Homology of configuration-section spaces

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Purdue Topology Seminar

joint work with

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Physical system | Model

particles moving in a manifold M (no collisions)

$$C_n(M) = \left\{ (x_1, \dots x_n) \in M^n \;\middle|\; \begin{matrix} x_i \neq x_j \\ \text{for } i \neq j \end{matrix} \right\} \Big/ \Sigma_n$$

.... with internal parameters in a space X

$$C_n(M, X) = \frac{\left\{ (x_1, \dots x_n) \in M^n \;\middle|\; \begin{matrix} x_i \neq x_j \\ \text{for } i \neq j \end{matrix} \right\} \times X^n}{\Sigma_n}$$

.... coupled with a field on M that may be singular (undefined) at the particles' locations

$$CMap_n(M, X) = \left\{ \begin{matrix} z \subseteq M \\ M \setminus z \xrightarrow{f} X \end{matrix} \;\middle|\; |z| = n \right\}$$

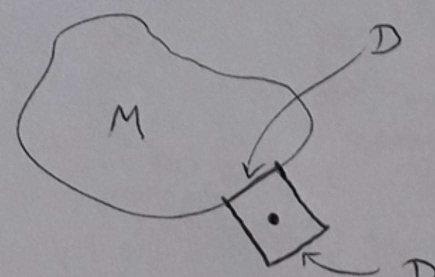
$$C\Gamma_n\left(M, \begin{matrix} E \\ \downarrow \\ M \end{matrix}\right) = \left\{ \begin{matrix} z \subseteq M \\ M \setminus z \xrightarrow{s} E \end{matrix} \;\middle|\; \begin{matrix} |z| = n \\ s \text{ section} \end{matrix} \right\}$$

Stabilisation

if $\partial M \neq \emptyset$ (and we impose a condition that the field is prescribed on a disc $D \subseteq \partial M$)

there are stabilisation maps

$$C_n(M) \longrightarrow C_{n+1}(M)$$
$$C_n(M, X) \longrightarrow C_{n+1}(M, X)$$
$$CMap_n(M, X) \longrightarrow CMap_{n+1}(M, X)$$
$$C\Gamma_n(M, \xi) \longrightarrow C\Gamma_{n+1}(M, \xi)$$



Stability

M —— connected manifold with $\partial M \neq \emptyset$

Thm [McDuff, Segal, '70s]

The sequence $\cdots \to C_n(M) \xrightarrow{(*)} C_{n+1}(M) \to \cdots$ is homologically stable : $(*)$ induces $\cong$ on H_i for all $i \leq \frac{n}{2}$.

With internal parameters:

Thm [Randal-Williams, '13]

The sequence $\cdots \to C_n(M,X) \longrightarrow C_{n+1}(M,X) \to \cdots$ is homologically stable iff X is path-connected.

Rmks

- $C_n(M,X)$ is not hom. stable if $\pi_0(X) \neq *$

  $\hookrightarrow$ H_0 grows unboundedly with n.

- The same will generally be true of $CMap_n(M,X)$ and $C\Gamma_n(M,\xi)$ without control over the "monodromy" / "charge" of the field near the particles.

Definition (CMap) [Ellenberg–Venkatesh–Westerland]

(CΓ similar; see next page for a hint)

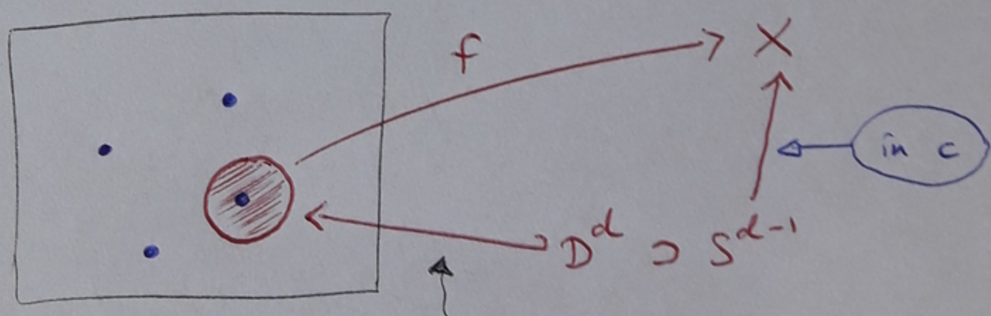
$$\mathrm{CMap}_n(M,X) = \left\{ \begin{array}{l} z \subseteq M \\ M \setminus z \xrightarrow{f} X \end{array} \;\middle|\; \begin{array}{l} |z| = n \\ f(D) = * \end{array} \right\}$$

$$\cong \frac{\mathrm{Diff}_\partial(M) \times \mathrm{Map}_D(M \setminus z, X)}{\mathrm{Diff}_\partial(M, z)}$$

$z \subseteq M$
$|z| = n$
fixed

For a subset $c \subseteq [S^{d-1}, X]$

$\mathrm{CMap}_n^c(M,X)$ = subspace where:



orientation-preserving if M is orientable

I.e. the "charge" of each particle is constrained to lie in c.

Idea / hint for $C\Gamma$

$$\left\{ p \in M,\ \text{germ of section of } \xi \text{ near } p \right\}$$
$$\|$$

$$\begin{tikzcd} E \arrow[d, "\xi"] & & \Sigma(\xi) \arrow[d, "\eta"] \\ M \arrow[rr, phantom, "\rightsquigarrow"] & & M \end{tikzcd}$$

bundle

manifold

covering space

$$C\Gamma(M, \xi) \longrightarrow \Gamma(\eta)$$

"*charge*" of a system $:=$ induced section of η

system $=$ points $+$ field

(Details in §3 of arXiv: 2007.11607 ...)

Examples

Bundle	Interpretation of $C\Gamma$
1. $\begin{array}{c} TM \setminus 0 \\ \downarrow \\ M \end{array}$	non-vanishing vector field on the complement.
2. $\begin{tikzcd} \tau^*(B) \arrow[r] \arrow[d] & B \arrow[d, "\theta"] \\ M \arrow[r, "\tau"'] & BO(d) \end{tikzcd}$	<u>θ-structure</u> on the complement → spin framing almost complex ...
3. $\begin{tikzcd} f^*(BG^\delta) \arrow[r] \arrow[d] & BG^\delta \arrow[d] \\ M \arrow[r, "f"'] & BG \end{tikzcd}$	<u>flat connection</u> (on the complement) for the principal G-bundle $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ classified by f.
4. $\begin{array}{c} D^2 \times BG \\ \downarrow \\ D^2 \end{array}$	$c \subseteq [S^1, BG] = \mathrm{Conj}(G)$ branched coverings of D^2 with monodromy $\subseteq c$ <u>Hurwitz spaces</u>
5. $\begin{array}{c} M^d \times K(\mathbb{Z}, d) \\ \downarrow \\ M^d \end{array}$	$\underbrace{\text{configuration \& } S^1\text{-parameter for each particle}}_{\text{non-locally!}}$ $\longleftrightarrow$ magnetic monopoles

Theorem (P.-Tillmann '20)

If the allowed charges are *fully constrained*, then

$$\cdots \to C\Gamma^c_n(M,\xi) \longrightarrow C\Gamma^c_{n+1}(M,\xi) \longrightarrow \cdots$$

is *homologically stable*.

Def^n (c is *fully constrained*):

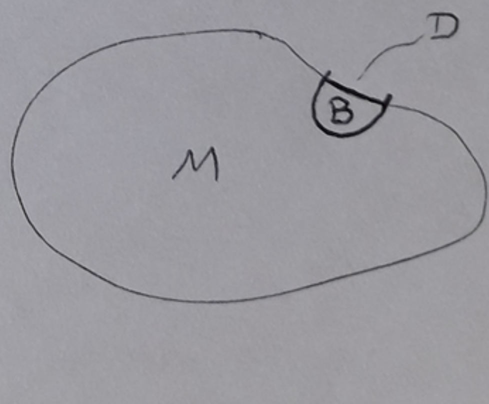
(CMap) •

$$\begin{array}{c} [S^{d-1}, X] \cong \pi_{d-1}(X)/\pi_1(X) \twoheadleftarrow \pi_{d-1}(X) \\ \cup\!\mid \\ c \end{array}$$

$$\boxed{c = \text{a single } \pi_1(X)\text{-orbit of size } 1}$$

(CΓ) •

$$\begin{tikzcd} X \arrow[r] & E \arrow[d, "\xi"] & \arrow[r, squiggly] & {} & \Sigma(\xi) \arrow[d, "\eta"] & \supset & \Sigma(\xi|_B) \arrow[d, "\eta|_B"] & \cong B \times [S^{d-1}, X] \\ & M & & & M & \supset & B & \end{tikzcd}$$



$$\begin{array}{ccc} \Gamma(\eta) & \xhookrightarrow{\text{restriction}} & \Gamma(\eta|_B) \\ \cup\!\mid & & \cong \\ c & & [S^{d-1}, X] \end{array}$$

Rmk If $\pi_1(X) = 0$, then the condition is just $\boxed{|c| = 1}$.

Examples

1 (non-vanishing vector fields)

$$X = \mathbb{R}^d - \{0\} \simeq S^{d-1} \qquad \pi_{d-1}(X) \cong \mathbb{Z}$$

$c \longleftrightarrow$ prescribing the winding # of the vector field around each singularity.

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2 (θ-structures)

$$\begin{array}{c} X \\ \downarrow \\ BO(d)\langle k \rangle \\ \downarrow \\ BO(d) \end{array}$$

k-connected cover

$k=1$	$BSO(d)$	orientations
$k=2$	$BSpin(d)$	spin
$k=4$	$BString(d)$	string

$$\pi_{d-1}(X) = 0 \quad \text{if} \quad k \leq d-2.$$

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3 (flat connections on a principal G-bundle $\begin{array}{c} P \\ \downarrow \\ M \end{array}$)

$$\begin{array}{c} X \\ \downarrow \\ BG^\delta \\ \downarrow \\ BG \end{array} \qquad \pi_i(X) \cong \pi_i(G) \quad i \geq 2$$

$\boxed{\text{e.g.}}$ if $d=3$, $G=$ Lie group then $\pi_{d-1}(X) \cong \pi_2(G) = 0$.

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4 (Hurwitz spaces) $\quad c \subseteq Conj(G)$

"fully constrained" $\longleftrightarrow$ c = single conj. class of size 1

$\longleftrightarrow$ $c \in ZG$

[Ellenberg – Venkatesh – Westerland '16] proved $\mathbb{Q}$-hom. stability for Hurwitz spaces with a <u>weaker</u> condition on c.

$\longrightarrow$ Cohen – Lenstra conjecture.

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5 (Monopoles) $\quad X = K(\mathbb{Z}, d)$

so $\pi_{d-1}(X) = 0$.

Q: What is the stable homology?

Already answered by [EVW] for CMap:

(for simplicity suppose that M is parallelisable)

Thm [EVW] The stable H_* of $CMap^c_n(M, X)$ is isomorphic to the H_* of the space of maps:

$$\begin{tikzcd}
M \arrow[r, dashed] & A_d(X,c) \arrow[dr, phantom, "\lrcorner"] & D^d \times Map(S^{d-1}, X) \arrow[l] \\
\partial M \arrow[u, hook] \arrow[r, dashed] & X \arrow[u] & S^{d-1} \times Map^c(S^{d-1}, X) \arrow[l, "ev"] \arrow[u, hook]
\end{tikzcd}$$

Thm [P.-Tillmann '20] The stable H_* of $C\Gamma^c_n(M, \xi)$ is isomorphic to the H_* of the space of sections:

$$\begin{tikzcd}
M \arrow[r, dashed] & E_d(\xi, c) \arrow[l, bend right] \\
\partial M \arrow[u, hook] \arrow[r, dashed] & E|_{\partial M} \arrow[u] \arrow[l, bend left, "\xi|_{\partial M}"]
\end{tikzcd}$$

$$\left(\begin{tikzcd}
E \arrow[r] \arrow[dr, "\xi"'] & E_d(\xi, c) \arrow[d] \\
& M
\end{tikzcd} \right)$$

Idea of proof of stability (CMap)

$$\begin{tikzcd}
\mathrm{Map}^c_*(M \setminus z_n, X) \arrow[r] \arrow[d] & \mathrm{CMap}^c_n(M,X) \arrow[r] \arrow[d] & C_n(\mathring{M}) \arrow[d] & \ni z_n \arrow[d, mapsto] \\
\mathrm{Map}^c_*(M \setminus z_{n+1}, X) \arrow[r] & \mathrm{CMap}^c_{n+1}(M,X) \arrow[r] & C_{n+1}(\mathring{M}) & \ni z_{n+1}
\end{tikzcd}$$

Induces (*) map of Serre spectral sequences

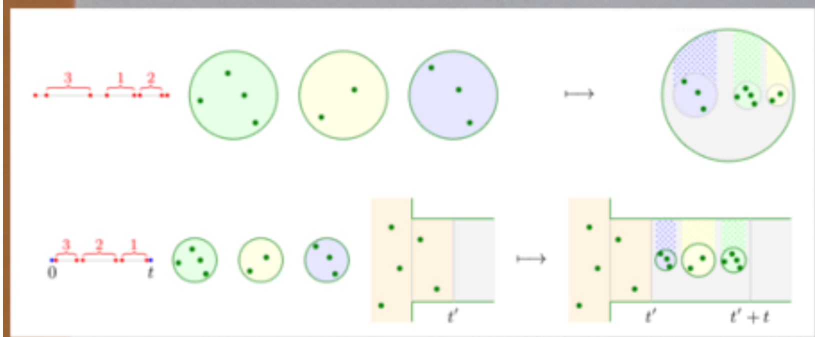
(**) monodromy actions $\pi_1 C_n(M) \curvearrowright \mathrm{Map}^c_*(M \setminus z_n, X)$

Steps (1) (**) extend to a functor

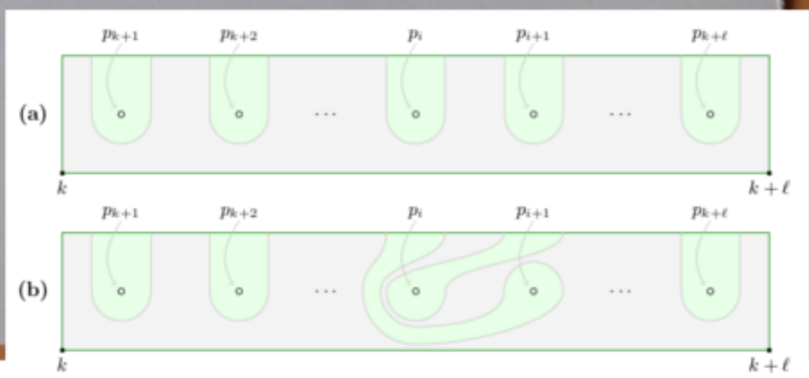
$$\mathcal{C}(M) \xrightarrow{F} hTop$$

braid category on M

- $\coprod_n \mathrm{CMap}^c_n(M,X)$ forms an E_0-module over an E_{d-1}-algebra.



- more subtle in dimension $d = 2$.



(2) The coefficient system

$$\mathcal{C}(M) \xrightarrow{F} hTop \xrightarrow{H_i(-;\mathbb{K})} Vect_{\mathbb{K}}$$

is <u>polynomial of degree $\leq i$</u>

- ⚠ Uses the restriction on the charges of the particles!

(3) [Kvannich '19] or [P. '18] under extra conditions $\Rightarrow$ twisted hom. stability for $C_n(M)$ with coefficients in $H_i(-;\mathbb{K}) \circ F$

$\Updownarrow$

stability for E^2 pages of $(*)$

$\Downarrow$

stability for limits of $(*)$ $\square$.

Q: When is

$$H_*\left(\mathrm{CMap}^c_n(M,X)\right) \longrightarrow H_*\left(\mathrm{CMap}^c_{n+1}(M,X)\right)$$

injective? (Outside of the stable range.)

Rmk

- $H_*(C_n(M)) \longrightarrow H_*(C_{n+1}(M))$ is split-injective ($\partial M \neq \emptyset$)
  [McDu 88, Segal]

- $H_*(C_n^+(M)) \longrightarrow H_*(C_{n+1}^+(M))$ is not injective in general
  oriented config-space
  [Guest-Kozlowsky-Yamaguchi '96]
  [P. '12]

- $H_*([B_n, B_n]) \longrightarrow H_*([B_{n+1}, B_{n+1}])$ is not injective in general
  [Frenkel '88,
  Markaryan '96,
  De Concini-Procesi-Salvetti '01,
  Callegaro '06]

Partial answer

Thm [P.-Tillmann '20]

If $d = \dim(M) \geq 3$ and $\left[\pi_1(M) = 0 \text{ or } \text{handle-}\dim(M) \leq d-2\right]$

then the map of E^2 pages of $(*)$ is split-injective.

Sketch:

(1) Explicit formulas for the monodromy actions

$$\begin{array}{ccc} \pi_1 C_n(M) & \curvearrowright & \mathrm{Map}_*^c(M \setminus z_n, X) \\ \cong & & \cong \\ \pi_1(M)^n \rtimes \Sigma_n & & \mathrm{Map}_*(M,X) \times \left(\Omega_c^{d-1} X\right)^n \end{array}$$

(2) Show that

$$\begin{tikzcd} \mathcal{C}(M) \arrow[r, "F"] \arrow[d] & h\mathrm{Top} \\ B_\#(M) \arrow[ur, dashed, "\hat{F}"'] & \end{tikzcd}$$

partial braid category

(3) [P.'18] $\Rightarrow$ stabilisation maps induce <u>split-injections</u> on H_* with coeffs. in any coeff. system defined on $B_\#(M)$. $\square$

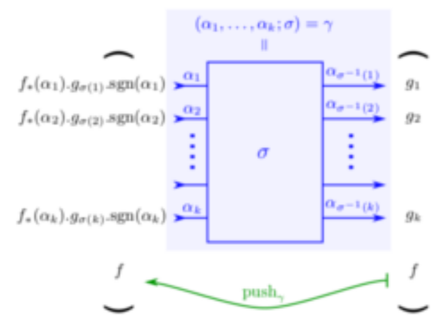


Figure 8.1 The action of the point-pushing map associated to $\gamma = (\alpha_1, \ldots, \alpha_k; \sigma) \in \pi_1(C_k(M))$ on the mapping space $\mathrm{Map}_*(M, X) \times (\Omega_c^{d-1} X)^k$. The loop γ is represented in blue, the elements of the mapping space in black and the point-pushing map is represented in green.