Homology of configuration-section spaces

30 Oct. 2020

Topology seminar

IMAR

joint work with Ulrike Tillmann

arXiv: 2007.11607
2007.11613

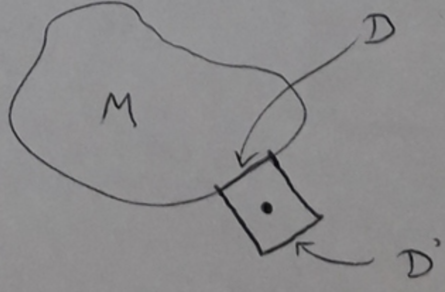
Physical system	Model
particles moving in a manifold M (no collisions)	$C_n(M) = \left\{ (x_1, \dots x_n) \in M^n \,\middle|\, \begin{matrix} x_i \neq x_j \\ \text{for } i \neq j \end{matrix} \right\} \Big/ \Sigma_n$
.... with internal parameters in a space X	$C_n(M,X) = \dfrac{\left\{ (x_1, \dots x_n) \in M^n \,\middle|\, \begin{matrix} x_i \neq x_j \\ \text{for } i \neq j \end{matrix} \right\} \times X^n}{\Sigma_n}$
.... coupled with a <u>field</u> on M that may be singular (undefined) at the particles' locations	$CMap_n(M,X) = \left\{ \begin{matrix} z \subseteq M \\ M \setminus z \xrightarrow{f} X \end{matrix} \,\middle|\, \lvert z \rvert = n \right\}$ $C\Gamma_n\left(M, \begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}\right) = \left\{ \begin{matrix} z \subseteq M \\ M \setminus z \xrightarrow{s} E \end{matrix} \,\middle|\, \begin{matrix} \lvert z \rvert = n \\ s \text{ section} \end{matrix} \right\}$

Stabilisation

if $\partial M \neq \emptyset$ (and we impose a condition that the field is prescribed on a disc $D \subseteq \partial M$)

there are <u>stabilisation maps</u>

$$C_n(M) \longrightarrow C_{n+1}(M)$$
$$C_n(M,X) \longrightarrow C_{n+1}(M,X)$$
$$CMap_n(M,X) \longrightarrow CMap_{n+1}(M,X)$$
$$C\Gamma_n(M,\xi) \longrightarrow C\Gamma_{n+1}(M,\xi)$$



Stability

M —— <u>connected</u> manifold with $\partial M \neq \emptyset$

<u>Thm</u> [McDuff, Segal, '70s]

The sequence $\cdots \to C_n(M) \xrightarrow{(*)} C_{n+1}(M) \to \cdots$ is <u>homologically stable</u> : $(*)$ induces $\cong$ on H_i for all $i \leq \frac{n}{2}$.

With internal parameters:

<u>Thm</u> [Randal-Williams, '13]

The sequence $\cdots \to C_n(M,X) \longrightarrow C_{n+1}(M,X) \to \cdots$ is <u>homologically stable</u> <u>if</u> X is path-connected.

<u>Rmks</u>

- $C_n(M,X)$ is <u>not</u> hom. stable if $\pi_0(X) \neq *$

  $\hookrightarrow$ H_0 grows unboundedly with n.

- The same will generally be true of $CMap_n(M,X)$ and $C\Gamma_n(M,\xi)$ without control over the "monodromy" / "charge" of the field near the particles.

Definition (CMap) [Ellenberg–Venkatesh–Westerland]

(CΓ similar; see next page for a hint)

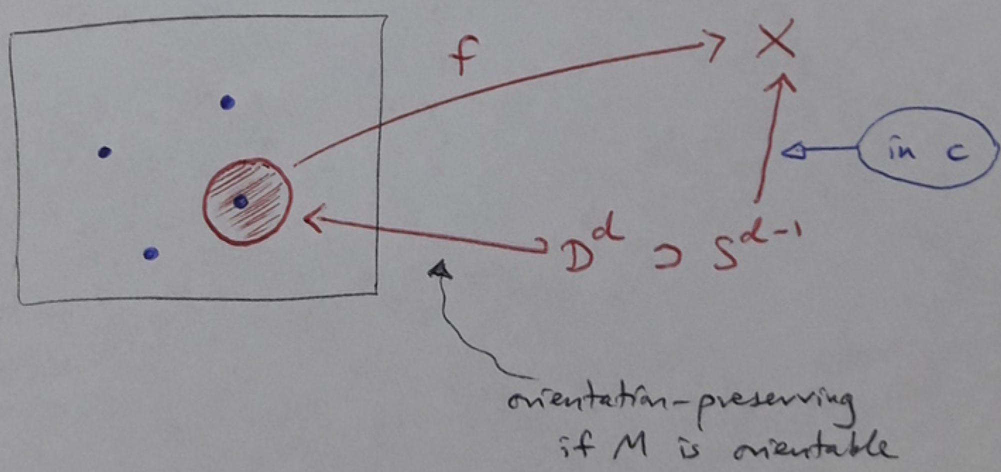
$$\mathrm{CMap}_n(M,X) = \left\{ \begin{array}{l} z \subseteq M \\ M \smallsetminus z \xrightarrow{f} X \end{array} \;\middle|\; \begin{array}{l} |z| = n \\ f(D) = * \end{array} \right\}$$

$$\cong \frac{\mathrm{Diff}_\partial(M) \times \mathrm{Map}_D(M \smallsetminus z, X)}{\mathrm{Diff}_\partial(M, z)}$$

$z \subseteq M$
$|z| = n$
fixed

For a subset $c \subseteq [S^{d-1}, X]$

$\mathrm{CMap}^c_n(M,X)$ = subspace where :



I.e. the "charge" of each particle is constrained to lie in c.

Idea / hint for $C\Gamma$

$$\{ p \in M,\ \text{germ of section of } \xi \text{ near } p \}$$
$$\| $$

$$\begin{tikzcd} E \arrow[d, "\xi"] & & \Sigma(\xi) \arrow[d, "\eta"] \\ M & & M \end{tikzcd}$$

bundle

manifold

$\rightsquigarrow$

covering space

$$C\Gamma(M, \xi) \longrightarrow \Gamma(\eta)$$

"charge" of a system := induced section of η

$\|$

points + field

(Details in §3 of arXiv: 2007.11607 ...)

Examples

Bundle	Interpretation of $C\Gamma$

1.
$$\begin{tikzcd} TM \setminus 0 \arrow[d] \\ M \end{tikzcd}$$

non-vanishing vector field on the complement.

2.
$$\begin{tikzcd} \tau^*(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\theta"] \\ M \arrow[r, "\tau"'] & BO(d) \end{tikzcd}$$

<u>θ-structure</u> on the complement

↳ spin
framing
almost complex ...

3.
$$\begin{tikzcd} f^*(BG^\delta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & BG^\delta \arrow[d] \\ M \arrow[r, "f"'] & BG \end{tikzcd}$$

<u>flat connection</u> (on the complement)

for the principal G-bundle $\begin{smallmatrix} P \\ \downarrow \\ M \end{smallmatrix}$

classified by f.

4.
$$\begin{tikzcd} D^2 \times BG \arrow[d] \\ D^2 \end{tikzcd}$$

$$c \subseteq [S^1, BG] = \mathrm{Conj}(G)$$

branched coverings of D^2 with monodromy $\subseteq c$

<u>Hurwitz spaces</u>

5.
$$\begin{tikzcd} M^d \times K(\mathbb{Z}, d) \arrow[d] \\ M^d \end{tikzcd}$$

$$\underbrace{\text{configuration \& } S^1\text{-parameter for each particle}}_{\text{non-locally!}}$$

$\longleftarrow\cdots\cdots\longrightarrow$ magnetic monopoles

Theorem (P.-Tillmann '20)

If the allowed charges are <u>fully constrained</u>,

then $\cdots \to C\Gamma_n^c(M,\xi) \longrightarrow C\Gamma_{n+1}^c(M,\xi) \longrightarrow \cdots$

is <u>homologically stable</u>.

Def^n (c is <u>fully constrained</u>):

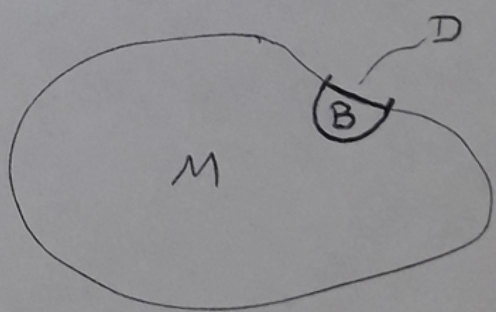
(CMap) • $$\begin{array}{c} [S^{d-1},X] \cong \pi_{d-1}(X)/\pi_1(X) \twoheadleftarrow \pi_{d-1}(X) \\ \cup\!\mid \\ c \end{array}$$

$$\boxed{c = \text{a single } \pi_1(X)\text{-orbit of size } 1}$$

(CΓ) •

$$\begin{array}{ccccc} X \to E & & \Sigma(\xi) & \supset & \Sigma(\xi|_B) \cong B \times [S^{d-1},X] \\ \downarrow \xi & \rightsquigarrow & \downarrow \eta & & \downarrow \eta|_B \\ M & & M & \supset & B \end{array}$$

$$\begin{array}{ccc} \Gamma(\eta) & \xrightarrow{\text{restriction}} & \Gamma(\eta|_B) \\ \cup\!\mid & & \wr\wr \\ c & & [S^{d-1},X] \end{array}$$



Rmk If $\pi_1(X)=0$, then the condition is just $\boxed{|c|=1}$.

Examples

1 (non-vanishing vector fields)

$$X = \mathbb{R}^d - \{0\} \simeq S^{d-1} \qquad \pi_{d-1}(X) \cong \mathbb{Z}$$

$c \longleftrightarrow$ prescribing the winding # of the vector field around each singularity.

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2 (θ-structures)

$$\begin{array}{c} X \\ \downarrow \\ BO(d)\langle k\rangle \\ \downarrow \\ BO(d) \end{array}$$

k-connected cover

$k=1$	$BSO(d)$	orientations
$k=2$	$BSpin(d)$	spin
$k=4$	$BString(d)$	string

$\pi_{d-1}(X) = 0$ if $k \leq d-2$.

---

3 (flat connections on a principal G-bundle $\begin{array}{c} P \\ \downarrow \\ M \end{array}$)

$$\begin{array}{c} X \\ \downarrow \\ BG^{\delta} \\ \downarrow \\ BG \end{array} \qquad \pi_i(X) \cong \pi_i(G) \quad i \geqslant 2$$

$\boxed{\text{e.g.}}$ if $d=3$, $G =$ Lie group, then $\pi_{d-1}(X) \cong \pi_2(G) = 0$.

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4 (Hurwitz spaces) $\quad c \subseteq \mathrm{Conj}(G)$

"fully constrained" $\longleftrightarrow$ $c =$ single conj. class of size 1

$\longleftrightarrow$ $c \in ZG$

[Ellenberg – Venkatesh – Westerland '16] proved $\mathbb{Q}$-hom. stability for Hurwitz spaces with a <u>weaker</u> condition on c. $\longrightarrow$ Cohen – Lenstra conjecture.

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5 (Monopoles) $\quad X = K(\mathbb{Z}, d)$

so $\pi_{d-1}(X) = 0$.

Q: What is the stable homology?

Already answered by [EVW] for CMap:

(for simplicity suppose that M is parallelisable)

Thm [EVW] The stable H_* of $CMap^c_n(M,X)$ is isomorphic to the H_* of the space of maps:

$$\begin{tikzcd} M \arrow[r, dashed] & A_d(X,c) & D^d \times Map(S^{d-1},X) \arrow[l] \\ \partial M \arrow[u, hook] \arrow[r, dashed] & X \arrow[u] \arrow[ur, phantom, "\lrcorner"] & S^{d-1} \times Map^c(S^{d-1},X) \arrow[l, "ev"'] \arrow[u, hook] \end{tikzcd}$$

Thm [P.-Tillmann '20] The stable H_* of $C\Gamma^c_n(M,\xi)$ is isomorphic to the H_* of the space of sections:

$$\begin{tikzcd} M \arrow[r, dashed] & E_d(\xi,c) \arrow[l, bend right] \\ \partial M \arrow[u, hook] \arrow[r, dashed] & E|_{\partial M} \arrow[u] \arrow[l, bend left, "\xi|_{\partial M}"] \end{tikzcd}$$

$$\left(\begin{tikzcd} E \arrow[r] \arrow[dr, "\xi"'] & E_d(\xi,c) \arrow[d] \\ & M \end{tikzcd} \right)$$

Idea of proof of stability (CMap)

$$
\begin{tikzcd}
\mathrm{Map}^c_*(M \setminus z_n, X) \arrow[r] \arrow[d] & \mathrm{CMap}^c_n(M, X) \arrow[r] \arrow[d] & C_n(\mathring{M}) \arrow[d] & \ni z_n \arrow[d, mapsto] \\
\mathrm{Map}^c_*(M \setminus z_{n+1}, X) \arrow[r] & \mathrm{CMap}^c_{n+1}(M, X) \arrow[r] & C_{n+1}(\mathring{M}) & \ni z_{n+1}
\end{tikzcd}
$$

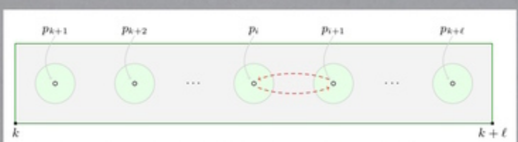
Induces (*) map of Serre spectral sequences

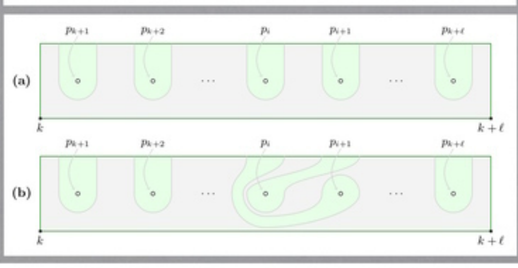
(**) monodromy actions $\pi_1 C_n(M) \curvearrowright \mathrm{Map}^c_*(M \setminus z_n, X)$

Steps (1) (**) extend to a functor

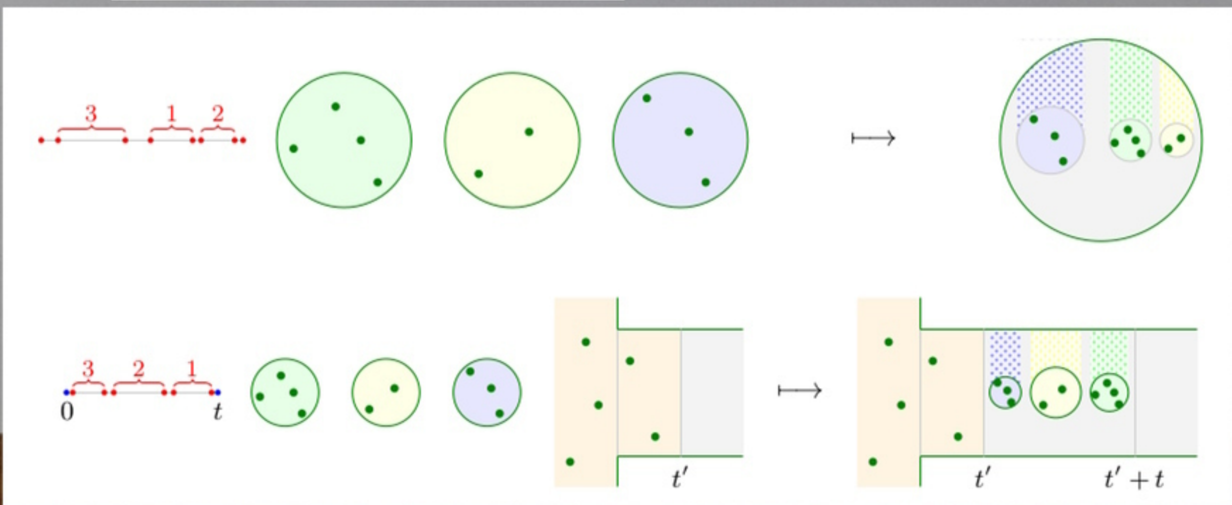
$$\mathcal{C}(M) \xrightarrow{F} h\mathrm{Top}$$

braid category on M





- $\coprod_n \mathrm{CMap}^c_n(M, X)$ forms an E_0-module over an E_{d-1}-algebra.
- more subtle in dimension $d = 2$.



(2) The coefficient system

$$\mathcal{C}(M) \xrightarrow{F} h\mathrm{Top} \xrightarrow{H_i(-;\mathbb{K})} \mathrm{Vect}_{\mathbb{K}}$$

is <u>polynomial of degree $\leq i$</u>

• ⚠ Uses the restriction on the charges of the particles!

(3) [Kranich '19] or [P. '18] under extra conditions $\Rightarrow$ twisted hom. stability for $C_n(M)$ with coefficients in $H_i(-;\mathbb{K})\circ F$

$\Updownarrow$

stability for E^2 pages of $(*)$

$\Downarrow$

stability for limits of $(*)$ $\square$.

Q: When is

$$H_*\left(CMap^c_n(M,X)\right) \longrightarrow H_*\left(CMap^c_{n+1}(M,X)\right)$$

injective? (Outside of the stable range.)

Rmk

- $H_*(C_n(M)) \longrightarrow H_*(C_{n+1}(M))$ is split-injective $(\partial M \neq \emptyset)$
[McDu 88, Segal]

- $H_*(C^+_n(M)) \longrightarrow H_*(C^+_{n+1}(M))$ is not injective in general
oriented config. space
[Guest-Kozlowsky-Yamaguchi '96]
[P. '12]

- $H_*([B_n, B_n]) \longrightarrow H_*([B_{n+1}, B_{n+1}])$ is not injective in general
[Frenkel '88,
Markaryan '96,
De Concini-Procesi-Salvetti '01,
Callegaro '06]

Partial answer

Thm [P.-Tillmann '20]
If $d = \dim(M) \geq 3$ and $\left[\pi_1(M) = 0 \text{ or handle-}\dim(M) \leq d-2\right]$

then the map of E^2 pages of (*) is split-injective.

Sketch:

(1) Explicit formulas for the monodromy actions

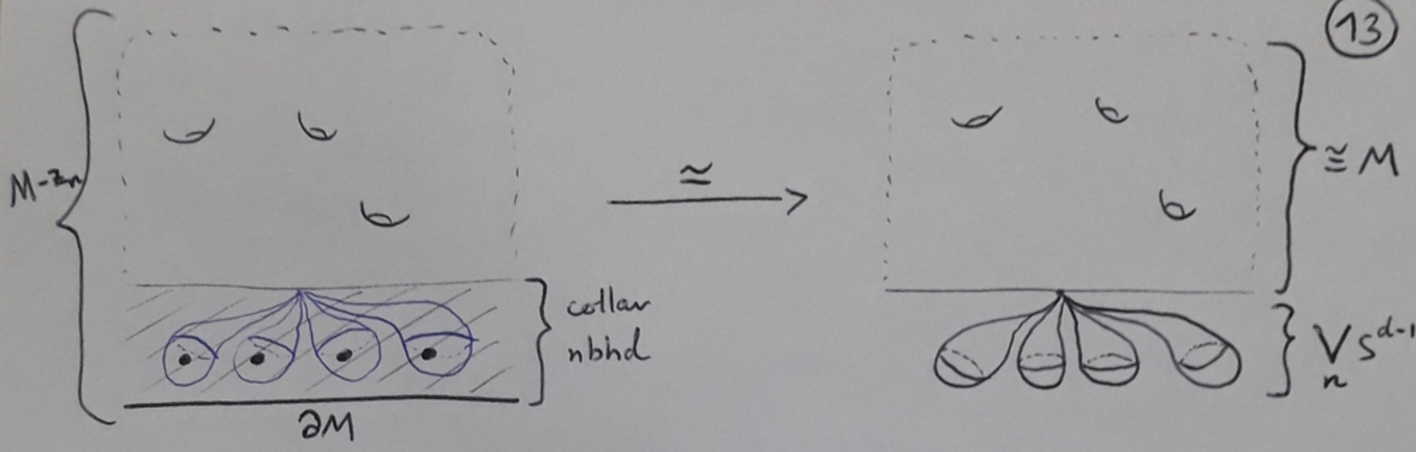
$$\begin{tikzcd} \pi_1 C_n(M) \arrow[r, phantom, "\curvearrowright"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Map}_*^c(M \setminus z_n, X) \arrow[d, phantom, "\cong" rotate=90] \\ \pi_1(M)^n \rtimes \Sigma_n & \mathrm{Map}_*(M, X) \times \left(\Omega_c^{d-1} X\right)^n \end{tikzcd}$$

(2) Show that

$$\begin{tikzcd} \mathcal{C}(M) \arrow[r, "F"] \arrow[d, hook] & h\mathrm{Top} \\ B_\#(M) \arrow[ur, dashed, "\hat{F}"'] & \end{tikzcd}$$

partial braid category

(3) [P.'18] $\Rightarrow$ stabilisation maps induce <u>split-injections</u> on H_* with coeffs. in any coeff. system defined on $B_\#(M)$. $\square$.

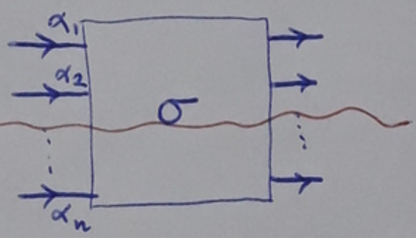


$$\pi_1 C_n(M) \curvearrowright Map_*^c(M \setminus Z_n, X)$$

$$\pi_1 C_n(M) \cong \pi_1(M)^n \rtimes \Sigma_n \qquad Map_*^c(M \setminus Z_n, X) \cong Map_*(M,X) \times \left(\Omega_c^{d-1} M\right)^n$$

$$(\alpha_1, \ldots, \alpha_n) \quad \sigma \qquad\qquad f \quad (g_1, \ldots, g_n)$$

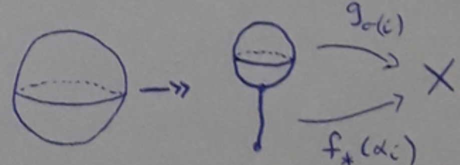
$$\begin{bmatrix} f_*(\alpha_1)\cdot g_{\sigma(1)} \cdot sgn(\alpha_1) \\ \vdots \\ f_*(\alpha_n)\cdot g_{\sigma(n)} \cdot sgn(\alpha_n) \\ f \end{bmatrix} \longleftarrow \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \\ f \end{bmatrix}$$



where $sgn(\alpha_i) = \begin{cases} +1 & \text{if } \alpha_i \text{ is orientable} \\ -1 & \text{if } \alpha_i \text{ is non-orientable.} \end{cases}$

$$f_*(\alpha_i) \in \pi_1(X) \curvearrowright \Omega_c^{d-1} X \ni g_{\sigma(i)}$$

"standard" action up to homotopy



$\hookrightarrow$ Purely diagrammatic formula for the action.

- $B_{\#}(M)$ is a diagrammatically-defined category, so it is easy to extend this formula to define the functor $B_{\#}(M) \xrightarrow{\hat{F}} hTop$.

<u>Note</u>: if $\pi_1(M) \neq 0$ & handle-dim $(M) = d-1$ (still $\dim(M) \geqslant 3$)

then the formula is much more complicated!

It depends on the <u>precise sequence</u> of (transverse) intersections between (smooth, embedded representatives of) the <u>loops $\alpha_i \in \pi_1(M)$</u> and the <u>cores of the $(d-1)$-handles of M.</u>

<u>Picture</u>: