Kontsevich characteristic classes à la Watanabe

(V)

GeMAT seminar
IMAR
17 July 2020

Following: T. Watanabe

"Some exotic nontrivial elements of the rational homotopy groups of $\mathrm{Diff}(S^4)$"

arXiv: 1812.02448

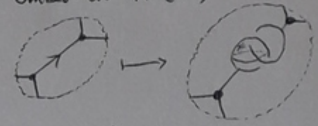
Recap

Γ trivalent labelled graph
(no self-loops or multiple edges)

[+ choose edge orientations so that each vertex looks like

]

embed in $\mathrm{int}(D^4)$



take regular neighbourhood $\rightsquigarrow W$

$\partial W \cong \partial\left(\begin{array}{c}\text{complement of}\\ \text{Borromean braid}\end{array}\right)$

$\rightsquigarrow \alpha_\Gamma$

handlebody $W \hookrightarrow \mathrm{int}(D^4)$

$$W = \coprod_{i=1}^{2k} V_i$$

$$\alpha_\Gamma : B_\Gamma \longrightarrow \mathrm{Diff}(\partial W)$$

$$B_\Gamma = (S^0)^k \times (S^1)^k$$

glue $W \times B_\Gamma$ to $(D^4 - \mathrm{int}(W)) \times B_\Gamma$ along α_Γ

$$\begin{array}{c} E^\Gamma \\ \big\downarrow \pi^\Gamma \\ B_\Gamma \end{array}$$

(D^4, ∂)-bundle

Last week $\boxed{\pi^\Gamma \text{ is bordant to some } (D^4,\partial)\text{-bundle over } S^k}$

$\hookrightarrow$ hence $\hat{Z}_k(\pi^\Gamma) \in \mathcal{A}_k$ is <u>defined</u>

Today THEOREM (Watanabe) $\boxed{\hat{Z}_k(\pi^\Gamma) = [\Gamma]}$

COROLLARY (Watanabe)
$$\pi_k B\mathrm{Diff}_\partial(D^4) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_k$$
$$\pi_k B\mathrm{Diff}_\partial(D^4) \otimes \mathbb{Q} \cong \pi_{k-1} \mathrm{Diff}_\partial(D^4) \otimes \mathbb{Q}$$

$\hookrightarrow$ <u>Part 1</u> : "<u>Graph-counting formula</u>" for $\hat{Z}_k(\pi)$ using (fibrewise) Morse functions

<u>Part 2</u> : $\begin{cases} \text{"Key lemma" — simplification of the graph-counting formula for } \pi = \pi^\Gamma \\ \text{"Homological lemma"} \rightsquigarrow \text{conclusion of the proof} \end{cases}$

Consider $\begin{array}{c} E \\ \downarrow \pi \\ B \end{array}$

$(\mathbb{R}^4, \mathbb{R}^4 - \mathrm{int}(D^4))$ - bundle $\equiv$ (D^4, ∂) - bundle

Choose $E \xrightarrow{f} \mathbb{R}$ <u>fibrewise Morse function</u> (restriction to each fibre is Morse)

ξ <u>v-gradient</u> (vertical vector field $\begin{array}{c} T^vE \\ \downarrow\uparrow \\ E \end{array}$

such that ξ_b is gradient-like for f_b)

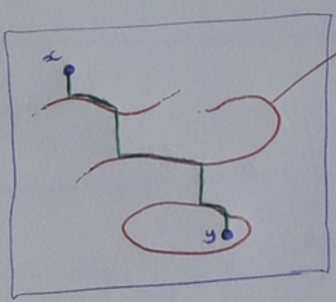
restriction to fibre over $b \in B$

$B \xrightarrow{h} \mathbb{R}$ <u>Morse function</u> with exactly one max (index k)
& min (index 0)
on each component of B

η <u>gradient-like vector field</u> for h

<u>Z-path</u>:

Crit(f)

$E =$ 

$\xrightarrow{f} \mathbb{R}$

$\downarrow \pi$

$B \xrightarrow{h} \mathbb{R}$

Finite concatenation of
- vertical
- horizontal

pieces.

vertical: contained in one fibre of π ($\Rightarrow h \circ \pi$ constant); descending w.r.t. f; flow-line of $-\xi$

horizontal: contained in Crit(f) ($\Rightarrow f$ constant); descending w.r.t. $h \circ \pi$; its projection to B is a flow-line of $-\eta$

Assumptions

- $\forall\, p, q$ components of $\mathrm{Crit}(f) \subseteq E$, $\quad \underbrace{D_p}_{\text{desc. mfld}} \cap \underbrace{A_q}_{\text{asc. mfld.}}$ is transverse

- $T^\vee D_p\big|_p$ is orientable

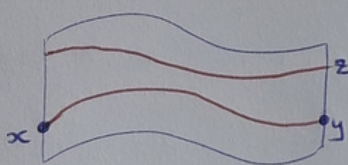
- $D_p \cap A_q = \emptyset \quad$ if $\quad |p| < |q|$.

- $\mathrm{CritValues}(f_b) \subseteq \mathbb{R}$ is independent of $b \in B$

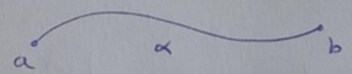
- if α is a flow-line of $-\eta$ between $a, b \in B$ critical points of h with consecutive indices,

then α is transverse to the conical stratification $B = \bigsqcup_{i=0}^{k} B^{(i)}$ induced by f

and:



$= E|_\alpha$ $\qquad$ $\mathrm{Crit}(f)$



signed # of Z-paths $x \rightsquigarrow y$ in $E|_\alpha$ $= 1$

signed # of Z-paths $x \rightsquigarrow z$ in $E|_\alpha$ $= 0$

Combinatorial propagator

For each maximum point $b \in B$ $\quad$ ($\exists$ one per component)

a chain map $\quad g : \underbrace{C_*^{\mathrm{Morse}}(\xi_b)}_{\text{gen. by critical pts of } f_b} \longrightarrow C_{*+1}^{\mathrm{Morse}}(\xi_b)$

with $\quad g\partial + \partial g = \mathrm{id}$.

Now choose:

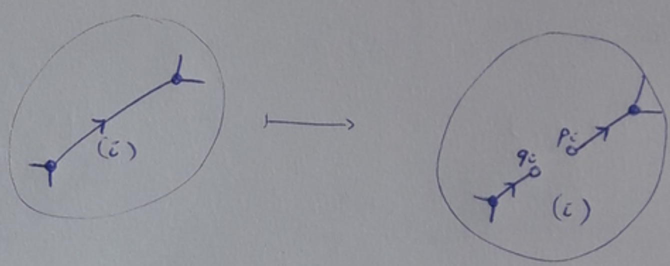
$f^{(1)} \ldots\ldots f^{(3k)}$ fibrewise Morse functions on E
$\xi^{(1)} \ldots\ldots \xi^{(3k)}$ v-gradients
h Morse function on B
η gradient-like for h

satisfying the assumptions above.

$g^{(1)} \ldots\ldots g^{(3k)}$ combinatorial propagators

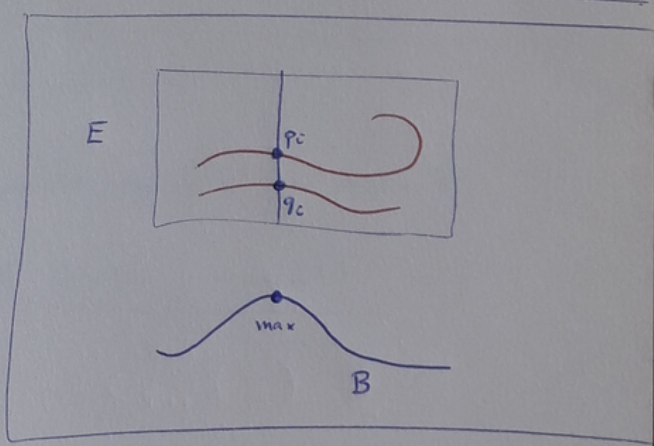
C-graph

- Γ connected trivalent labelled graph $\left(\begin{array}{l} 3k \text{ edges} \\ 2k \text{ vertices}\end{array}\right)$
- orientation of each edge
- for <u>some</u> of its edges:



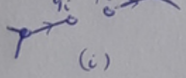
$p_i, q_i \in \mathrm{Crit}(f) \cap \pi^{-1}(\max(h))$

$$\left(\deg_i(\Gamma) = |p_i| - |q_i| \right)$$



Z-graph

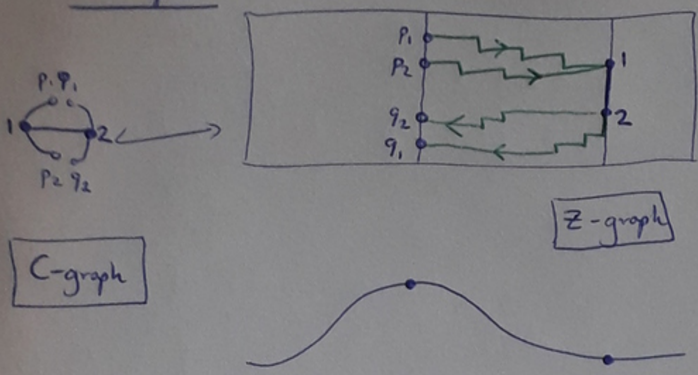
The above, plus:

- $\Gamma \hookrightarrow E$ embedding:
- univalent vertex labelled by $p \longmapsto p$
- trivalent vertices $\longmapsto$ <u>same</u> fibre of E
- non-split edge $i \longmapsto$ flow-line for $-\xi^{(i)}$
-  $\longmapsto \left[\begin{array}{l} \underline{Z\text{-path}} \text{ to } p_i \\ \text{inverse } \underline{Z\text{-path}} \text{ from } q_i \end{array}\right]$ that project to the <u>same</u> flow-line for $-\eta$.

w.r.t. $(f^{(i)}, \xi^{(i)}, h, \eta)$

(up to piecewise smooth reparametrisation)

Example:



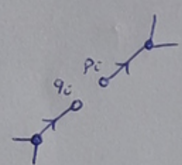
Trace of a C-graph

assume that each separated edge has degree = 1

Γ C-graph

$$\operatorname{trace}(\Gamma) := \prod_{\substack{\text{separated edges} \\ i \text{ of } \Gamma}} \left(- \text{coeff of } p_i \text{ in } g^{(i)}(q_i) \right) . [\Gamma] \in \mathcal{A}_k$$

with its separated edges glued back together again!



$$\begin{array}{ccc} C_*^{Morse}(\xi_{max}) & \xrightarrow{g^{(i)}} & C_{*+1}^{Morse}(\xi_{max}) \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ q_i & & p_i \end{array}$$

$\xi_{max} = \xi_b$ for $b \in B$ a <u>maximum</u> of h.

Lemma. $\mathcal{M}_\Gamma^Z := \{ \text{all } Z\text{-graphs with underlying } C\text{-graph} = \Gamma \}$

if $f^{(\cdot)}, \xi^{(\cdot)}$ <u>generic</u> and each separated edge of Γ has degree = 1

then $\mathcal{M}_\Gamma^Z$ is a <u>compact, oriented 0-dim. manifold</u>

$$\rightsquigarrow \# \mathcal{M}_\Gamma^Z \in \mathbb{Z}.$$

Def.

$$Z_k^{Morse}(\pi, f^{(\cdot)}, \xi^{(\cdot)}, h, \eta, g^{(\cdot)}) := \lambda_k \sum_{\substack{C\text{-graphs } \Gamma \text{ with} \\ \text{all separated edges} \\ \text{of degree } 1}} \overbrace{\# \mathcal{M}_\Gamma^Z}^{\text{signed sum of } Z\text{-graphs}} . \operatorname{trace}(\Gamma) \in \mathcal{A}_k$$

$$\lambda_k = \frac{1}{2^{3k}(2k)!\,(3k)!}$$

Given a choice of $f^{(\cdot)}$ fiberwise Morse functions

$\xi^{(\cdot)}$ v-gradients

$g^{(\cdot)} : C_*^{Morse}(\xi^{(\cdot)}_{max}) \longrightarrow C_{*+1}^{Morse}(\xi^{(\cdot)}_{max})$ combinatorial propagators

we may "turn upside-down" some of them:

$$f^{(i)} \longmapsto -f^{(i)}$$

$$\xi^{(i)} \longmapsto -\xi^{(i)}$$

$$g^{(i)} \longmapsto C_*^{Morse}(-\xi^{(i)}_{max}) \cong C_{4-*}^{Morse}(\xi^{(i)}_{max})^{\vee} \xrightarrow{\ g^{(i)\vee}\ } C_{3-*}^{Morse}(\xi^{(i)}_{max})^{\vee} \cong C_{*+1}^{Morse}(-\xi^{(i)}_{max})$$

call this $-g^{(i)}$

<u>Def</u>

$$Z_k^{av.\,Morse}(\pi, f^{(\cdot)}, \xi^{(\cdot)}, h, \eta, g^{(\cdot)}) := \frac{1}{2^{3k}} \sum_{(\varepsilon_1, \dots, \varepsilon_{3k}) \in \{\pm 1\}^{3k}} Z_k^{Morse}(\pi, \varepsilon . f^{(\cdot)}, \varepsilon . \xi^{(\cdot)}, h, \eta, \varepsilon . g^{(\cdot)})$$

"averaged Morse"

<u>Theorem</u> (Graph-counting formula)

$\exists$ admissible propagators $\theta^{(1)} \dots \theta^{(3k)}$ for π such that:

$$\boxed{Z_k(\pi, \theta^{(\cdot)}) = (-1)^{3k} \cdot Z_k^{av.\,Morse}(\pi, f^{(\cdot)}, \xi^{(\cdot)}, h, \eta, g^{(\cdot)})}$$

"Key lemma" of §5

For the bundles $\begin{array}{c} E^{\Gamma} \\ \downarrow \pi^{\Gamma} \\ B_{\Gamma} = (S^0)^k \times (S^1)^k \end{array}$ constructed earlier,

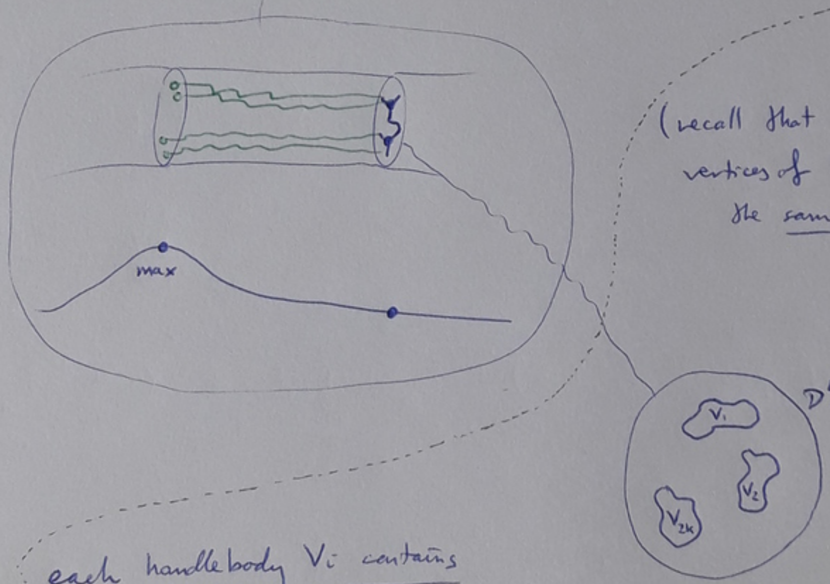
if we take (h, η) to be the "standard" Morse function & gradient on $\coprod_{2^k} (S^1)^k$

then one may construct
- fibrewise Morse functions $f^{(1)}, \ldots, f^{(3k)} : E^{\Gamma} \longrightarrow \mathbb{R}$
- v-gradients $\xi^{(1)}, \ldots, \xi^{(3k)}$
- combinatorial propagators $g^{(1)}, \ldots, g^{(3k)}$

such that

(a) $$\hat{Z}_k(\pi^{\Gamma}) = (-1)^{3k} \cdot \underbrace{Z_k^{\text{av. Morse}}(\pi, f^{(\cdot)}, \xi^{(\cdot)}, h, \eta, g^{(\cdot)})}_{(*)}$$

(b) a Z-graph $\Gamma' \hookrightarrow E^{\Gamma}$ may only contribute non-trivially to the sum $(*)$ if:



(recall that all trivalent vertices of Γ' lie in the same fibre of E^{Γ})

fibre of E^{Γ} containing the trivalent vertices of Γ'

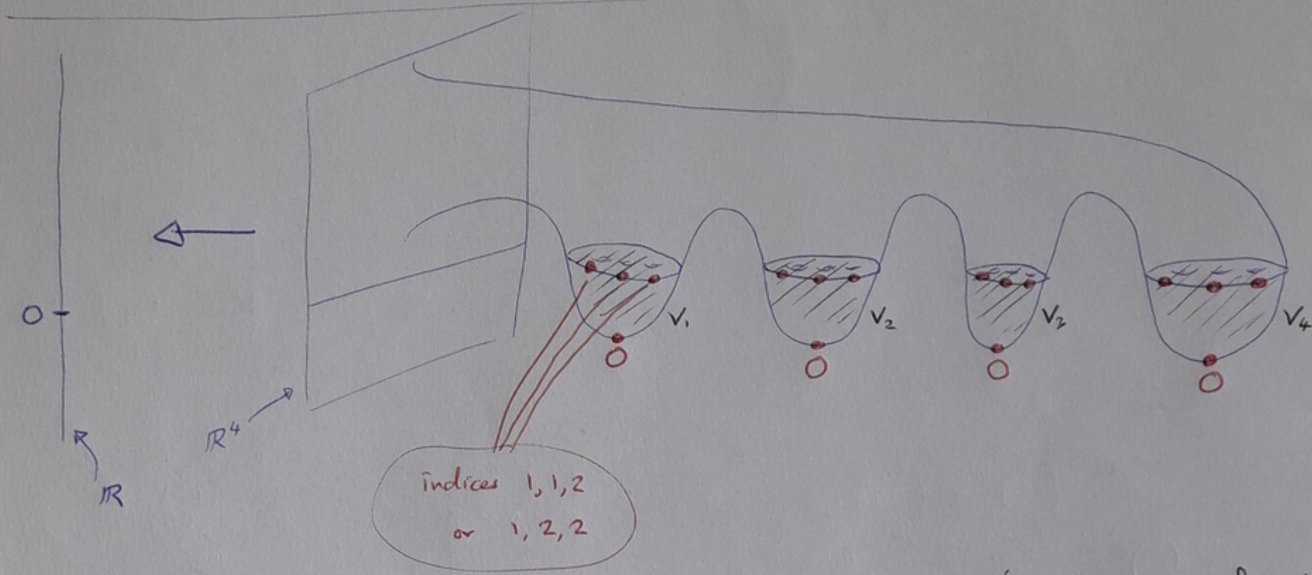
each handlebody V_i contains exactly one trivalent vertex of Γ'

in a better picture, the handlebodies V_i should be linked by Hopf links according to Γ

Sketch proof of (a) & details of Morse functions

$$h : (S^1)^k \longrightarrow \mathbb{R}$$
$$(z_1, \ldots, z_k) \longmapsto \sum_i \mathrm{Re}(z_i)$$

$$f^{(c)}\big|_{\text{fibre}} : \mathbb{R}^4 \longrightarrow \mathbb{R}$$



(no critical pts of order 4)

- construct bordism (of manifolds not of bundles!) from E^Γ to $D^4 \times S^k$

  $\Rightarrow$ by Proposition last week,

$$\hat{Z}_k(\pi^\Gamma) = Z_k(\pi^\Gamma, \ldots\ldots) - \alpha_k(\ldots\ldots) + P_k(\ldots\ldots)$$

any choice of propagators on π^Γ

- show that the Morse functions above (+ v-gradients) satisfy the assumptions of §3

  $\Rightarrow$ by the Graph-counting formula,

$$Z_k(\pi^\Gamma, \ldots\ldots) = (-1)^{3k} \, Z_k^{\text{av. Morse}}(\pi^\Gamma, f^{(\cdot)}, \xi^{(\cdot)}, h, \eta, g^{(\cdot)})$$

some choice of propagators

- check that, with these choices,

$$\alpha_k(\ldots\ldots) = 0,$$
$$\& \quad P_k(\ldots\ldots) = 0.$$

$\square$.

Lemma (actually proven a bit later)

$Z_k^{\text{Morse}}(\pi, f^{(\cdot)}, \xi^{(\cdot)}, h, \eta, g^{(\cdot)})$ doesn't change if we multiply some of the $f^{(i)}$ by (-1).

Upshot

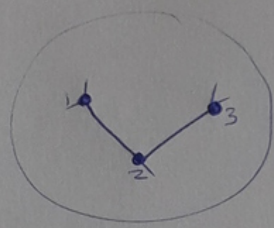
$$\hat{Z}_k(\pi^\Gamma) = \frac{(-1)^{3k}}{2^{3k}(2k)!(3k)!} \sum_{\substack{\text{labelled} \\ \text{trivalent} \\ \text{graphs } H}} \begin{pmatrix} \#\text{ of } Z\text{-graphs with} \\ \text{underlying } H \end{pmatrix} \cdot [H]$$

(counted with sign & with a coefficient $\in \mathbb{Q}$ depending on the combinatorial propagators $g^{(1)} \cdots g^{(3k)}$)

$\Rightarrow$ Our goal is:

$$\begin{pmatrix} \text{signed } \# \text{ of } Z\text{-graphs} \\ \text{with underlying} \\ \text{trivalent graph} \cong H \end{pmatrix} = \begin{cases} 0 & H \not\cong \Gamma \\ \text{non-}0 & H = \Gamma \end{cases}$$

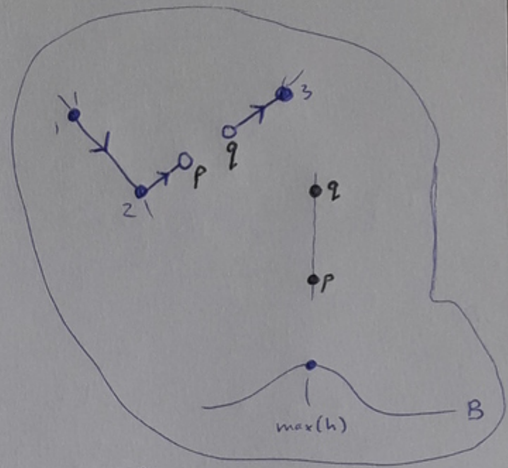
Picture.



labelled trivalent graph $\qquad H$

$\downarrow$ edge orientations (any!)
split some edges
label with critical points of $E^\Gamma|_{\max(h)}$

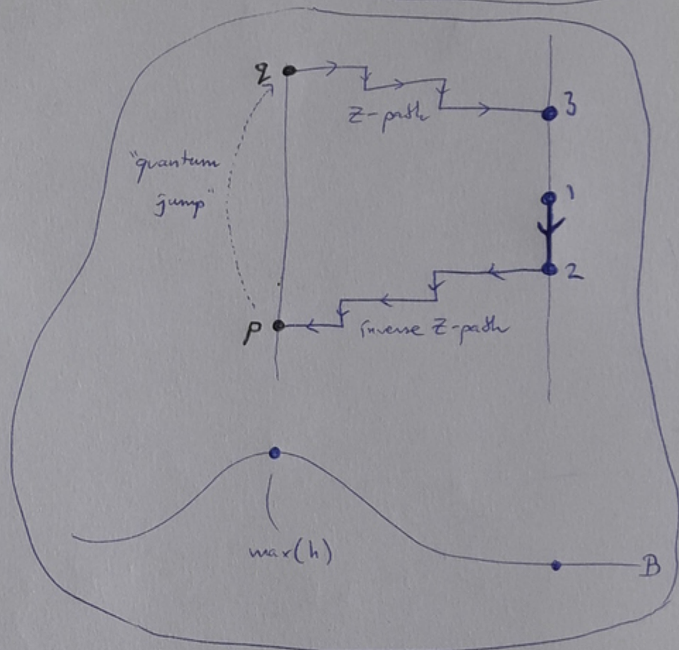
C-graph $\qquad H$



$\downarrow$ embed into E^Γ

edges $\longmapsto$ (inverse) Z-paths

Z-graph $\qquad I : H \hookrightarrow E^\Gamma$



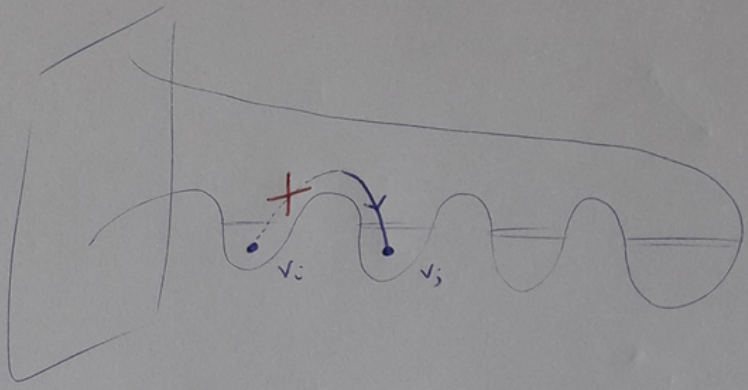
Observation

Suppose H is a C-graph counting non-trivially towards the sum $\hat{Z}_\kappa(\pi^\Gamma)$

Then
- all edges of H must be split
- all 'outgoing' univalent vertices are labelled by point of index 0, 1, 2
- all 'incoming' —— 1, 2, 3.

proof by picture

(i)



non-split edge $\longmapsto$ vertical flow line in fibre $\cong \mathbb{R}^4$
between $\underbrace{\text{points in distinct } V_i, V_j}$ $(i \neq j)$

by "key lemma" (b)

This can't exist.

(ii) the only critical points of $f^{(i)}|_{\text{fibre}}$ in $\coprod_{i=1}^{2k} V_i$ have index 0, 1, 2.

(iii) - no c. points of index 4
- Z-paths must have a non-trivial vertical segment $\Rightarrow$ index cannot be 0.

$\longrightarrow$ combinatorial restriction on possible C-graphs H in the sum......

Homological lemma

(black box)

$$\hat{Z}_k(\pi^\Gamma) = \frac{1}{2^{3k}(2k)!(3k)!} \sum_H \sum_{\sigma\in\Sigma_{2k}} \left((\pm) \prod_{\substack{\text{edges} \\ (i,j) \text{ of } H^\sigma}} \sum_{a,b=1}^{3} lk\left(\beta_a^{(i)}, \beta_b^{(j)}\right)\right) \cdot [H] \quad \in \mathcal{A}_k$$

labelled trivalent graph with $2k$ vertices, + edge-orientations

we may assume no self-loops or multiple edges (otherwise $[H]=0$)

depending on the edge-labelling of H

H with its vertex-labelling permuted by σ

where $\beta_1^{(i)}, \beta_2^{(i)}, \beta_3^{(i)}$ = cores of (positive-index) handles of $\underbrace{V_i \subset \mathrm{int}(D^4)}_{\text{determined by } \Gamma}$

Calculate this in stages:

(1) $$\sum_{a,b=1}^{3} lk\left(\beta_a^{(i)}, \beta_b^{(j)}\right) = \begin{cases} 1 & \text{if } \exists \text{ edge } i \text{---} j \text{ in } \Gamma \\ 0 & \text{o/w} \end{cases}$$

Note: The sign is always $+1$ independently of the orientation of the edge i --- j of H, since the linking number is symmetric if at least one of the manifolds involved is odd-dimensional.

(2) $$\prod_{\substack{\text{edges} \\ (i,j) \text{ of } H^\sigma}} \left(\text{---}\,''\,\text{---} \right) = \begin{cases} 1 & \text{if } H^\sigma \cong \Gamma \text{ modulo edge-labellings (preserving vertex-labellings)} \\ 0 & \text{o/w} \end{cases}$$

(3) $\displaystyle\prod_{\substack{\text{edges}\\ (i,j) \text{ of } H^\sigma}} \sum_{a,b=1}^{3} \mathrm{lk}\left(\beta_a^{(i)}, \beta_b^{(j)}\right) \cdot [H] = \begin{cases} [H] = [H^\sigma] = (\pm)[\Gamma] & \text{if } H^\sigma \cong \Gamma \text{ modulo edge-labellings} \\ 0 & \text{o/w} \end{cases}$

depending on the edge-labelling of H

cancels with the other $(\pm)$ in the formula!

(4) $\displaystyle\sum_{\sigma \in \Sigma_{2k}} \left((\pm) \text{———}\,''\,\text{———} \right) = \begin{cases} |\mathrm{Aut}(\Gamma)| \cdot [\Gamma] & \text{if } H \cong \Gamma \text{ as trivalent graphs} \\ 0 & \text{o/w} \end{cases}$

(5) $\displaystyle\sum_{H} \left(\text{———}\,''\,\text{———} \right) = \underbrace{\frac{2^{3k}(2k)!\,(3k)!}{|\mathrm{Aut}(\Gamma)|}}_{\substack{\#\text{ of choices of}\\ \text{edge-labelling, vertex-labelling, edge-orientations}\\ \text{for } \Gamma}} \cdot |\mathrm{Aut}(\Gamma)| \cdot [\Gamma]$

$$= 2^{3k}(2k)!\,(3k)! \cdot [\Gamma]$$

(6) $\displaystyle\hat{Z}_k(\pi^\Gamma) = \frac{1}{2^{3k}(2k)!\,(3k)!} \sum_{H} \left(\text{———}\,''\,\text{———} \right) = [\Gamma].$

$\square$.