On the different flavours of Lawrence-Bigelow representations

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Motivation (relevance of the representations that we study)

1 Quantum invariants

Quantum group $\mathcal{H}$ + representation V $\longrightarrow$ $B_n \curvearrowright V^{\otimes n}$ $\}$ $(*)$

(e.g. $\mathcal{H} = \mathcal{U}_q(\mathfrak{sl}(2))$
$V =$ Verma module)

$\downarrow$ Witten-Reshetikhin-Turaev (algebraic)

Invariants of links in $\mathbb{R}^3$ and of 3-manifolds

(e.g. coloured Jones poly.
coloured Alexander poly.)

Via certain isomorphisms between [Kohno, Ito, Martel]

- quantum representations of B_n (*)

& - topologically-defined rep's of B_n,

several topological interpretations of certain WRT invariants have been found.

[Ito, Anghel, Martel]

② Linearity

- The (reduced) Burau representation of B_n is

faithful (i.e. injective)	for $n \leq 3$	(easy)
?	for $n = 4$	
unfaithful	for $n \geq 5$	[Moody, Long-Paton, Bigelow]

- The "Lawrence-Krammer-Bigelow" (LKB) representation of B_n is

faithful for all n [Bigelow, Krammer]

They are also finite-dimensional, hence B_n is linear for all n.

isomorphic to a subgroup of $GL_k(\mathbb{F})$, $\mathbb{F}$ field, k finite.

- Linearity is open for almost all other MCGs of surfaces & surface braid groups.

(It is known for

- $Mod(\Sigma_1) \cong SL_2(\mathbb{Z}) \leqslant GL_2(\mathbb{C})$
- $Mod(\Sigma_2) \leqslant GL_{64}(\mathbb{C})$ [Bigelow-Budney '01])

Constructions

Lawrence – Bigelow representations

- (reduced) Burau repr. [Burau '35]

$$\sigma_i \longmapsto \begin{pmatrix} I & & \\ & \begin{matrix} 1 & x & 0 \\ 0 & -x & 0 \\ 0 & 1 & 1 \end{matrix} & \\ & & I \end{pmatrix} \in GL_{n-1}(\mathbb{Z}[x^{\pm 1}])$$

- [Lawrence '90, Bigelow '00/'04]

$$B_n \curvearrowright H_k(C_k(D^2 \setminus n); \mathbb{Z}[x^{\pm1}, d^{\pm1}]) \qquad (k \geqslant 2)$$

$$H_1(D^2 \setminus n; \mathbb{Z}[x^{\pm1}]) \qquad (k=1)$$

or H_k^{lf}, $H_k(-,\partial)$, $H_k(\text{covering})$, …

General setting (homological repr's of MCGs)

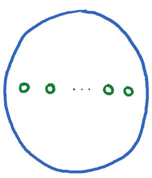
Σ surface (connected, compact, orientable)

$\partial\Sigma = \partial_{out} \cup \partial_{in}$ partition into two non-ϕ submanifolds : $\partial_{out} \cap \partial_{in} = \partial\partial_{out} = \partial\partial_{in}$.

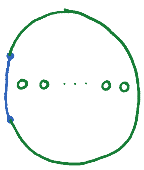
Def $\Gamma := \pi_0(\mathrm{Diff}(\Sigma, \partial_{out}))$

restriction to ∂_{out} is the identity

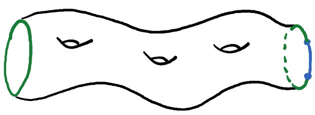
Examples



$\Gamma \cong B_n$



$\Gamma \cong B_n$



$\Gamma \cong \Gamma'_{3,1}$

fix $k \geqslant 1$ integer

Def

$$\left.\begin{aligned} M_{in} &= C_k(\Sigma \setminus \partial_{out}) \\ M_{out} &= C_k(\Sigma \setminus \partial_{in}) \end{aligned}\right\} \text{connected, orientable } 2k\text{-manifolds}$$

Choose

$$\begin{array}{c} \mathcal{L} \\ \downarrow \\ M_{in} \simeq C_k(\Sigma) \simeq M_{out} \end{array}$$

rank-1 local system

$\equiv$ bundle of R-modules with fibres $\cong R$

$\equiv$ $\pi_1 C_k(\Sigma) \longrightarrow GL_1(R) = R^\times$

Assume that $\mathcal{L}$ is <u>invariant</u> under $\mathrm{Diff}(\Sigma, \partial_{out})$

$$\begin{tikzcd} \pi_1 C_k(\Sigma) \arrow[dr] \arrow[d, "\varphi"'] & \\ \pi_1 C_k(\Sigma) \arrow[r] & GL_1(R) \end{tikzcd} \qquad \mathrm{Diff}(\Sigma, \partial_{out}) \ni \varphi$$

Observation

Γ acts on

$$\underbrace{\begin{array}{ll} H_k(M_{in}; \mathcal{L}) & H_k(M_{out}; \mathcal{L}) \\ H_k^{lf}(M_{in}; \mathcal{L}) & H_k^{lf}(M_{out}; \mathcal{L}) \\ H_k(M_{in}, \partial M_{in}; \mathcal{L}) & H_k(M_{out}, \partial M_{out}; \mathcal{L}) \end{array}}_{\text{These are the homological representations that we study}}$$

One more flavour

$$\begin{array}{c} \hat{C}_k(\Sigma) \\ \downarrow \\ C_k(\Sigma) \end{array}$$

regular covering with deck transf. group $= G$ $\rightsquigarrow$

$$\begin{array}{c} \mathbb{Z}[\hat{C}_k(\Sigma)] \\ \downarrow \\ C_k(\Sigma) \end{array}$$

rank-1 local system over $R = \mathbb{Z}[G]$

$$\left[\begin{tikzcd} \pi_1 C_k(\Sigma) \arrow[d, two heads] \\ G \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \pi_1 C_k(\Sigma) \arrow[d, two heads] & \\ G \arrow[r, hook] & \mathbb{Z}[G]^\times \cong \{\pm 1\}\times G \end{tikzcd}\right]$$

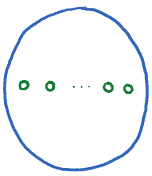
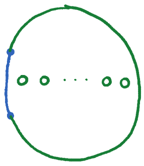
If $\mathcal{L}$ arises in this way, we may also consider the Γ-representations

$$H_k^{lf}(\hat{M}_{in}) \qquad\qquad H_k^{lf}(\hat{M}_{out})$$

where
$$\begin{tikzcd} \hat{M}_{in} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \hat{C}_k(\Sigma) \arrow[d] \\ M_{in} \arrow[r, hook] & C_k(\Sigma) \end{tikzcd}$$
and similarly for $\hat{M}_{out}$.

*We could also do this for ordinary (not lf) homology, but this is the same as $H_k(M_\circ; \mathcal{L})$ by Shapiro's Lemma.

<u>Motivating example</u> $(n \geqslant 2)$

$\Sigma =$  or  $\quad (\text{so } \Gamma \cong B_n)$

$\mathcal{L}$ arises from
$$\begin{tikzcd} \pi_1 C_k(D^2 \setminus n) \arrow[r, hook] \arrow[dr, bend right, "\varphi"'] & B_{k,n} \arrow[d, two heads, "ab"] \\ & \begin{Bmatrix} \mathbb{Z}^2 & k=1 \\ \mathbb{Z}^3 & k\geqslant 2 \end{Bmatrix} \end{tikzcd}$$
$= \langle$  c, (if $k\geqslant 2$)  d,  x $\rangle$

$\underbrace{d, x}_{\parallel}$: image$(\varphi) =: G$

So $R = \mathbb{Z}[G] \cong \begin{cases} \mathbb{Z}[x^{\pm 1}] & k=1 \\ \mathbb{Z}[x^{\pm 1}, d^{\pm 1}] & k\geqslant 2 \end{cases}$

We may also consider
$$\mathcal{L}' = \mathcal{L} \underset{R}{\otimes} S \qquad \text{for any } R \xrightarrow{\theta} S \qquad (\text{"specialisation"})$$

For example
$$R \cong \begin{Bmatrix} \mathbb{Z}[x^{\pm 1}] & k=1 \\ \mathbb{Z}[x^{\pm 1}, d^{\pm 1}] & k\geqslant 2 \end{Bmatrix} \longrightarrow \mathbb{C}, \quad \text{determined by } x, d \in \mathbb{C}^*$$

Theorem [Anghel-P. '20]

These Γ-representations are related in the following way:

$$
\begin{tikzcd}
 & \mathcal{H}^{\partial}_k(M_{in};\mathcal{L}) \arrow[d, dash, dashed, "dual"] \arrow[dr] & \mathcal{H}^{\partial}_k(M_{out};\mathcal{L}) \arrow[d, dash, dashed, "dual"'] \arrow[dl] & \\
\mathcal{H}_k(M_{in};\mathcal{L}) \arrow[r, phantom, "\cong"] & \mathcal{H}^{lf}_k(M_{in};\mathcal{L}) \arrow[d, "(\hat{\ })"] & \mathcal{H}^{lf}_k(M_{out};\mathcal{L}) \arrow[r, phantom, "\cong"] \arrow[d, "(\hat{\ })"'] & \mathcal{H}_k(M_{out};\mathcal{L}) \\
 & \mathcal{H}^{lf}_k(\widehat{M}_{in}) & \mathcal{H}^{lf}_k(\widehat{M}_{out}) &
\end{tikzcd}
$$

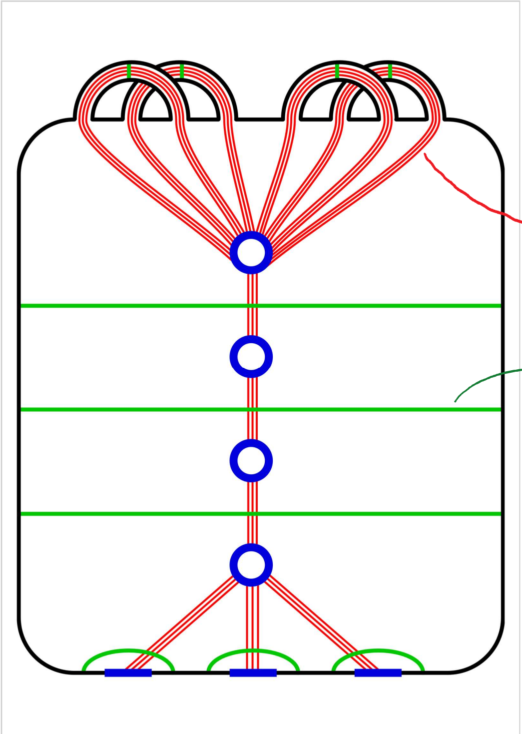
free R-modules (explicit bases)

free $\hat{R}$-modules

where
- $\mathcal{H}^{\partial}_k(M_\bullet;\mathcal{L}) \subseteq \mathcal{H}_k(M_\bullet,\partial M_\bullet;\mathcal{L})$ is an explicit subrepresentation
- "dual" means that there is a non-degenerate pairing
- the isomorphisms $\cong$ hold if $\mathcal{L}$ is "generic" [Kohno]
- the diagonal embeddings exist if $\mathcal{L}$ is "homogeneous" (below...)
- the embeddings $(\hat{\ })$ are certain completions of R-modules ($R = \mathbb{Z}[G]$).

Explicit bases :

(assume that $\partial_{out} \subseteq$ a single component of $\partial\Sigma$)

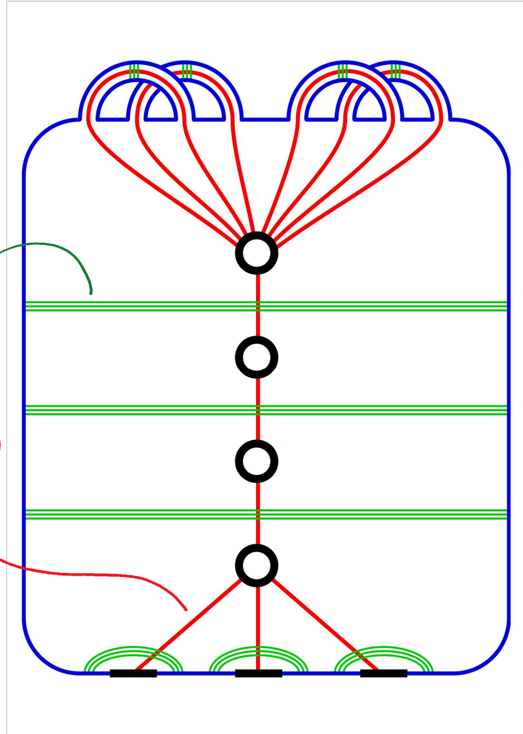


$\mathcal{H}^{\partial}_k(M_{in};\mathcal{L})$

$\mathcal{H}^{lf}_k(M_{in};\mathcal{L})$

$\mathcal{H}^{\partial}_k(M_{out};\mathcal{L})$

$\mathcal{H}^{lf}_k(M_{out};\mathcal{L})$



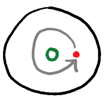
M_{in} (∂_{in} blue) $\qquad\qquad$ M_{out} (∂_{out} blue)

configurations in
- int(Σ)
- blue part of $\partial\Sigma$

Smallprint:

$$\mathcal{L} : \pi_1 C_k(\Sigma) \longrightarrow R^\times$$

is generic if  $\longmapsto$ an element u : $1-u \in R^\times$

 $\longmapsto$ ——— // ———

o = part of ∂_{out} if we consider M_{in}
$\qquad\quad\ \partial_{in}$ $\qquad\qquad\qquad\quad M_{out}$

is homogeneous if  $\longmapsto$ the same element $u \in R^\times$, which has ∞ order. [R int. dom.]

The completion ($\widehat{\ }$) is given by $- \otimes_{\mathbb{Z}[G]} \widehat{\mathbb{Z}[G]}$,

where $\widehat{\mathbb{Z}[G]} = \prod_G \mathbb{Z}$

$$\mathbb{Z}[G] = \bigoplus_G \mathbb{Z} \hookrightarrow \prod_G \mathbb{Z}$$

Ideas of the proof

Pairings

$$\begin{array}{c} H_k^{lf}(M_\bullet) \otimes H_k(M_\bullet, \partial M_\bullet) \\ \Big\downarrow \begin{pmatrix}\text{Poincaré} \\ \text{duality}\end{pmatrix} \otimes \mathrm{id} \\ H^k(M_\bullet, \partial M_\bullet) \otimes H_k(M_\bullet, \partial M_\bullet) \\ \Big\downarrow \text{relative cap product} \\ H_0(M_\bullet) \cong R \end{array}$$

depends on choice of $[M_\bullet] \in H_{2k}^{lf}(M_\bullet) \cong R$; unique modulo $R^\times$

$\left.\begin{array}{l} \tilde{H}_k^\partial(M_\bullet) \subseteq H_k(M_\bullet, \partial M_\bullet) \\ \tilde{H}_k^{lf}(M_\bullet) \subseteq H_k^{lf}(M_\bullet) \end{array}\right\}$ submodules generated by the elements pictured above (Definition)

Calculation $\Rightarrow$ evaluating the pairing on all pairs of elements pictured above $\longrightarrow$ identity matrix !

$\Rightarrow$ (1) these sets of elements are linearly independent (generate free submodules)
(2) restricted to these submodules, the pairing is non-degenerate

Proposition $\tilde{H}_k^{lf}(M_\bullet) = H_k^{lf}(M_\bullet)$
i.e. the elements pictured above span the lf-homology.

$\longrightarrow$ Idea of proof (extension of [Bigelow '04], [An-Ko '10])

- It is enough to prove that:
  $C_k(\text{red arcs}) \hookrightarrow C_k(\Sigma \setminus \partial_{in}) = M_{out}$ induces an $\cong$ on H_*^{lf}
- Σ deformation retracts onto $(\text{red arcs}) \cup \partial_{in}$
  through self-embeddings except at time $t=1$
- run this deformation retraction until time $t = 1-\varepsilon$ ($\varepsilon > 0$ very small)
- (!) excision argument $\Rightarrow$ pass to the subspace of configurations with the property that if you continue the def. retraction up to time $t=1$, then no collisions occur
- finish the deformation retraction.

Completions

An analogous deformation retraction argument also works on the covering space $\begin{array}{c}\widehat{M}_{out}\\ \downarrow\\ M_{out}\end{array}$ and proves the statement about completions ($\widehat{\ }$).

[Instead of $C_k(\text{red arcs}) \hookrightarrow C_k(\Sigma \setminus \partial_{in}) = M_{out}$

we have $\left(\begin{array}{c}\text{all lifts of configs}\\ \text{in the red arcs}\end{array}\right) \hookrightarrow \widehat{M}_{out}$.]

Embeddings

$$\begin{tikzcd}
H_k^{\partial}(M_{in}) \arrow[r, dashed] \arrow[d, hook] & H_k^{lf}(M_{out}) \cong \widetilde{H}_k((M_{out})^+) \arrow[d, equal] \\
H_k(M_{in}, \partial M_{in}) \arrow[r] & H_k((M_{out})^+, \{\infty\})
\end{tikzcd}$$

one-point compactification

$$\begin{array}{ccc}
(M_{in}, \partial M_{in}) & & ((M_{out})^+, \{\infty\}) \\
\| & & \cong \\
\left(C_k(\Sigma \setminus \partial_{out}), \begin{array}{c}\text{collisions with}\\ \partial_{in}\end{array}\right) & & \left(C_k(\Sigma), \begin{array}{c}\text{collisions of two points}\\ \text{collisions with } \partial_{in}\end{array}\right)
\end{array}$$

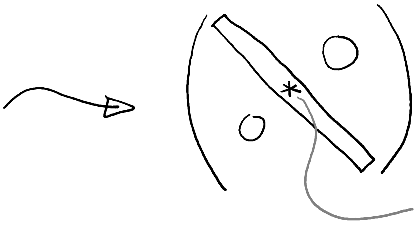
$$\left(C_k(\Sigma \setminus \partial_{out}), \text{collisions with } \partial_{in}\right) \hookrightarrow \left(C_k(\Sigma), \text{collisions with } \partial_{in}\right) \hookrightarrow \left(C_k(\Sigma), \begin{array}{c}\text{collisions of two points}\\ \text{collisions with } \partial_{in}\end{array}\right)$$

- $\mathrm{Diff}(\Sigma, \partial_{out})$ - equivariant

  $\Rightarrow$ Γ-equivariant on H_*.

- why injective??

Use our explicit bases above & compute the matrix of this map



product of quantum factorials $(u \in R^\times)$

$[i]_u! = [1]_u [2]_u \cdots [i]_u$

$[i]_u = 1 + u + u^2 + \cdots + u^{i-1}$

$\Rightarrow$ non-(zero-divisors).

∎

<u>Remark</u> (1) We know explicitly the images of the embeddings

$$H_k^\partial(M_{in}; \mathcal{L}) \hookrightarrow H_k^{lf}(M_{out}; \mathcal{L})$$
$$H_k^\partial(M_{out}; \mathcal{L}) \hookrightarrow H_k^{lf}(M_{in}; \mathcal{L})$$

(2) If $k \geq 2$, they are <u>not</u> surjective, so $H_k^{lf}(M_{in}; \mathcal{L})$ and $H_k^{lf}(M_{out}; \mathcal{L})$ are <u>reducible</u>.

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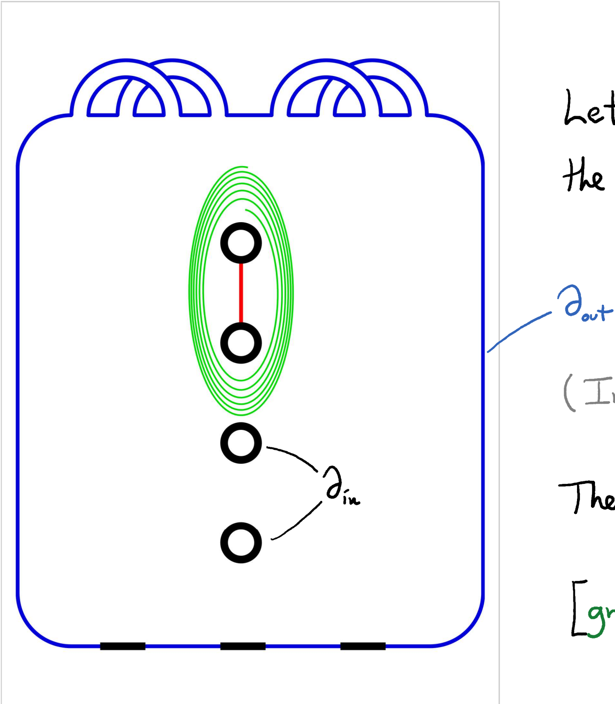
An "exotic" element of $H_1^{lf}(\widehat{M}_{out})$ (with $k=1$)



Let $y, z \in G$ be the monodromy of the covering
$$\begin{array}{ccc} \widehat{M}_{out} & = & \widehat{\Sigma \setminus \partial_{in}} \\ \downarrow & & \downarrow \\ M_{out} & = & \Sigma \setminus \partial_{in} \end{array}$$
around the top two punctures.

(In our motivating example, $y = z = x \in G = \mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$.)

Then:

$$[\text{green helix}] = [\text{red arc}] \cdot \sum_{i \in \mathbb{Z}} (1-y)(yz)^i \quad \in H_1^{lf}(\widehat{M}_{out})$$