Representations of the Torelli group via the Heisenberg group

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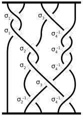
Braid groups – definitions & viewpoints

Many definitions of B_n:

- $\pi_1(C_n(D^2))$
- $\{$diffeomorphisms of $S = D^2 \setminus \{n \text{ points}\}\}/$isotopy
- $\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2 \end{array} \right\rangle$
- $\Bigg\{$ 

$\Bigg\}$

Braid groups – applications & connections

Knot theory:

- $\bigsqcup_{n\geqslant 1} B_n \twoheadrightarrow \{\text{knots/links in } \mathbb{R}^3\}$ [Alexander, Markov]
- *Burau representation* $\longmapsto$ *Alexander polynomial*

Algebraic geometry:

- [Moishezon]: alg. curve in $\mathbb{CP}^2 \longmapsto$ *braid monodromy* $F_N \to B_d$
- [Libgober]: alg. curve in $\mathbb{CP}^2 \longmapsto$ invariant *using a representation of* B_d

Homotopy theory:

- [Berrick-Cohen-Wong-Wu, 2006]:

$$\pi_*(S^2) \;\cong\; \frac{\{\text{Brunnian braids in } S^2 \times [0,1]\}}{\{\text{Brunnian braids in } D^2 \times [0,1]\}}$$

Mapping class groups of surfaces

Definition

$$\mathrm{Map}(S) = \{\text{diffeomorphisms of } S\}/\text{isotopy}$$

Example: $\mathrm{Map}(D_n) = B_n$ where $D_n = D^2 \smallsetminus \{n \text{ points}\}$

Applications & connections:

- $\mathrm{Map}(S) = \pi_1(\mathcal{M}_S)$
  $\mathcal{M}_S =$ *moduli space of algebraic curves* of topological type S
- 3-manifold topology — via *Heegaard splittings*
- 4-dimensional symplectic topology [Donaldson]

Representations of braid groups – Burau

[Burau] representation (1935):

$$\sigma_i \quad \longmapsto \quad \mathrm{I}_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus \mathrm{I}_{n-i-1}$$

- This defines $B_n \longrightarrow GL_n(\mathbb{Z}[t^{\pm 1}]) \subset GL_n(\mathbb{R})$
- Q($\leqslant$[Birman'74]): *Is this representation injective?* ($\equiv$'faithful')
- A($n \leqslant 3$): Yes [Magnus-Peluso'69]
- A($n \geqslant 5$): No [Moody'91,Long-Paton'93,Bigelow'99]
- A($n = 4$): ??
- Q: *Are the braid groups linear?*
  — Does B_n embed into some $GL_N(\mathbb{F})$?

Representations of braid groups – Lawrence

[Lawrence] representation (1990) — geometric definition.

- $\mathrm{Diff}(D_n)$ acts on $C_k(D_n)$ (unordered configuration space)
- $B_n = \mathrm{Map}(D_n) = \mathrm{Diff}(D_n)/\sim$ acts on $H_*(C_k(D_n);\mathbb{Z})$
- Two modifications:
    - Choose $\pi_1(C_k(D_n)) \twoheadrightarrow Q$ invariant under the action. Then B_n acts on $H_*(C_k(D_n);\mathbb{Z}[Q])$
    - Replace H_* with H_*^{bm} (*Borel-Moore homology*) Then $H_*^{bm}(C_k(D_n);\mathbb{Z}[Q])$ is a free $\mathbb{Z}[Q]$-module concentrated in degree $* = k$

$$\mathrm{Lawrence}_k \colon B_n \longrightarrow GL_N(\mathbb{Z}[Q]) = \mathrm{Aut}_{\mathbb{Z}[Q]}\left(H_*^{bm}(C_k(D_n);\mathbb{Z}[Q])\right)$$

Representations of braid groups – Lawrence

How is the quotient Q defined?

- $\pi_1(D_n) = F_n \longrightarrow \mathbb{Z} = Q$ "total winding number"
- $\pi_1(C_k(D_n)) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ ("total winding number", "self-winding number")

Lemma

This quotient is $\mathrm{Map}(D_n)$-invariant, and hence

$$\mathrm{Lawrence}_k \colon B_n \longrightarrow GL_N(\mathbb{Z}[Q])$$

is well-defined. Moreover, we have $\mathrm{Lawrence}_1 = \mathrm{Burau}$.

Theorem [Bigelow'00,Krammer'00]

$\mathrm{Lawrence}_2$ is faithful (injective). Hence B_n embeds into $GL_N(\mathbb{R})$.

Representations of mapping class groups

- Q: Does $\mathrm{Map}(S)$ embed into $GL_N(\mathbb{F})$ for other surfaces S?
- $\mathrm{Map}(\text{torus}) \cong SL_2(\mathbb{Z}) \subset GL_2(\mathbb{R})$
- $\mathrm{Map}(\Sigma_2) \subset GL_{64}(\mathbb{C})$ [Bigelow-Budney'01]
- In general, wide open!
  - Kontsevich (2006): proposal of a sketch of a construction of a faithful finite-dimensional representation of $\mathrm{Map}(\Sigma_g)$
  - Dunfield (cf. [Margalit'18]): computational evidence suggesting that this will *not* actually be faithful
- From now on, focus on $\Sigma = \Sigma_{g,1}$ (orientable, genus g, one boundary component)

Main result [Blanchet-P.-Shaukat'21]

A new representation of (a central extension of) $\mathrm{Tor}(\Sigma) \subset \mathrm{Map}(\Sigma)$.

Representations of MCGs – Moriyama

Simplest analogue of the Lawrence representations:

$$\mathrm{Map}(\Sigma) \quad \circlearrowleft \quad H_k^{bm}(F_k(\Sigma'); \mathbb{Z})$$

- $F_k(\quad) =$ *ordered* configuration space
- $\Sigma' = \Sigma \setminus (\text{interval in } \partial\Sigma)$
- *untwisted* $\mathbb{Z}$ coefficients
- $H_k^{bm}(F_k(\Sigma'); \mathbb{Z})$ is a free abelian group of finite rank

Theorem [Moriyama'07]

The kernel of this representation is $\mathfrak{J}(k) \subset \mathrm{Map}(\Sigma)$.

- $\mathfrak{J}(k)$ is the k-th term of the *Johnson filtration* of $\mathrm{Map}(\Sigma)$
- What is this?

The Johnson filtration

- Lower central series: $\pi_1(\Sigma) = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \cdots$
- $\Gamma_i = [\pi_1(\Sigma), \Gamma_{i-1}]$ (commutators of length $i+1$)

Definition [Johnson'81]

$\mathfrak{J}(k)$ = kernel of the action of $\mathrm{Map}(\Sigma)$ on $\pi_1(\Sigma)/\Gamma_k$.

- $\mathrm{Map}(\Sigma) = \mathfrak{J}(0) \supset \mathfrak{J}(1) \supset \mathfrak{J}(2) \supset \mathfrak{J}(3) \supset \cdots$
- $\mathfrak{J}(1) = \mathrm{Tor}(\Sigma) = \ker\left(\mathrm{Map}(\Sigma) \circlearrowleft H_1(\Sigma;\mathbb{Z})\right)$ *Torelli group*

Theorem [Johnson'81]

$$\bigcap_{k=1}^{\infty} \mathfrak{J}(k) = \{1\}$$

Corollary [Moriyama'07]

$\bigoplus_{k=1}^{\infty} H_k^{bm}(F_k(\Sigma');\mathbb{Z})$ is a faithful (∞-dim.!) $\mathrm{Map}(\Sigma)$-representation.

Rep. of MCGs – abelian twisted coefficients

- Idea: Enrich the representation by taking homology with *twisted coefficients* $\mathbb{Z}[Q]$, where $\pi_1(C_k(\Sigma')) = B_k(\Sigma) \twoheadrightarrow Q$.
- $Q = \mathfrak{S}_k$ corresponds to the Moriyama representations:
$$H_k^{bm}(F_k(\Sigma'); \mathbb{Z}) = H_k^{bm}(C_k(\Sigma'); \mathbb{Z}[\mathfrak{S}_k]).$$
- First try abelian quotients Q.

Fact ($k \geqslant 2$)

$$B_k(S)^{ab} \cong \pi_1(S)^{ab} \oplus \begin{cases} \mathbb{Z} & S \text{ planar} \\ \mathbb{Z}/(2k-2) & S = S^2 \\ \mathbb{Z}/2 & \text{otherwise.} \end{cases}$$

- If S is non-planar, we can only "count" the *self-winding number* of braids on S **mod 2**. (or mod $2k-2$ if $S = S^2$)
- In $\mathbb{Z}[B_k(S)^{ab}]$, the corresponding "variable" t will have order two: $t^2 = 1$.
  $\longmapsto$ We get a much "weaker" representation...

Rep. of MCGs – non-abelian twisted coefficients

Theorem [Bellingeri'04]

$$B_k(\Sigma_{g,1}) \cong \left\langle \sigma_1, \ldots, \sigma_{k-1}, \begin{matrix} a_1, \ldots, a_g \\ b_1, \ldots, b_g \end{matrix} \;\middle|\; \cdots \text{ some relations } \cdots \right\rangle$$

Adding the relations saying that σ_1 is *central* (commutes with every element), we obtain:

$$B_k(\Sigma_{g,1})/\langle\!\langle\, [\sigma_1, x] \,\rangle\!\rangle \cong \left\langle \sigma, \begin{matrix} a_1, \ldots, a_g \\ b_1, \ldots, b_g \end{matrix} \;\middle|\; \begin{matrix} \text{all pairs commute except} \\ a_i b_i = \sigma^2 b_i a_i \end{matrix} \right\rangle$$

Definition

$$\mathcal{H}_g = B_k(\Sigma_{g,1})/\langle\!\langle\, [\sigma_1, x] \,\rangle\!\rangle$$

This is the *genus-g discrete Heisenberg group*. Note that:

$$\mathcal{H}_1 \cong \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3(\mathbb{Z})$$

Rep. of MCGs – non-abelian twisted coefficients

Lemma

The action $\mathrm{Map}(\Sigma) \circlearrowleft B_k(\Sigma)$ descends to a well-defined action on the quotient $\mathcal{H}_g$.

Proof

- Aim: $\ker(B_k(\Sigma) \twoheadrightarrow \mathcal{H}_g)$ is preserved by the $\mathrm{Map}(\Sigma)$-action.
- This is $\langle\!\langle\, [\sigma_1, x] \,\rangle\!\rangle$, so it is enough to show that σ_1 is fixed by the $\mathrm{Map}(\Sigma)$-action.
- Let $[\varphi] \in \mathrm{Map}(\Sigma) = \mathrm{Diff}(\Sigma)/\!\sim$ be represented by a diffeo. φ that fixes *pointwise* a collar neighbourhood of $\partial\Sigma$.
- The loop of configurations $\sigma_1 \in B_k(\Sigma) = \pi_1(C_k(\Sigma))$ can be homotoped to stay inside this collar neighbourhood.

The Heisenberg group fits into a central extension:

$$1 \to \mathbb{Z} \longrightarrow \mathcal{H}_g \longrightarrow H_1(\Sigma;\mathbb{Z}) \to 1$$

and the $\mathrm{Map}(\Sigma)$-action on $\mathcal{H}_g$ lifts the natural action on $H_1(\Sigma;\mathbb{Z})$.

Rep. of MCGs – non-abelian twisted coefficients

Proposition [Blanchet-P.-Shaukat'21]

(a) $\ker(\mathrm{Map}(\Sigma) \circlearrowleft \mathcal{H}_g) = \mathrm{Chill}(\Sigma)$

(b) $\{\varphi \in \mathrm{Map}(\Sigma) \mid \varphi$ acts on $\mathcal{H}_g$ by conjugations$\} = \mathrm{Tor}(\Sigma)$

$$\mathrm{Map}(\Sigma) \supset \mathrm{Tor}(\Sigma) \supset \mathrm{Chill}(\Sigma) \supset \mathfrak{J}(2) \supset \mathfrak{J}(3) \supset \cdots$$

Idea of proof

$$\Phi \colon \mathrm{Map}(\Sigma) \longrightarrow \mathrm{Aut}^+(\mathcal{H}_g) \cong H \rtimes Sp(H) \qquad H = H_1(\Sigma; \mathbb{Z})$$

(a) Under this identification, $\Phi =$ "Trapp representation"
(= the action of $\mathrm{Map}(\Sigma)$ on {unit vector fields on Σ})

(b) alg.: $\mathrm{Inn}(\mathcal{H}_g) \longleftrightarrow 2H$
top.: $\mathrm{image}(\Phi) \subseteq 2H \rtimes Sp(H)$

Rep. of MCGs – non-abelian twisted coefficients

Theorem [Blanchet-P.-Shaukat'21]

We obtain well-defined representations, defined over $\mathbb{Z}[\mathcal{H}_g]$:

(a) $\mathrm{Chill}(\Sigma) \quad \circlearrowleft \quad H_k^{bm}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$

(b) $\widetilde{\mathrm{Tor}}(\Sigma) \quad \circlearrowleft \quad H_k^{bm}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$

where $\widetilde{\mathrm{Tor}}(\Sigma)$ is a $\mathbb{Z}$-central extension of $\mathrm{Tor}(\Sigma)$.

Idea of proof

Lemma $\Rightarrow$ we obtain a *twisted* representation, defined over $\mathbb{Z}[\mathcal{H}_g]$:

$$\mathrm{Map}(\Sigma) \quad \circlearrowleft \quad H_k^{bm}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$$

"Twisted" means:

- the action is $\mathbb{Z}$-linear
- there is also an action on the ground ring $\mathbb{Z}[\mathcal{H}_g]$
- these are compatible: $\varphi(\lambda.v) = \varphi(\lambda).\varphi(v)$.

Rep. of MCGs – non-abelian twisted coefficients

Idea of proof (cont.)

Proposition (a) $\Rightarrow$ the action of $\mathrm{Chill}(\Sigma)$ on $\mathbb{Z}[\mathcal{H}_g]$ is trivial
$\Rightarrow$ the representation is *untwisted* on $\mathrm{Chill}(\Sigma)$.

Untwisting Lemma: A twisted representation of Γ over $\mathbb{Z}[G]$, where the action $\Gamma \circlearrowleft G$ is by conjugations, can be *untwisted* by passing to a central extension of Γ by $\mathcal{Z}(G) = \mathsf{Centre}(G)$.

Proposition (b) $\Rightarrow$ we can apply the Untwisting Lemma to $\Gamma = \mathrm{Tor}(\Sigma)$ and $G = \mathcal{H}_g$.

Q: What is the kernel of this representation?
Q′: Is it smaller than $\mathfrak{J}(k) = \ker(\mathrm{Moriyama}_k)$?

Upper bound on the kernel

$(k = 2)$
Quotient of $\widetilde{\mathrm{Tor}}(\Sigma)$-representations / twisted $\mathrm{Map}(\Sigma)$-representations:

$$H_2^{bm}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g]) \twoheadrightarrow H_2^{bm}(C_2(\Sigma'); \mathbb{Z}[\mathfrak{S}_2])$$

induced by $\mathcal{H}_g \twoheadrightarrow (\mathcal{H}_g)^{ab} = \mathbb{Z}/2 \oplus H_1(\Sigma; \mathbb{Z}) \twoheadrightarrow \mathbb{Z}/2 = \mathfrak{S}_2$.

- The right-hand side is precisely $H_2^{bm}(F_2(\Sigma'); \mathbb{Z}) = \mathrm{Moriyama}_2$, hence
$$\ker\left(H_2^{bm}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g])\right) \quad \subseteq \quad \mathfrak{J}(2)$$
- Let $\gamma \subset \Sigma$ be a simple closed curve that separates off a genus-1 subsurface. Then the *Dehn twist* T_γ lies in $\mathfrak{J}(2)$.
- Calculations $\Rightarrow$ T_γ acts *non-trivially* in our representation.

Corollary [Blanchet-P.-Shaukat'21]

The kernel of the $\widetilde{\mathrm{Tor}}(\Sigma)$-representation $H_2^{bm}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$ is **strictly smaller** than $\mathfrak{J}(2)$.

Example calculation

Set $k = 2$ and $g = 1$. In this case the representation

$$H_2^{bm}\left(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_1]\right)$$

is free of rank 3 over $\mathbb{Z}[\mathcal{H}_1] = \mathbb{Z}[\sigma^{\pm 1}]\langle a^{\pm 1}, b^{\pm 1}\rangle / (ab = \sigma^2 ba)$

Let γ be a curve isotopic to $\partial\Sigma = \partial\Sigma_{1,1}$. Then T_γ acts via:

$$\begin{bmatrix}
\begin{matrix}\sigma^{-8}b^2+\sigma^{-4}a^{-2}-\sigma a^{-2}b^2+(\sigma^{-1}-\sigma^{-2})a^{-2}b+\\(\sigma^{-3}-\sigma^{-4})a^{-1}b^2+(\sigma^{-4}-\sigma^{-5})a^{-1}b\end{matrix} &
\begin{matrix}(\sigma^2+1-2\sigma^{-1}+\sigma^{-2}+\sigma^{-4})a^{-2}b^2-\sigma a^{-2}b^4+\\(-\sigma^2+\sigma+\sigma^{-1}-\sigma^{-2})a^{-2}b^3-\sigma^{-3}a^{-2}+\\(-1+\sigma^{-1}+\sigma^{-3}-\sigma^{-4})a^{-2}b\end{matrix} &
\begin{matrix}(-1+2\sigma^{-1}-\sigma^{-2}-\sigma^{-4}+\sigma^{-5})a^{-2}b+(\sigma-1)a^{-2}b^3+\\(\sigma^2-\sigma-\sigma^{-1}+2\sigma^{-2}-\sigma^{-3})a^{-2}b^2+(-\sigma^{-3}+\sigma^{-4})a^{-1}b+\\(\sigma^{-4}-\sigma^{-5})a^{-1}b^3+(-\sigma^{-2}+\sigma^{-3}+\sigma^{-5}-\sigma^{-6})a^{-1}b^2+(-\sigma^{-3}+\sigma^{-4})a^{-2}\end{matrix} \\
\begin{matrix}-\sigma^{-1}-\sigma^{-3}+2\sigma^{-4}-\sigma^{-5}-\sigma^{-7}+\sigma^{-2}a^2+\\(\sigma^{-1}-\sigma^{-2}-\sigma^{-4}+\sigma^{-5})a+\sigma^{-6}a^{-2}+\\(\sigma^{-3}-\sigma^{-4}-\sigma^{-6}+\sigma^{-7})a^{-1}\end{matrix} &
\begin{matrix}1+\sigma^{-2}-\sigma^{-3}+\sigma^{-6}+\sigma^{-6}a^{-2}b^2-\sigma^{-1}b^2+\\(\sigma^{-3}-\sigma^{-4})a^{-1}b^2+(-1+\sigma^{-1}+\sigma^{-3}-\sigma^{-4})b+\\(\sigma^{-2}-2\sigma^{-3}+\sigma^{-4}+\sigma^{-6}-\sigma^{-7})a^{-1}b-\sigma^{-5}a^{-2}+\\(-\sigma^{-2}+\sigma^{-3}+\sigma^{-5}-\sigma^{-6})a^{-1}+(\sigma^{-5}-\sigma^{-6})a^{-2}b\end{matrix} &
\begin{matrix}(-\sigma^{-6}+\sigma^{-7})a^{-2}b+(\sigma^{-1}-\sigma^{-2}-\sigma^{-4}+2\sigma^{-5}-\sigma^{-6})b+\\(-\sigma^{-3}+2\sigma^{-4}-\sigma^{-5}-\sigma^{-7}+\sigma^{-8})a^{-1}b+1-\sigma^{-1}+\sigma^{-2}-\\3\sigma^{-3}+2\sigma^{-4}+\sigma^{-6}-\sigma^{-7}+(-\sigma^{-2}+2\sigma^{-3}-\sigma^{-4}+\sigma^{-5}-2\sigma^{-6}+\sigma^{-7})a^{-1}\\+(\sigma^{-2}-\sigma^{-3})ab+(-1+\sigma^{-1}+\sigma^{-3}-\sigma^{-4})a+(-\sigma^{-5}+\sigma^{-6})a^{-2}\end{matrix} \\
\begin{matrix}-\sigma^{-6}ab+(-\sigma^{-3}+\sigma^{-4}-\sigma^{-7})b-\sigma^{-4}+\\(\sigma^{-1}-\sigma^{-4}+\sigma^{-5})a^{-1}b+\sigma^{-2}a^{-2}b+\\(-\sigma^{-3}+\sigma^{-6})a^{-1}+\sigma^{-5}a^{-2}\end{matrix} &
\begin{matrix}(-1-\sigma^{-2}+2\sigma^{-3}-\sigma^{-6})a^{-1}b+\sigma^{-1}a^{-1}b^3+\\\sigma^{-2}a^{-2}b^3+(1-\sigma^{-1}-\sigma^{-3}+\sigma^{-4})a^{-1}b^2+\\(\sigma^{-1}-\sigma^{-2}+\sigma^{-5})a^{-2}b^2+(-\sigma^{-1}+\sigma^{-4}-\sigma^{-5})a^{-2}b+\\(\sigma^{-2}-\sigma^{-5})a^{-1}-\sigma^{-4}a^{-2}\end{matrix} &
\begin{matrix}\sigma^{-3}+(\sigma^{-2}-\sigma^{-3}-\sigma^{-5}+\sigma^{-6})a^{-1}+\\(-\sigma^{-1}+\sigma^{-2}-\sigma^{-5}+\sigma^{-6})a^{-1}b^2+(-\sigma^{-2}+\sigma^{-3})a^{-2}b^2+\\(-1+\sigma^{-1}+2\sigma^{-3}-3\sigma^{-4}+\sigma^{-7})a^{-1}b+\\(-\sigma^{-1}+\sigma^{-2}-\sigma^{-5}+\sigma^{-6})a^{-2}b+(-\sigma^{-4}+\sigma^{-5})b^2+\\(\sigma^{-2}-\sigma^{-3}-\sigma^{-5}+\sigma^{-6})b+(-\sigma^{-4}+\sigma^{-5})a^{-2}\end{matrix}
\end{bmatrix}$$

Thank you!