Representations of the Torelli group via the Heisenberg group

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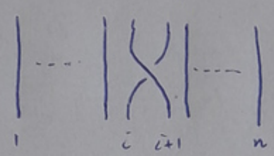
Representations of braid groups

1935 [Burau] representation $\sigma_i \longmapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$

$\parallel$



$$B_n \xrightarrow{(*)} GL_n(\mathbb{Z}[t^{\pm 1}]) \hookrightarrow GL_n(\mathbb{R})$$

$t \longmapsto$ transcendental number

Q: ($\leq$ [Birman '74]) Is $(*)$ injective?
(Is the Burau representation faithful?)

A:
$n=2$ ✓ (easy)
$n=3$ ✓ [Magnus–Peluso '69]
$n \geq 5$ ✗ [Moody '91]
[Long–Paton '93]
[Bigelow '99]
$n=4$??

Q: Are the braid groups B_n linear?
(Does B_n embed in some $GL_N(\mathbb{F})$?)

1990 [Lawrence] representations

$\hookrightarrow$ more geometric definition.

Idea: $B_n = \mathrm{Map}(D_n)$

$\quad = \pi_0 \mathrm{Diff}_\partial(D_n)$ $\qquad D_n = D^2 \setminus \{n \text{ interior pts}\}$.

$\mathrm{Diff}_\partial(D_n)$ acts on $C_k(D_n)$ ← unordered configuration space

$\Rightarrow$ $\mathrm{Map}(D_n)$ acts on $H_*(C_k(D_n))$

2 modifications:

① choose $\pi_1 C_k(D_n) \twoheadrightarrow Q$
that is invariant under the action.

$\rightsquigarrow$ $\mathrm{Map}(D_n)$ acts on $H_*\left(C_k(D_n);\ \mathbb{Z}[Q]\right)$

(by $\mathbb{Z}[Q]$-module automorphisms)

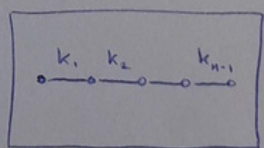
② replace H_* with $H_*^{BM} = H_*^{lf}$

Lemma (Bigelow) $H_*^{BM}\left(C_k(D_n);\ R\right)$

any local coeff

- is concentrated in degree $= k$
- is free as an R-module.

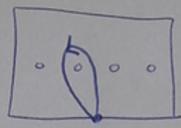
basis:
$k = k_1 + \cdots + k_{n-1}$



① + ② $\Rightarrow$ obtain a representation

$$B_n = \mathrm{Map}(D_n) \longrightarrow GL_N(\mathbb{Z}[Q])$$

How to choose Q?

$\underline{k=1}$ $\quad \pi_1(D_n) = F_n \twoheadrightarrow \mathbb{Z}$ $\quad$ send each  to 1

"total winding number"

$\underline{k \geq 2}$

$$\begin{array}{ccc} \pi_1(C_k(D_n)) & & \\ \Big\downarrow \text{abelianise} & & \\ \mathbb{Z} \times \mathbb{Z}^n & \xrightarrow{\ \mathrm{id} \times \mathrm{add}\ } & \mathbb{Z} \times \mathbb{Z} \end{array}$$

self-winding number

winding numbers around the n punctures

Lemma.
- This quotient is $\mathrm{Map}(D_n)$-invariant.
- For $k=1$, this recovers the (reduced) Burau representation:

$$\begin{array}{ccc} B_n & \xrightarrow{\text{red. Burau}} & GL_{n-1}(\mathbb{Z}[t^{\pm 1}]) \\ \| & & \| \\ \mathrm{Map}(D_n) & \xrightarrow[\text{Lawrence } k=1]{} & GL_{n-1}(\mathbb{Z}[\mathbb{Z}]) \end{array}$$

Q: Are these faithful?

A: [Bigelow '00, Krammer '00] Yes! for $k=2$.

Hence $B_n \hookrightarrow GL_N(\mathbb{R})$ for all n (N depending on n)

"Braid groups are linear"

Q: What about <u>mapping class groups</u> (more generally)?

<u>Very partial answer</u>:

- $\Sigma_1 = T^2 \qquad \mathrm{Map}(T^2) \cong SL_2(\mathbb{Z})$

- hyperelliptic MCG: $\mathrm{Map}^{hyp}(\Sigma_g)$ is linear [Bigelow - Budney '01]

  trick: $\begin{array}{c} \Sigma_g \\ \Big\downarrow {\scriptstyle /hyp} \\ D^2 \end{array}$ ← 2:1 except at $2g+2$ fixed pts

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Map}^{hyp}(\Sigma_g) \longrightarrow \mathrm{Map}(D_{2g+2}) \longrightarrow 1$$

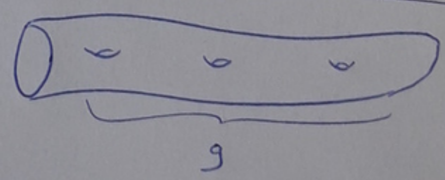
- $\mathrm{Map}^{hyp}(\Sigma_2) = \mathrm{Map}(\Sigma_2)$.

<u>In general, very open!</u>

<u>2006</u> Kontsevich —— proposed a sketch of a construction of a faithful fin-dim repr. of $\mathrm{Map}(\Sigma_g)$

Dunfield —— computational evidence suggesting that this will <u>not</u> actually be faithful

[cf. survey by Margalit '18]

From now on, we'll focus on $\Sigma = \Sigma_{g,1} =$ 

<u>Preview of main result</u> (Blanchet - P - Shaukat)

A new representation of $\widetilde{\mathrm{Tor}}(\Sigma)$ that is a candidate to be faithful.

↑ central extension of Torelli group of Σ.

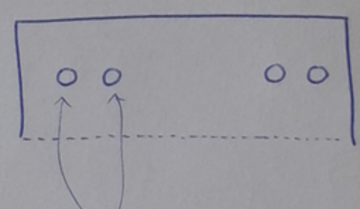
Simplest analogue of Lawrence repr.

$$\mathrm{Map}(\Sigma) \overset{\rho_k}{\curvearrowright} H_k^{BM}\big(F_k(\Sigma');\, \mathbb{Z}\big) \cong H_k^{BM}\big(C_k(\Sigma');\, \mathbb{Z}[\Sigma_k]\big)$$

ordered configurations

trivial coeffs!

$\Sigma' = \Sigma \setminus$ interval on boundary

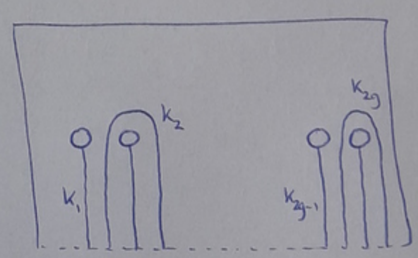
$=$ 

identify via reflection

Analogue of Bigelow's lemma:

This is a free $\mathbb{Z}[\Sigma_k]$-module on the basis:

$k = k_1 + \cdots + k_{2g}$



Thm (Moriyama '07)

$$\ker(\rho_k) = J(k) \subseteq \mathrm{Map}(\Sigma)$$

k^{th} term in the Johnson filtration of $\mathrm{Map}(\Sigma)$

what is this?

Johnson filtration:

$$\begin{array}{ccccccc} \Gamma_0(\pi_1(\Sigma)) & \supseteq & \Gamma_1(\pi_1(\Sigma)) & \supseteq & \Gamma_2(\pi_1(\Sigma)) & \supseteq & \cdots\cdots \quad \text{etc.} \\ \| & & \| & & \| & & \\ \pi_1(\Sigma) & & [\pi_1(\Sigma), \pi_1(\Sigma)] & & [\pi_1(\Sigma), \Gamma_1(\cdots)] & & \end{array}$$

- "lower central series"
- Fact: each $\Gamma_i(\cdots)$ is a characteristic subgroup.

$$Map(\Sigma) \curvearrowright \pi_1(\Sigma) / \Gamma_k(\pi_1(\Sigma))$$

Def (Johnson) $J(k) :=$ the kernel of this action

Note: $$\begin{array}{ccccccc} J(0) & \supseteq & J(1) & \supseteq & J(2) & \supseteq & \cdots\cdots \\ \| & & \| & & & & \\ Map(\Sigma) & & Tor(\Sigma) & & & & \end{array}$$

Thm (Johnson) $$\bigcap_{k=1}^{\infty} J(k) = \{1\}.$$

Corollary (Moriyama '07) $$\underbrace{\bigoplus_{k=1}^{\infty} H_k^{BM}(F_k(\Sigma'); \mathbb{Z})}_{\text{infinite rank.}}$$ is a faithful $Map(\Sigma)$-repr.

$\boxed{\text{Side remark:}}$ [Bianchi-Miller-Wilson '21]

study $Map(\Sigma) \curvearrowright H_*(F_k(\Sigma'); \mathbb{Z})$

Thm kernel $\supsetneq J(k)$

Conj. kernel $= \langle J(k) \cup \{T_{\partial\Sigma}\} \rangle$

Twisted coeffs

Abelian:

$$\pi_1 C_k(\Sigma') = B_k(\Sigma) \twoheadrightarrow ?$$

Prop: $B_k(S)^{ab} \cong \pi_1(S)^{ab} \oplus \begin{cases} \mathbb{Z} & S \text{ planar} \\ \mathbb{Z}/(2k-2) & S = S^2 \\ \mathbb{Z}/2 & \text{o/w} \end{cases}$ $(k \geq 2)$

- So if S is non-planar, we can only "count" the <u>self-winding number</u> of braids on S <u>mod 2</u> (or mod $2k-2$ if $S = S^2$)

- In $\mathbb{Z}[B_k(S)^{ab}]$, the corresponding "variable" t has order two: $t^2 = 1$.
  ↳ get a much "weaker" representation....

Non-abelian:

[Bellingeri]

$$B_k(\Sigma) \cong \Big\langle \underbrace{\sigma_1 \ldots \sigma_{k-1}}_{\text{braid}} \ \underbrace{\begin{matrix} a_1 \ldots a_g \\ b_1 \ldots b_g \end{matrix}}_{\text{handles}} \ \Big| \ \ldots\ldots \Big\rangle$$

$\boxed{\sigma_1 \text{ central}}$

$(\Rightarrow \sigma_1 = \sigma_2 = \cdots = \sigma_{n-1})$
$= \sigma$

$$\downarrow$$

$$\mathcal{H}_g \cong \Big\langle \sigma, \begin{matrix} a_1 \ldots a_g \\ b_1 \ldots b_g \end{matrix} \ \Big| \ \begin{matrix} \text{all pairs commute} \\ \text{except } [a_i, b_i] = \sigma^2 \end{matrix} \Big\rangle$$

<u>"discrete Heisenberg group"</u>

Properties of $\mathcal{H}_g$:

- $\mathcal{H}_g \cong \mathbb{Z}^{g+1} \rtimes \mathbb{Z}^g$
- → central extension $1 \to \mathbb{Z} \to \mathcal{H}_g \to H_1(\Sigma;\mathbb{Z}) \to 1$
- For $k \geq 3$, (not for $k=2$) $\ker\left(B_k(\Sigma) \twoheadrightarrow \mathcal{H}_g\right) = \Gamma_3\left(B_k(\Sigma)\right)$

[Bellingeri - Gervais - Guaschi '08]

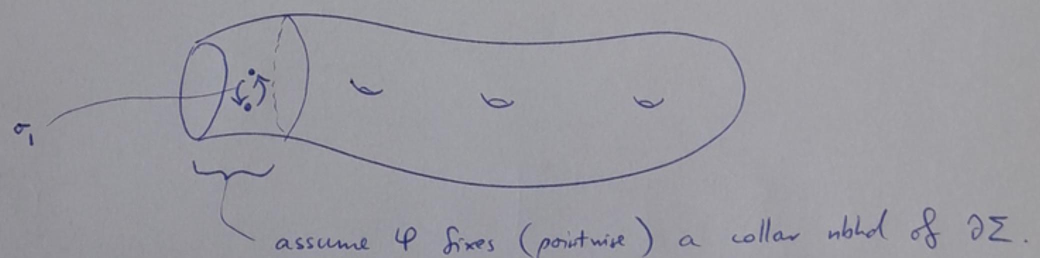
Lemma $\mathrm{Map}(\Sigma) \curvearrowright B_k(\Sigma)$ descends to a well-defined action on $\mathcal{H}_g$.

proof We need to prove that $N := \ker\left(B_k(\Sigma) \twoheadrightarrow \mathcal{H}_g\right)$ is fixed (setwise) by the action.

[For $k \geq 3$ this holds because $\Gamma_3(B_k(\Sigma))$ is characteristic.]

Note: $N = \langle\langle [\sigma_1, x] \mid \text{all } x \rangle\rangle$

So it is enough to check that each $\varphi \in \mathrm{Map}(\Sigma)$ fixes σ_1.
This is geometrically clear:



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Note: The induced action $\mathrm{Map}(\Sigma) \curvearrowright \mathcal{H}_g$ is a lift of the symplectic action $\mathrm{Map}(\Sigma) \curvearrowright H_1(\Sigma;\mathbb{Z})$.

Prop: (a) $\ker\left(\mathrm{Map}(\Sigma) \curvearrowright \mathcal{H}_g \right) = \mathrm{Chill}(\Sigma)$ ~ "Chillingworth subgroup"

(b) $\left\{ \varphi \in \mathrm{Map}(\Sigma) \;\middle|\; \begin{array}{c} \varphi \text{ acts on } \mathcal{H}_g \text{ by} \\ \underline{\text{inner}} \text{ automorphisms} \end{array} \right\} = \mathrm{Tor}(\Sigma)$

$$\begin{array}{lcl} J(0) & = & \mathrm{Map}(\Sigma) \\ \cup\!\!\mid & & \\ J(1) & = & \mathrm{Tor}(\Sigma) \\ \cup\!\!\mid & & \\ \mathrm{Chill}(\Sigma) & & \\ \cup\!\!\mid & & \\ J(2) & & \\ \cup\!\!\mid & & \\ \vdots & & \end{array}$$

Idea:

$$\mathrm{Map}(\Sigma) \xrightarrow{\Phi} \mathrm{Aut}^+(\mathcal{H}_g) \cong H \rtimes Sp(H)$$

fix $\sigma \in \mathcal{H}_g$

$H = H_1(\Sigma; \mathbb{Z})$.

(a) Under this id^n, Φ = "Trapp representation" [Trapp, Morita, ...]

$\qquad\qquad\qquad\qquad\qquad\quad\| $

action on $\{$unit vector fields on $\Sigma\}$

[Trapp] $\Rightarrow$ $\ker(\Phi) = \mathrm{Chill}(\Sigma)$.

(b) alg: $\mathrm{Inn}(\mathcal{H}_g) \longleftrightarrow 2H$

top: $\mathrm{image}(\Phi) \subseteq 2H \rtimes Sp(H)$.

//.

Theorem (Blanchet - P - Shaukat '21)

We obtain a well-defined representation

(a) $\mathrm{Chill}(\Sigma) \curvearrowright H_k^{BM}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$

(b) $\widetilde{\mathrm{Tor}}(\Sigma) \curvearrowright H_k^{BM}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$

by $\mathbb{Z}[\mathcal{H}_g]$-module automorphisms, where $\widetilde{\mathrm{Tor}}(\Sigma)$ is a central extension of $\mathrm{Tor}(\Sigma)$ by $\mathbb{Z}$.

proof (a) immediate.

In fact we get $\mathrm{Map}(\Sigma) \curvearrowright H_k^{BM}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$

by twisted $\mathbb{Z}[\mathcal{H}_g]$-module rep's
- action by $\mathbb{Z}$-module aut's
- \+ action on $\mathbb{Z}[\mathcal{H}_g]$
- \+ compatibility.

Lemma ("untwisting trick")

A twisted repr. of Γ over $\mathbb{Z}[G]$, where the action on G is inner, can be untwisted by passing to a central ext. of Γ by $\underbrace{Z(G)}_{\text{centre of } G}$.

Apply this to $\Gamma = \mathrm{Tor}(\Sigma)$
$G = \mathcal{H}_g$.

//.

Comparison to Moriyama:

$(k=2)$ Quotient of $\left[\begin{array}{l}\text{twisted } \mathrm{Map}(\Sigma)\text{-rep's} \\ \widetilde{\mathrm{Tor}}(\Sigma) \text{ - rep's}\end{array}\right.$

$$\begin{array}{c} H_2^{BM}\left(C_2(\Sigma');\, \mathbb{Z}[\mathcal{H}_g]\right) \\ \big\downarrow \\ H_2^{BM}\left(C_2(\Sigma');\, \underbrace{\mathbb{Z}[\mathbb{Z}/2]}_{=\,\mathbb{Z}[\Sigma_2]}\right) \\ \| \\ 2^{nd} \text{ Moriyama repr.} \end{array} \qquad \text{ind. by} \qquad \begin{tikzcd} B_k(\Sigma) \arrow[d, two heads] \arrow[dd, bend right, two heads, "ab"'] \\ \mathcal{H}_g \arrow[d, two heads, "ab"] \\ \mathbb{Z}/2 \oplus H_1(\Sigma;\mathbb{Z}) \arrow[d, two heads] \\ \mathbb{Z}/2 \end{tikzcd}$$

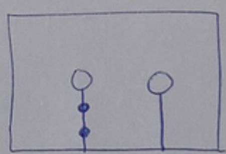
Cor $\ker\left(\mathrm{Map}(\Sigma) \curvearrowright H_2^{BM}\left(C_2(\Sigma');\, \mathbb{Z}[\mathcal{H}_g]\right)\right) \subseteq J(2)$

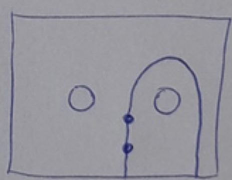
[Johnson]: the subgroup of $\mathrm{Map}(\Sigma)$ generated by T_γ, γ separating curve

Calc^n Sage

$\underline{g=1}$

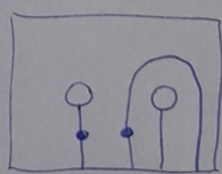
Repr. is 3-dim over $\mathbb{Z}[\mathcal{H}_g]$





$T_{\partial\Sigma} \in J(2)$

$\left(T_{\partial\Sigma}\right)_* = \ldots\ldots$ (P.T.O.)



Cor. $\ker(\cdots) \subsetneq J(2)$.

$$
\begin{bmatrix}
\begin{matrix} u^{-8}b^2+u^{-4}a^{-2}-ua^{-2}b^2+(u^{-1}-u^{-2})a^{-2}b+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(u^{-4}-u^{-5})a^{-1}b \end{matrix} &
\begin{matrix} (u^2+1-2u^{-1}+u^{-2}+u^{-4})a^{-2}b^2-ua^{-2}b^4+ \\ (-u^2+u+u^{-1}-u^{-2})a^{-2}b^3-u^{-3}a^{-2}+ \\ (-1+u^{-1}+u^{-3}-u^{-4})a^{-2}b \end{matrix} &
\begin{matrix} (-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-2}b+(u-1)a^{-2}b^3+ \\ (u^2-u-u^{-1}+2u^{-2}-u^{-3})a^{-2}b^2+(-u^{-3}+u^{-4})a^{-1}b+ \\ (u^{-4}-u^{-5})a^{-1}b^3+(-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}b^2+(-u^{-3}+u^{-4})a^{-2} \end{matrix} \\
\begin{matrix} -u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^2+ \\ (u^{-1}-u^{-2}-u^{-4}+u^{-5})a+u^{-6}a^{-2}+ \\ (u^{-3}-u^{-4}-u^{-6}+u^{-7})a^{-1} \end{matrix} &
\begin{matrix} 1+u^{-2}-u^{-3}+u^{-6}+u^{-6}a^{-2}b^2-u^{-1}b^2+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(-1+u^{-1}+u^{-3}-u^{-4})b+ \\ (u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})a^{-1}b-u^{-5}a^{-2}+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}+(u^{-5}-u^{-6})a^{-2}b \end{matrix} &
\begin{matrix} (-u^{-6}+u^{-7})a^{-2}b+(u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b+ \\ (-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})a^{-1}b+1-u^{-1}+u^{-2}- \\ 3u^{-3}+2u^{-4}+u^{-6}-u^{-7}+(-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a^{-1} \\ +(u^{-2}-u^{-3})ab+(-1+u^{-1}+u^{-3}-u^{-4})a+(-u^{-5}+u^{-6})a^{-2} \end{matrix} \\
\begin{matrix} -u^{-6}ab+(-u^{-3}+u^{-4}-u^{-7})b-u^{-4}+ \\ (u^{-1}-u^{-4}+u^{-5})a^{-1}b+u^{-2}a^{-2}b+ \\ (-u^{-3}+u^{-6})a^{-1}+u^{-5}a^{-2} \end{matrix} &
\begin{matrix} (-1-u^{-2}+2u^{-3}-u^{-6})a^{-1}b+u^{-1}a^{-1}b^3+ \\ u^{-2}a^{-2}b^3+(1-u^{-1}-u^{-3}+u^{-4})a^{-1}b^2+ \\ (u^{-1}-u^{-2}+u^{-5})a^{-2}b^2+(-u^{-1}+u^{-4}-u^{-5})a^{-2}b+ \\ (u^{-2}-u^{-5})a^{-1}-u^{-4}a^{-2} \end{matrix} &
\begin{matrix} u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a^{-1}+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-1}b^2+(-u^{-2}+u^{-3})a^{-2}b^2+ \\ (-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})a^{-1}b+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-2}b+(-u^{-4}+u^{-5})b^2+ \\ (u^{-2}-u^{-3}-u^{-5}+u^{-6})b+(-u^{-4}+u^{-5})a^{-2} \end{matrix}
\end{bmatrix}
$$

- $u \equiv \sigma$
- $a \equiv a_1$
- $b \equiv b_1$