' E \end{array}$

(!) but not a bordism of smooth D^4-bundles.

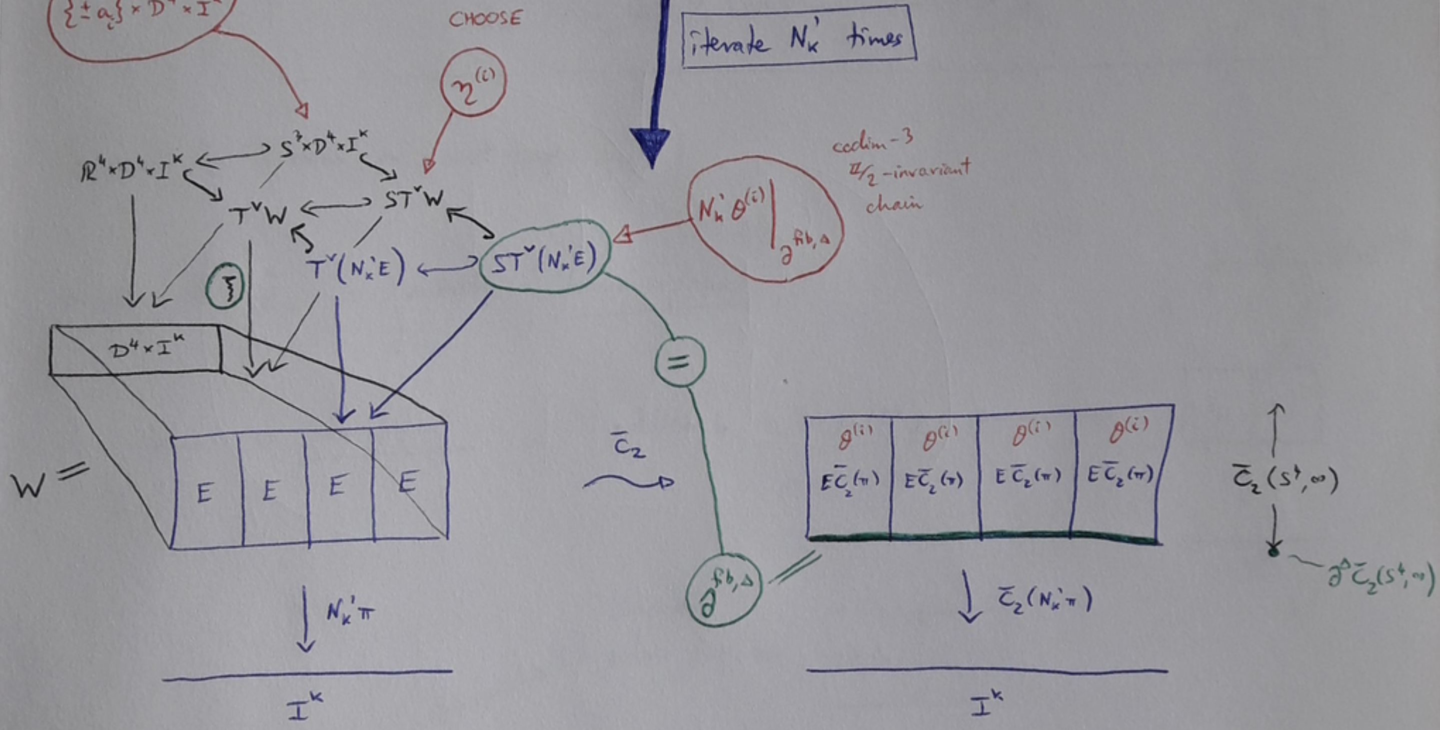
This is because (a) is a diffeo. (rel. ∂) of manifolds, not of smooth D^4-bundles.

Construction of $\hat{Z}_k$:

Step ②



iterate N_k' times



$\eta^{(i)}$ codim-3, $\mathbb{Z}/2$-invariant chain such that

$$\partial\eta^{(i)}\big|_{\text{front face}} = N_k'\theta^{(i)}\big|_{\partial^{fb,\Delta}}$$

$$\partial\eta^{(i)}\big|_{\text{back face}} = \{\pm a_i\}\times D^4\times I^k \qquad (a_i\in S^3)$$

Def: $$\boxed{\hat{Z}_k(\pi,\theta,\eta) := Z_k(\pi,\theta) - \alpha_k(\xi,\eta) \in \mathcal{A}_k}$$

Theorem [Watanabe, Thm 2.7]

suppose that
$$\begin{tikzcd} E \arrow[rr, "\cong_\partial"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & I^k & \end{tikzcd}$$
and θ, η = choices of adm. prop. etc for π

θ', η' = choices of adm. prop. etc for π'

then $\hat{Z}_k(\pi, \theta, \eta) = \hat{Z}_k(\pi', \theta', \eta') \in A_k$.

↳ **Coro**

$\hat{Z}_k(\pi, \theta, \eta)$ depends only on the isom. class of π $\xleftrightarrow{1:1}$ $\pi_k BDiff_\partial(D^4)$

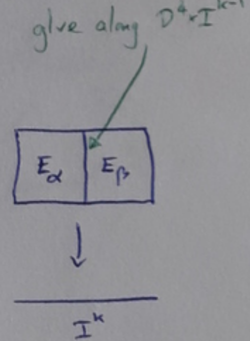
Thus we get a well-defined function
$$\hat{Z}_k : \pi_k BDiff_\partial(D^4) \longrightarrow A_k$$

(proof of Theorem on next pages ......)

Lemma: $\hat{Z}_k$ is actually a homomorphism

↳ idea of proof:

(i) addition in $\pi_k BDiff_\partial(D^4) \ni \alpha + \beta$ $\longleftrightarrow$



(ii) choice of adm. prop. on $E\bar{C}_2(\pi_\alpha) \mid E\bar{C}_2(\pi_\beta)$

restricts to an adm prop on $E\bar{C}_2(\pi_\alpha)$ and on $E\bar{C}_2(\pi_\beta)$.

(iii) $\hat{Z}_k(\alpha+\beta) = \sum$ intersections of pullbacks of adm. prop.'s on $\square\square$

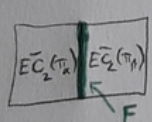
$= \underbrace{\sum \text{intersections ... on } \square\,\square}_{\hat{Z}_k(\alpha)} + \underbrace{\sum \text{intersection ... on } \square\,\square}_{\hat{Z}_k(\beta)}$

NB: conversely, if

θ_α adm prop on $E\bar{C}_2(\pi_\alpha)$

θ_β ---------- $E\bar{C}_2(\pi_\beta)$

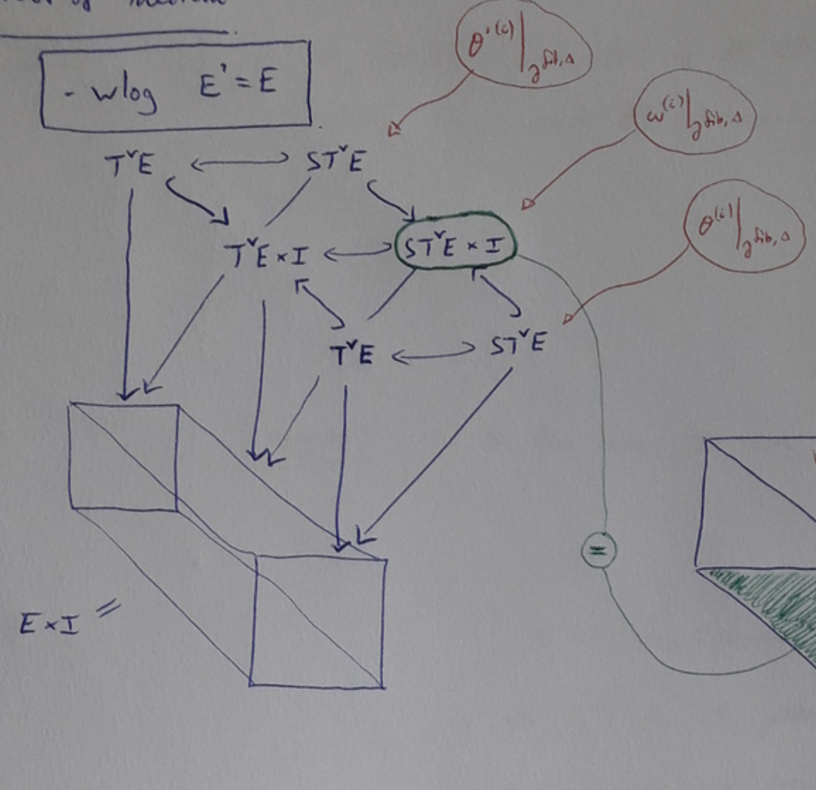
then $\theta_\alpha + \theta_\beta$ adm prop on



$\Leftrightarrow \partial\theta_\alpha|_F = -\partial\theta_\beta|_F$

proof of Theorem

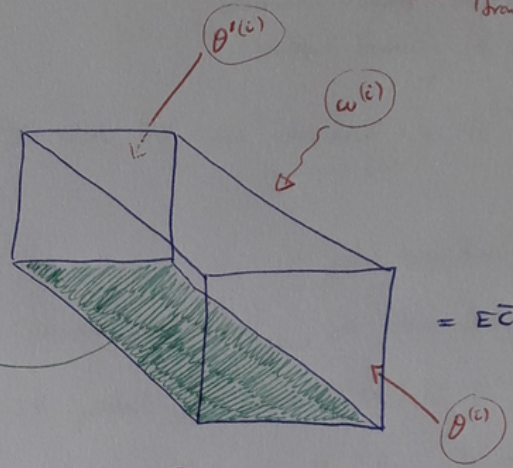
$\boxed{\text{- wlog } E' = E}$



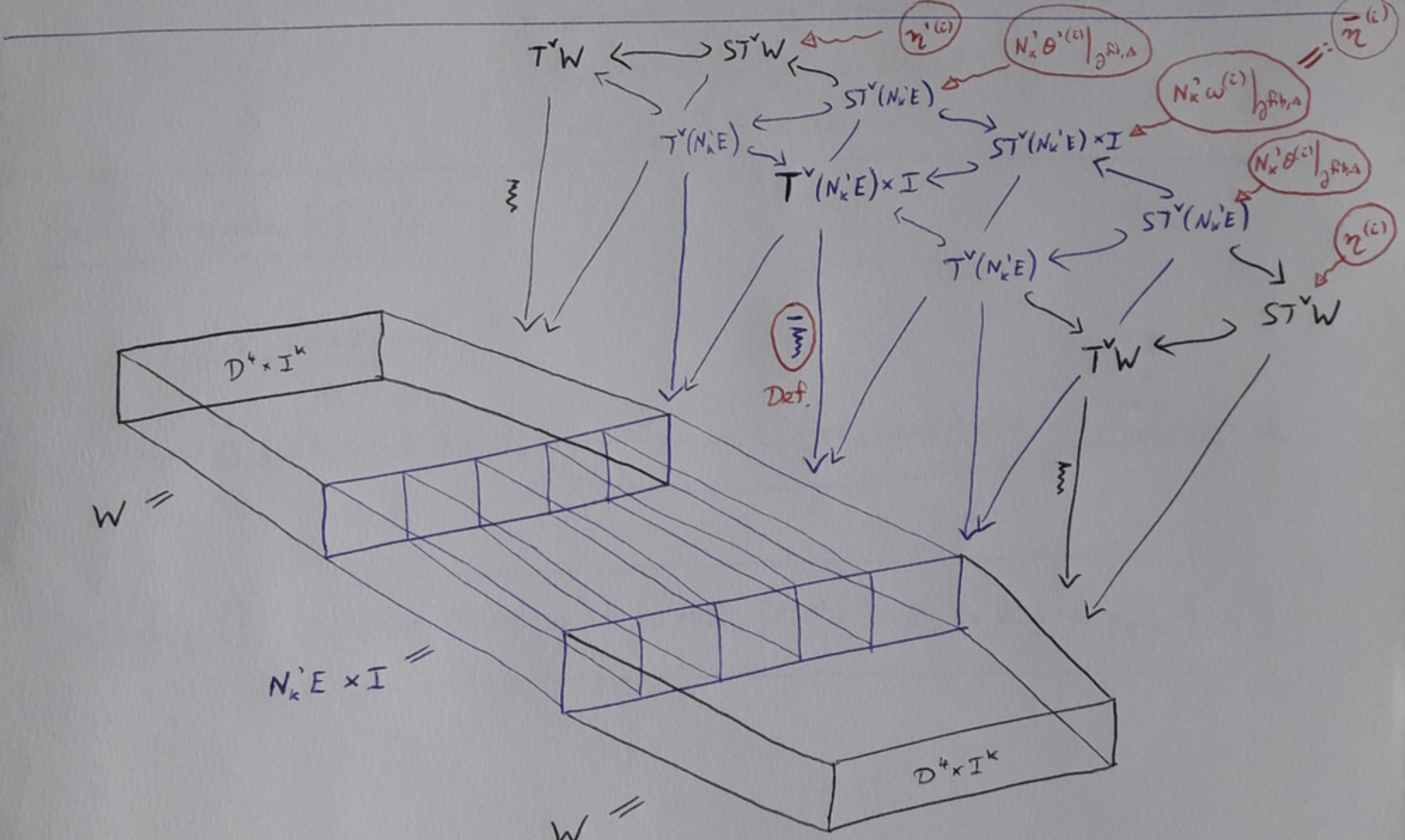
Lem: $\exists$ adm. prop. $\omega^{(i)}$ on $E\bar{C}_2(\pi) \times I$ with

$$\partial \omega^{(i)}\big|_{\text{back}} = \theta'^{(i)}$$

$$\partial \omega^{(i)}\big|_{\text{front}} = -\theta^{(i)}$$



$= E\bar{C}_2(\pi) \times I$



Claim:

(1) $Z_k(\pi, \theta) - Z_k(\pi, \theta') = \alpha_k(\bar{\xi}, \bar{\eta})$

(2) $\alpha_k(\xi, \eta) - \alpha_k(\xi, \eta') = \alpha_k(\bar{\xi}, \bar{\eta})$

Note that

(1) - (2)

$\Downarrow$

Theorem.

Aside: (1) $\Rightarrow$ $Z_k(\pi, \theta)$ is independent of $\theta = (\theta^{(1)} \dots \theta^{(3k)})$ if each $\theta^{(i)}$ is assumed to be a strong adm. prop.

depends on choice (in particular, $\exists$) vertical framing of π

argument: - in this case, $\partial\theta$ and $\partial\theta'$ are "standard" on $\partial^{fib,\Delta}$

pulled back from $\{\pm a\} \subseteq S^3$

& we may assume that $\partial\omega|_{\partial^{fib,\Delta}}$ is also standard

- $\Rightarrow$ $\bar{\eta}^{(i)}$ are all pulled back from $\{\pm a_i\} \subseteq S^3$
- generically, we may assume that $a_i \neq \pm a_j$ for $i \neq j$

$\Rightarrow$ $\bar{\eta}^{(i)}$ are pairwise disjoint

$\Rightarrow$ $\alpha_k(\bar{\xi}, \bar{\eta}) = 0.$

Proof of equation (1):

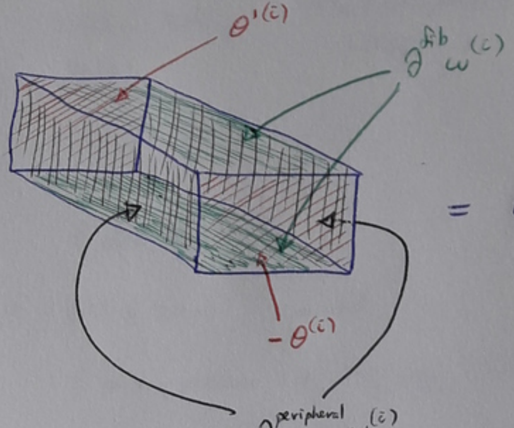
$$\begin{tikzcd} & Z_k(\pi \times I, \omega) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \# \arrow[d, phantom, "\rotatebox{90}{$=$}"] & \\ \cdots \arrow[r] & C_1(E\bar{C}_{2k}(\pi) \times I) \otimes A_k \arrow[rr, "\partial"] \arrow[rrrr, bend right, "0"'] & & C_0(E\bar{C}_{2k}(0) \times I) \otimes A_k \arrow[rr, Rightarrow, "\text{aug.}"] & & A_k \end{tikzcd}$$

Claim (1)': $\boxed{\#\partial Z_k(\pi \times I, \omega) = -Z_k(\pi, \theta) + Z_k(\pi, \theta') + \alpha_k(\bar{\xi}, \bar{\eta})}$

$$\# \partial Z_k(\pi \times I, \omega) = \lambda_k \sum_{\Gamma} \underbrace{\partial I(\pi \times I, \omega, \Gamma)}\,.\,[\Gamma] \quad \in \mathcal{A}_k$$

$$\partial I(\pi \times I, \omega, \Gamma) = \sum_{\substack{X \subseteq \{1, \dots, 3k\} \\ |X| \geq 1}} I\left(\pi \times I, \left\{ \begin{matrix} \omega^{(i)} & i \notin X \\ \underbrace{\partial \omega^{(i)}} & i \in X \end{matrix} \right\}, \Gamma\right)$$

$$\partial \omega^{(i)} = \theta'^{(i)} - \theta^{(i)} + \partial^{fib} \omega^{(i)} + \partial^{peripheral} \omega^{(i)} \qquad (*)$$



$= E\bar{C}_2(\pi) \times I$

Hence $\# \partial Z_k(\pi \times I, \omega)$

$= \sum$ terms involving $\theta'^{(i)}$ $\longrightarrow$ $Z_k(\pi, \theta')$

$+ \sum$ terms involving $\theta^{(i)}$ $\longrightarrow$ $-Z_k(\pi, \theta)$

$+ \sum$ terms involving $\partial^{peripheral} \omega^{(i)}$ $\longrightarrow$ 0

since $\partial^{peripheral} \omega^{(i)}$ is supported in the faces of $E\bar{C}_2(\pi) \times I$ corresponding to a <u>trivial</u> D^4-bundle over $\partial I^k \times I$

$+ \sum$ terms involving $\partial^{fib} \omega^{(i)}$ $\longleftrightarrow$ ....

$+$ <u>no mixed terms</u>

since the terms of $(*)$ have support in different faces of $E\bar{C}_2(\pi) \times I$.

$$\partial^{fib}\omega^{(i)} = \sum_{\substack{A \subseteq \{1,\dots,2k,\infty\} \\ |A| \geq 2}} \underbrace{\partial^{fib,A}\omega^{(i)}}$$

the piece supported in the fibrewise-∂-stratum of $E\bar{C}_2(n)\times I$ corresponding to A.

$$\sum \text{terms involving } \partial^{fib}\omega^{(i)} = \lambda_k \sum_{\Gamma}\sum_{X}\sum_{A} I\left(\pi\times I, \begin{Bmatrix} \omega^{(i)} & i\notin X \\ \partial^{fib,A}\omega^{(i)} & i\in X \end{Bmatrix}, \Gamma\right).[\Gamma]$$

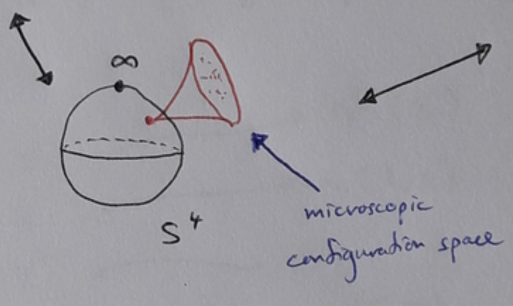
Γ: trivalent graph, vertices $= \{1,\dots,2k\}$, edges $= \{1,\dots,3k\}$

$X \subseteq \{1,\dots,3k\}$, $|X| \geq 1$

$A \subseteq \{1,\dots,2k,\infty\}$, $|A| \geq 2$

Facts:

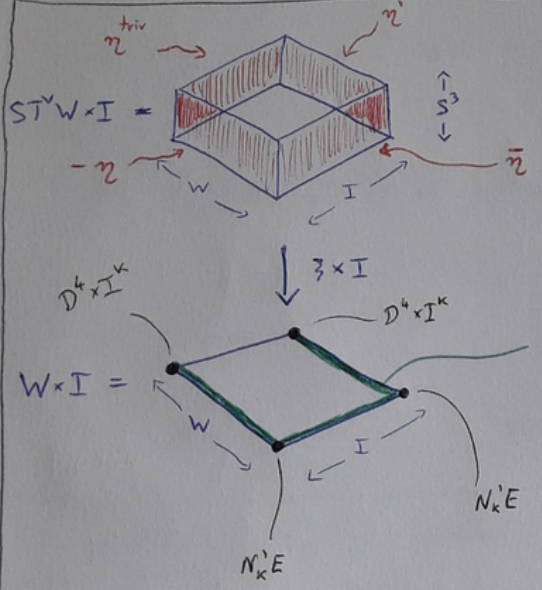
- terms with $\infty \in A$ cancel (dim. reasons)
- terms with $\infty \notin A$ and $3 \leq |A| \leq 2k-1$ cancel (dim. reasons + $\mathbb{Z}/2$-symmetry)
- terms with $\infty \notin A$ and $|A| = 2$ cancel (IHX relations)
- terms with $A = \{1,\dots,2k\}$



$\alpha_k(\bar{\xi},\bar{\eta})$.

Proof of equation (2) : (philosophically similar)

$$
\begin{array}{c}
\alpha_k(\xi\times I, \bar{\bar{\eta}}) \\
\cap \\
C_1(ST^vW\times I)\otimes \mathcal{A}_k \xrightarrow{\ \partial\ } C_0(ST^vW\times I)\otimes \mathcal{A} \xrightarrow[\ \ \#\ \ ]{\text{aug.}} \mathcal{A}_k
\end{array}
$$



Lem: $\exists\ \bar{\bar{\eta}}$ $\mathbb{Z}/_2$-inv. codim.-3-chain on $ST^vW\times I$ such $\partial\bar{\bar{\eta}}$ restricts to η, η^{triv}, η' and $\bar{\eta}$

$\eta^{triv} = \{\pm a\}\times D^4\times I^{k+1}$ on $S^3\times D^4\times I^{k+1}$

= diagram at bottom of page 12 (with 3 dim. collapsed to 1 dim.)

Claim (2)': $\boxed{\#\partial\alpha_k(\xi\times I, \bar{\bar{\eta}}) = -\alpha_k(\xi,\eta) + \alpha_k(\xi,\eta') + \alpha_k(\bar{\xi},\bar{\eta})}$

Note that $\partial\bar{\bar{\eta}} = \eta' - \eta + \eta^{triv} + \bar{\eta} + \partial^{fib}\bar{\bar{\eta}}$ $\Big]$ (**)

Hence $\#\partial\alpha_k(\xi\times I, \bar{\bar{\eta}})$

$= \sum$ terms involving η' $\longrightarrow \alpha_k(\xi,\eta')$

$+\sum$ terms involving η $\longrightarrow -\alpha_k(\xi,\eta)$

$+\sum$ terms involving η^{triv} $\longrightarrow 0$

$+\sum$ terms involving $\bar{\eta}$ $\longrightarrow \alpha_k(\bar{\xi},\bar{\eta})$

$+\sum$ terms involving $\partial^{fib}\bar{\bar{\eta}}$ $\longrightarrow$

all strata except $A=\{1,2,\dots,2k\}$ cancel as before (dim. reasons + $\mathbb{Z}/_2$-symmetry + IHX relation)

this time $\nexists$ stratum corresponding to $A=\{1,2,\dots,2k\}$ in $\bar{S}_{2k}$, since <u>not all</u> points may collide simultaneously in microscopic config spaces.

Oriented bordism-invariance of $\hat{Z}_k$

Oriented bordism:

$$\widetilde{\Omega}_k(X) = \{\text{closed } k\text{-manifolds with based map to } X\} \big/ \text{oriented bordism}$$

$$\rightsquigarrow \quad \widetilde{\Omega}_k(BG) \cong \{\text{closed } k\text{-manifolds with principal } G\text{-bundle trivialised over the basepoint}\} \big/ \text{oriented bordism}$$

$$\rightsquigarrow \quad \widetilde{\Omega}_k(B\mathrm{Diff}_\partial(D^4)) \cong \{\text{closed } k\text{-manifolds with smooth } D^4\text{-bundle (with trivial } \partial\text{) trivialised over the basepoint}\} \big/ \text{oriented bordism}$$

$\exists$ natural map

$$H = \text{“Hurewicz”}: \quad \pi_k B\mathrm{Diff}_\partial(D^4) \longrightarrow \widetilde{\Omega}_k(B\mathrm{Diff}_\partial(D^4))$$

Proposition: $\hat{Z}_k$ is oriented-bordism-invariant, in other words:

$$\begin{tikzcd}
\pi_k B\mathrm{Diff}_\partial(D^4) \arrow[rr, "H"] \arrow[dr, two heads] \arrow[dd, "\hat{Z}_k"'] & & \widetilde{\Omega}_k(B\mathrm{Diff}_\partial(D^4)) \\
& \mathrm{im}(H) \arrow[ur, hook] \arrow[dl, dashed] & \\
A_k & &
\end{tikzcd}$$

More generally, if $\tilde{h}_*$ is a (reduced) generalised homology theory, choosing $x \in \tilde{h}_0(S^0) = h_0(pt)$ determines a Hurewicz map

$$\tilde{h}_0(S^0) \cong \tilde{h}_k(S^k)$$

$$\pi_k(X) \xrightarrow{H} \tilde{h}_k(X)$$

$$[S^k \xrightarrow{f} X] \longmapsto f_*(x)$$

Proposition': if $x \in h_0(pt)$ generates a $\mathbb{Z}$-summand then:

$$\begin{tikzcd}
\pi_k(BG) \arrow[rr, "H"] \arrow[dr, two heads] \arrow[dd, "\varphi"'] & & \tilde{h}_k(BG) \\
& \mathrm{im}(H) \arrow[ur, hook] \arrow[dl, dashed] & \\
V & &
\end{tikzcd}$$

any homom. to a $\mathbb{Q}$-vector-space

G any topological group

Prop' $\Rightarrow$ Prop : take
$G = \mathrm{Diff}_\partial(D^d)$
$\varphi = \hat{Z}_n$
$\tilde{h}_* =$ oriented bordism

and note that $\Omega_0(pt) \cong \mathbb{Z}\langle \bullet^+ \rangle$ $\square$.

Rmk:
- We could take $\tilde{h}_* =$ K-theory
- But we cannot take $\tilde{h}_* =$ unoriented bordism, since $\Omega_0^{unor}(pt) \cong \mathbb{Z}/2\langle \bullet \rangle$

---

sketch proof of Prop':

$$\begin{tikzcd}
\pi_k BG \arrow[rr, "H"] \arrow[dr] \arrow[dd, "\varphi"'] & & \tilde{h}_k(BG) \arrow[dr] & \\
& \pi_k BG \otimes \mathbb{Q} \arrow[rr, "H\otimes\mathbb{Q}"] \arrow[dl] & & \tilde{h}_k(BG)\otimes\mathbb{Q} \\
V & & &
\end{tikzcd}$$

- By a diagram chase, it suffices to prove that $H\otimes\mathbb{Q}$ is injective.
- To see this:

class.^{n} of rational gen. hom. theories

$$\begin{tikzcd}
\pi_* BG \otimes \mathbb{Q} \arrow[r, "H\otimes\mathbb{Q}"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{h}_*(BG)\otimes\mathbb{Q} \arrow[r, phantom, "\cong"] & \tilde{H}_*(BG;\mathbb{Q}) \otimes h_*(pt) \\
& & H_*(BG;\mathbb{Q}) \otimes h_*(pt) \arrow[u, "(\cong \text{ in deg}>0)\otimes \mathrm{id}"'] \\
\mathcal{U}(\pi_* BG \otimes \mathbb{Q}) \underset{\mathbb{Z}}{\otimes} \mathbb{Z} \arrow[rr, "\mathrm{id}\otimes(1\mapsto x)"] & & \mathcal{U}(\pi_* BG \otimes \mathbb{Q}) \otimes h_*(pt) \arrow[u, "\cong"', "\text{Milnor-Moore}"]
\end{tikzcd}$$

split-injective iff $x \in h_0(pt)$ generates a $\mathbb{Z}$-summand.

universal enveloping Hopf algebra

$\square$.

Outlook:

$$\begin{tikzcd}
\pi_{k-1}\mathrm{Diff}_\partial(D^4) \cong \pi_k B\mathrm{Diff}_\partial(D^4) \arrow[rr, "\hat{Z}_k"] \arrow[dd, "H"'] \arrow[dr] & & \mathcal{A}_k \\
 & \mathrm{im}(H) \arrow[ur, "\hat{Z}_k"] \arrow[dl] \arrow[r, phantom, "(*)"] & \\
\Omega_k B\mathrm{Diff}_\partial(D^4) & & \mathbb{Z}\left\{\begin{array}{l}\text{labelled trivalent graphs}\\ \text{with no loops or}\\ \text{multiple edges}\end{array}\right\} \arrow[ul] \arrow[uu, "\Gamma \mapsto [\Gamma]"'] \arrow[ll, "\Gamma \mapsto \pi^\Gamma"]
\end{tikzcd}$$

red = next week

green = in 2 weeks

$(*)$ commutes

$$\Updownarrow$$

$$\hat{Z}_k(\pi^\Gamma) = [\Gamma]$$

$$\Big\downarrow {\scriptstyle - \otimes \mathbb{Q}}$$

$$\begin{tikzcd}
 & \mathcal{A}_k \\
\mathrm{im}(H) \otimes \mathbb{Q} \arrow[ur, "\hat{Z}_k \otimes \mathbb{Q}"] & \\
 & \mathbb{Q}\left\{\begin{array}{l}\text{labelled trivalent graphs}\\ \text{with no loops or}\\ \text{multiple edges}\end{array}\right\} \arrow[uu, two heads] \arrow[ul]
\end{tikzcd}$$

also commutes

$\Rightarrow$ $\hat{Z}_k \otimes \mathbb{Q} : \pi_{k-1}\mathrm{Diff}_\partial(D^4) \twoheadrightarrow \mathcal{A}_k$

is surjective.