Moduli spaces of high-dimensional discs

after Watanabe, Weiss, Kupers-Randal-Williams

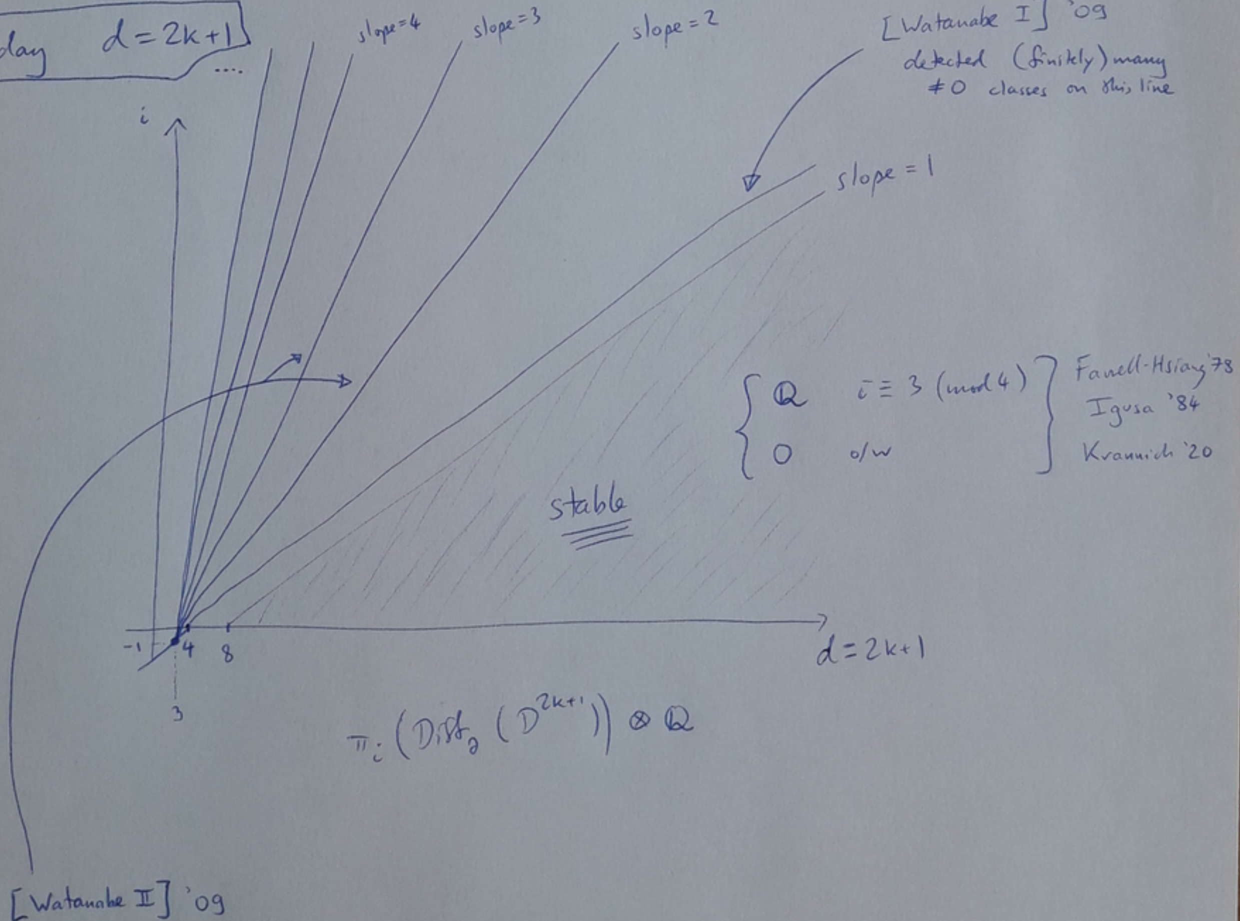
GeMAT seminar
IMAR
28 May 2021

III. Odd-dimensional discs, far above the stable range

Aim: Study the rational homotopy groups of

$$\mathrm{Diff}_\partial(D^d) = \left\{ D^d \xrightarrow{\varphi} D^d \text{ diffeomorphism fixing pointwise a neighbourhood of } \partial D^d \right\}$$

$\boxed{\text{Today} \quad d = 2k+1}$ ....



[Watanabe II] :

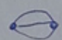
Every rational homotopy group lying on one of the lines of slope $n \geq 2$ above has dimension $\geq \dim(\underbrace{A_{2n,3n}}_{\{\text{trivalent graphs}\}/\text{relations}})$

when k is odd, i.e. $d \equiv 3 \pmod 4$

purely combinatorial (in principle)

Lower bounds on $\dim(A_{2n,3n})$ come from

- weight systems
  - Lie algebras
  - Rozansky - Witten
- brute force computations

n	$\dim(A_{2n,3n})$
1	1 
2	1
3	1
4	2
5	2
6	3
7	4
8	5
9	6
10	8
11	9

[Bar-Natan]
[Kneissler]

Plan

- Graph complexes
- Kontsevich integral I_π
- Watanabe's main theorem — detects non-trivial D^d-bundles
- Idea of construction of I_π
- Idea of construction of Ψ_{2n} — builds non-trivial D^d-bundles

Graph complexes

Def $\mathcal{G}_{n,m} := \mathbb{Q}\left\{ \begin{array}{l} \text{finite, connected graphs } \Gamma \text{ with valence} \geq 3 \\ V(\Gamma) \longleftrightarrow \{1,\dots,n\} \\ E(\Gamma) \longleftrightarrow \{1,\dots,m\} \\ \text{orientation} \end{array} \right\} \Big/ \sim$

$or(\Gamma) = \{\text{orientation of each edge of } \Gamma\} \big/$ reversing orientations of an even # of edges

$\boxed{\bar{\Gamma} \sim -\Gamma}$

where $\bar{\Gamma}$ means Γ with opposite orientation

Note: trivalent $\longleftrightarrow$ $n = 2\ell$, $m = 3\ell$.

Def $\mathcal{G}_{n,m} \xrightarrow{d} \mathcal{G}_{n-1,m-1}$

$\Gamma \longmapsto \sum_{e \in E(\Gamma)} \Gamma/e$ ⟵ induced labelling & orientation

Lemma $d^2 = 0$

Observation: $\gamma \in \mathcal{G}_{n,m}$

$d\gamma = 0 \longleftrightarrow \underbrace{\text{system of linear equations in the coeffs of } \gamma}_{R_{n,m}}$

$R_{n,m}$ ⟵ collection of <u>relations</u> on $\mathcal{G}_{n,m}$

Def $\quad \mathcal{I}_{n,m} := \mathcal{G}_{n,m} / \mathcal{R}_{n,m}$

Key property: this allows us to view $\ker(d)$ as a quotient of $\mathcal{G}_{n,m}$:

$$\begin{tikzcd}
{} \arrow[r, "0"] & \ker(d) \arrow[r, "0"] \arrow[d, hook] \arrow[dd, bend left, "\cong"] & \ker(d) \arrow[r, "0"] \arrow[d, hook] \arrow[dd, bend left, "\cong"] & {} \\
{} \arrow[r, "d"] & \mathcal{G}_{n,m} \arrow[r, "d"] \arrow[d, two heads] & \mathcal{G}_{n-1,m-1} \arrow[r, "d"] \arrow[d] & {} \\
{} \arrow[r, "0"] & \mathcal{I}_{n,m} \arrow[r, "0"] & \mathcal{I}_{n-1,m-1} \arrow[r, "0"] & {}
\end{tikzcd}$$

Rmk $\quad \mathcal{I}_{2\ell,3\ell} = \{\text{IHX relations}\}$

Def $\quad \Gamma' = \Gamma$ with two vertex labels swapped
$\quad\quad\;\; \Gamma'' = \Gamma$ with two edge labels swapped

$$\mathcal{A}^{odd}_{n,m} := \mathcal{I}_{n,m} \Big/ \begin{matrix} \Gamma' \sim -\Gamma \\ \Gamma'' \sim \Gamma \end{matrix}$$ target for Kontsevich char. classes of odd-dim. disc bundles

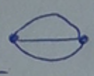
$$\mathcal{A}^{even}_{n,m} := \mathcal{I}_{n,m} \Big/ \begin{matrix} \Gamma' \sim \Gamma \\ \Gamma'' \sim -\Gamma \end{matrix}$$ target for Kontsevich char. classes of even-dim. disc bundles

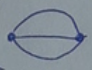
Today: $\mathcal{A}_{n,m} = \mathcal{A}^{odd}_{n,m}$.

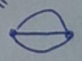
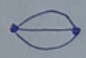
Comparison:

ℓ	1	2	3	4	5	6	7	8	9
$\dim(\mathcal{A}^{odd}_{2\ell,3\ell})$	Θ 1	1	1	2	2	3	4	5	6
$\dim(\mathcal{A}^{even}_{2\ell,3\ell})$	0	1 	0	0	1	0	0	0	1

In particular,

$[$  $] \neq 0 \in \mathcal{A}^{odd}_{2,3}$

$[$  $] = 0 \in \mathcal{A}^{even}_{2,3}$

- $\Gamma =$  has an auto. swapping two edges, so $\Gamma = -\Gamma$ in $\mathcal{A}^{even}_{2,3}$
- $\Gamma =$  has an auto. swapping two vertices, but this <u>also</u> reverses orientations of 3 edges, so $\Gamma = (-1)^4\,\Gamma$ in $\mathcal{A}^{odd}_{2,3}$.

<u>Why $\mathcal{A}_{n,m}$?</u> $\longrightarrow$ "universal" cycles!

$(\mathcal{G}_{n,m}, d)$ does not contain a <u>canonical</u> cycle.

$(\mathcal{A}_{n,m} \otimes \mathcal{G}_{n,m}, 1 \otimes d)$ has a canonical ("<u>universal</u>") cycle:

$$\tilde{\gamma}_{n,m} = \frac{1}{n!\,m!} \sum_{\Gamma} [\Gamma] \otimes \Gamma$$

oriented labelled graphs

Kontsevich integral I_π

Black box (Kontsevich)

Associated to any framed (D^d, ∂)-bundle $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$ over a smooth manifold,

there is a chain map

$$\begin{tikzcd}
\mathcal{G}_{n,m} \arrow[r, "I_\pi"] \arrow[d, "d"] & \Omega_{dR}^{m(d-1)-nd}(B) \arrow[d, "d"] \\
\mathcal{G}_{n-1,m-1} \arrow[r, "I_\pi"] & \Omega_{dR}^{m(d-1)-nd+1}(B)
\end{tikzcd}$$

(1) I_π depends on a choice of "propagator", but if $\gamma \in \mathcal{G}_{n,m}$ is a cycle, then $[I_\pi(\gamma)] \in H^{m(d-1)-nd}(B;\mathbb{R})$ does not.

(2) $[I_\pi(\gamma)]$ is a characteristic class of framed (D^d, ∂)-bundles (it behaves well under pullback)

(3) If $\dim(B) = m(d-1)-nd$, B closed & oriented, then $\int_B [I_\pi(\gamma)]$ depends on the bundle $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$ up to oriented cobordism of bundles.

<u>Corollary</u> Each cycle $\gamma \in \mathcal{G}_{n,m}$ defines a homomorphism

$$\begin{array}{ccc} FDisc^{d}_{m(d-1)-nd} & \longrightarrow & \mathbb{R} \\ \| & & \\ \left\{ \begin{array}{l} \text{framed } (D^d,\partial)\text{-bundles over closed} \\ \text{oriented } (m(d-1)-nd)\text{-manifolds} \end{array} \right\} \Big/ \text{cobordism} & & \\ \left[\begin{array}{c} E \\ \downarrow \pi \\ B \end{array} \right] & \longmapsto & \int_B \left[I_\pi(\gamma) \right] \end{array}$$

operation $= \amalg = \#$

<u>Obs</u> Properties (1)–(3) hold also if I_π is tensored with $\mathcal{A}_{n,m}$.

$$\rightsquigarrow \quad \left[\mathcal{A}_{n,m} \otimes I_\pi(\gamma) \right] \in H^{m(d-1)-nd}(B; \mathcal{A}_{n,m} \otimes \mathbb{R})$$

$\rightsquigarrow$ and integrating, we get:

<u>Corollary'</u> Each cycle $\gamma \in \mathcal{A}_{n,m} \otimes \mathcal{G}_{n,m}$ defines a homomorphism

$$FDisc^{d}_{m(d-1)-nd} \longrightarrow \mathcal{A}_{n,m} \otimes \mathbb{R} \qquad (*)$$

<u>Def</u> The <u>universal Kontsevich characteristic class</u> $KCC_{n,m}$ is the homomorphism $(*)$ associated to the "universal" cycle $\tilde{\gamma}_{n,m} = \frac{1}{n!m!} \sum_{\Gamma} [\Gamma] \otimes \Gamma$

$\in \mathcal{A}_{n,m} \otimes \mathcal{G}_{n,m}$.

Watanabe's main theorem

How are <u>framed</u> (D^d, ∂) - bundles related to $\pi_*(\mathrm{Diff}_\partial(D^d))$?

<u>Def</u> $\mathrm{BDiff}_\partial(D^d)$:= the space classifying smooth (D^d, ∂) - bundles

- it comes equipped with a smooth (D^d, ∂) -bundle
- for all (paracompact) B,

pullback : $\mathrm{Maps}(B, \mathrm{BDiff}_\partial(D^d)) / \mathrm{htpy} \longrightarrow \left\{ \begin{array}{l} \text{smooth } (D^d,\partial)\text{-} \\ \text{- bundles over } B \end{array} \right\} / \cong$

is a bijection.

(This <u>exists</u> & is <u>unique</u>)

[Milnor] — exists

up to htpy equivalence — unique

Lemma

<u>Lemma</u> $\pi_i(\mathrm{Diff}_\partial(D^d)) \cong \pi_{i+1}(\mathrm{BDiff}_\partial(D^d))$

<u>Def</u> $\widetilde{\mathrm{BDiff}}_\partial(D^d)$:= the space classifying smooth (D^d, ∂) - bundles that are <u>framed</u>

framed: trivialisation of the vertical tangent bundle

$$\begin{array}{ccc} \mathbb{R}^d \cong T_e E_{\pi(e)} & \longrightarrow & T^v E \\ & & \downarrow \\ D^d \cong E_b & \longrightarrow & E \ni e \\ & & \downarrow \pi \\ & & B \ni b \end{array}$$

$$\begin{array}{ccc} T^v E & \cong & \mathbb{R}^d \times E \\ \downarrow & \swarrow & \\ E & & \end{array}$$

$\exists$ canonical map (up to $\simeq$)

$$B\widetilde{\mathrm{Diff}}_\partial(D^d) \longrightarrow B\mathrm{Diff}_\partial(D^d)$$

<u>Prop</u> (Watanabe) The induced map on $\pi_i(-) \otimes \mathbb{Q}$ is an <u>isomorphism</u>

if
- d odd
- i even
- $i > 2k-2$.

$\exists$ canonical map

$$\begin{tikzcd}
\pi_i\left(B\widetilde{\mathrm{Diff}}_\partial(D^d)\right) \arrow[d, phantom, "\cong" rotate=90] & & \mathrm{FDisc}^d_i \arrow[d, equal] \\
\left\{\begin{matrix}\text{framed } (D^d,\partial)\text{-bundles} \\ \text{over } S^i\end{matrix}\right\} \Big/ \cong \arrow[rr, "H"] & & \left\{\begin{matrix}\text{framed } (D^d,\partial)\text{-bundles} \\ \text{over closed oriented } i\text{-mflds}\end{matrix}\right\} \Big/ \text{cob.}
\end{tikzcd}$$

Now set $(n,m) = (2\ell, 3\ell)$
$d = 2k+1$ $\quad\rightsquigarrow\quad m(d-1) - nd = 2\ell(k-1)$

<u>Theorem</u> (Watanabe) For all $\ell \geq 1$, k odd,
$\exists$ $\mathbb{Q}$-linear map $\mathcal{G}_{2\ell,3\ell} \xrightarrow{\Psi_{2\ell}} \mathrm{FDisc}^{2k+1}_{2\ell(k-1)} \otimes \mathbb{Q}$

such that

$$\begin{tikzcd}
 & \pi_{2\ell(k-1)}\left(B\widetilde{\mathrm{Diff}}_\partial(D^{2k+1})\right) \otimes \mathbb{Q} \arrow[d, "H \otimes \mathbb{Q}"] & \\
\mathcal{G}_{2\ell,3\ell} \arrow[r, "\Psi_{2\ell}"] \arrow[d, two heads] & \mathrm{FDisc}^{2k+1}_{2\ell(k+1)} \otimes \mathbb{Q} \arrow[r, "\mathrm{KCC}_{2\ell,3\ell}"] & \mathcal{A}_{2\ell,3\ell} \otimes \mathbb{R} \\
\mathcal{A}_{2\ell,3\ell} \arrow[r, hook] & \mathcal{A}_{2\ell,3\ell} \otimes \mathbb{R} \arrow[ur, bend right, "\text{mult}^{\text{n}} \text{ by a certain +ve integer}"'] &
\end{tikzcd}$$

<u>commutes</u> and <u>image$(\Psi_{2\ell}) \subseteq$ image$(H \otimes \mathbb{Q})$</u>

Corollary (Watanabe) For all $\ell \geq 2$, k odd,

$$\dim_{\mathbb{Q}}\left(\pi_{2\ell(k-1)-1}\left(\mathrm{Diff}_\partial(D^{2k+1})\right)\otimes\mathbb{Q}\right) \geq \dim_{\mathbb{Q}}\left(\mathcal{A}_{2\ell,3\ell}\right)$$

proof
- Tensor the big diagram with $\mathbb{R}$.
- Each $x \in \mathcal{A}_{2\ell,3\ell}\otimes\mathbb{R}$ lifts to $y \in \mathrm{image}(\Psi_{2\ell})\otimes\mathbb{R}$
- Each $y \in \mathrm{image}(\Psi_{2\ell})\otimes\mathbb{R}$ lifts to $z \in \pi_{2\ell(k-1)}\left(B\widetilde{\mathrm{Diff}}_\partial(D^{2k+1})\right)\otimes\mathbb{R}$
- Hence

$$\dim_{\mathbb{R}}\left(\pi_{2\ell(k-1)}\left(B\widetilde{\mathrm{Diff}}_\partial(D^{2k+1})\right)\otimes\mathbb{R}\right) \geq \dim_{\mathbb{R}}\left(\mathcal{A}_{2\ell,3\ell}\otimes\mathbb{R}\right)$$

$$\Rightarrow \quad \dim_{\mathbb{Q}}\left(\pi_{2\ell(k-1)}\left(B\widetilde{\mathrm{Diff}}_\partial(D^{2k+1})\right)\otimes\mathbb{Q}\right) \geq \dim_{\mathbb{Q}}\left(\mathcal{A}_{2\ell,3\ell}\right)$$

Since $\begin{pmatrix} d = 2k+1 & \text{odd} \\ i = 2\ell(k-1) & \text{even} \\ \ell \geq 2 \Rightarrow & i > 2k-2 \end{pmatrix}$ we have an isomorphism

$$\pi_{2\ell(k-1)}\left(B\widetilde{\mathrm{Diff}}_\partial(D^{2k+1})\right)\otimes\mathbb{Q} \cong \pi_{2\ell(k-1)}\left(B\mathrm{Diff}_\partial(D^{2k+1})\right)\otimes\mathbb{Q}$$

$$\cong \pi_{2\ell(k-1)-1}\left(\mathrm{Diff}_\partial(D^{2k+1})\right)\otimes\mathbb{Q}$$

$//$

Rmk: [Watanabe II] : lines of slope $\ell \geq 2$ $\longrightarrow$ all odd k. [when $\mathcal{A}_{2\ell,3\ell} \neq 0$]

[Watanabe I] : $\ell = 1$ $\longrightarrow$ only finitely many values of k (even & odd)

$d = 2k+1$

$k = \frac{1}{2}(d-1)$

Idea of construction of I_π.

Input

$$\begin{array}{c} E \\ \pi \downarrow \\ B \end{array} \quad \text{smooth } (D^d, \partial)\text{-bundle} + \text{framing of } T^vE$$

(d odd)

Output

$$\mathcal{G}_{n,m} \xrightarrow{I_\pi} \Omega_{dR}^{m(d-1)-nd}(B)$$

$$\Gamma \in \mathcal{G}_{n,m}, \quad \Gamma \longmapsto \text{differential form on } B$$

① Associated bundle

$$\begin{array}{c} E\overline{C}_n(\pi) \\ \downarrow \\ B \end{array}$$

whose fibre over $b \in B$ is

$$\overline{C}_n(E_b) \cong \overline{C}_n(D^d)$$

Fulton-MacPherson compactⁿ of the ordered config space of n pts in E_b

dim = nd.

②

$$\begin{tikzcd} E\overline{C}_2(\pi) \arrow[d] & \supseteq & \partial^{fib} E\overline{C}_2(\pi) \arrow[dll] \arrow[r, "p"] & S^{d-1} \\ B & & & \end{tikzcd}$$

defined using the topology of $\partial \overline{C}_2(D^d)$.

$$\overline{C}_2(D^d) = Bl_\Delta\left(Bl_{S^d \vee S^d}\left(Bl_{\{(\infty,\infty)\}}\left(S^d \times S^d\right)\right)\right)$$

$$\partial \overline{C}_2(D^d) \cong S^d \times S^{d-1} \xrightarrow{pr_2} S^{d-1}$$

Choose $\omega \in \Omega_{dR}^{d-1}\left(E\overline{C}_2(\pi)\right)$:

$$\omega|_{\partial^{fib}} = p^*(\mathrm{vol})$$

"propagator"

(!) The framing of T^vE is needed to make this definition on each fibre glue together well.

③ $e \in E(\Gamma) \longrightarrow$

$$\begin{tikzcd} E\bar{C}_n(\pi) \arrow[rr, "f_e"] \arrow[dr] & & E\bar{C}_2(\pi) \arrow[dl] \\ & B & \end{tikzcd}$$

forget all config. points except i^{th} + j^{th}

$i \overset{e}{\bullet\!\!-\!\!-\!\!\bullet} j$

$1 \leq i, j \leq n$

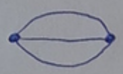
$$\bigwedge_{e \in E(\Gamma)} f_e^*(\omega) \quad \in \quad \Omega_{dR}^{m(d-1)}\left(E\bar{C}_n(\pi)\right)$$

④ Integrate along the fibre :

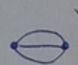
$$\underbrace{\int_{\text{fibre}} \bigwedge_{e \in E(\Gamma)} f_e^*(\omega)}_{=: I_\pi(\Gamma)} \quad \in \quad \Omega_{dR}^{m(d-1)-nd}(B).$$

<u>Simplest example</u> :

$n = 2$
$m = 3$

$\Gamma =$ 

3 edges , each $f_e = \mathrm{id}$

$I_\pi\Big($  $\Big) = \displaystyle\int_{\bar{C}_2(D^d)} \omega \wedge \omega \wedge \omega \quad \in \quad \Omega_{dR}^{d-3}(B)$

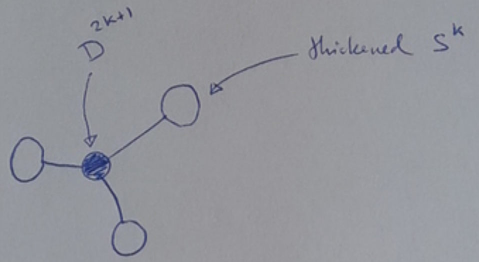
$\longrightarrow$ This is the special case that Watanabe used in Part I (cf. last talk) to detect classes in $\pi_{2k-3}\left(\mathrm{Diff}_\partial\left(D^{2k+1}\right)\right) \otimes \mathbb{Q}$.

Idea of construction of $\psi_{2\ell}$

Input $\Gamma \in \mathcal{G}_{2\ell, 3\ell}$

Output Framed (D^{2k+1}, ∂)-bundle over a closed, oriented $2\ell(k-1)$-manifold.

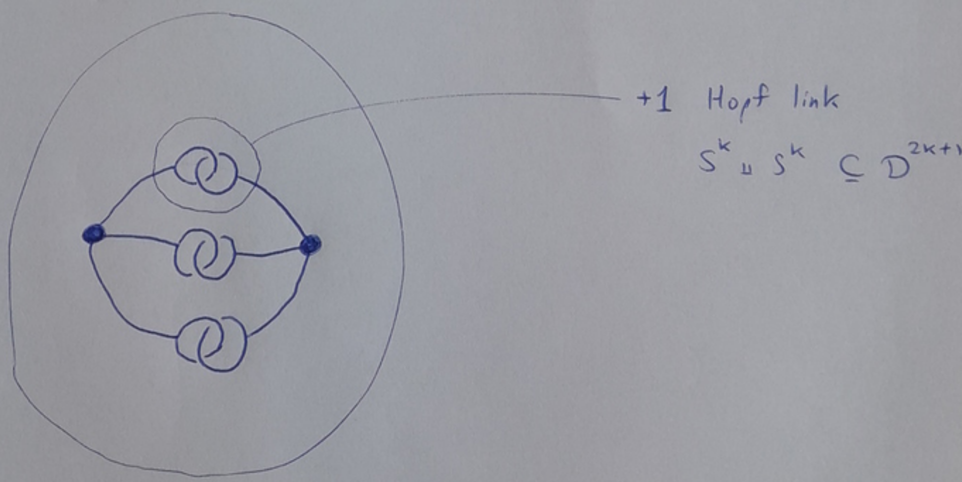
$B = (S^{k-1})^{2\ell}$.

① $V =$ 

$= (0\text{-handle}) \cup (\text{three } k\text{-handles})$ $\quad$ $(2k+1)$-dim. handlebody.

Take 2ℓ copies, named $V_1, \dots, V_{2\ell}$

Embed according to Γ into $\operatorname{int}(D^{2k+1})$:



$W := V_1 \sqcup \cdots \sqcup V_{2\ell} \subseteq \operatorname{int}(D^{2k+1})$.

(2) Construct a non-trivial framed $(V, \partial V)$-bundle over S^{k-1}.

$\boxed{\text{Black box}}$

via the "Borromean rings"

$$S^{2k-1} \sqcup S^k \sqcup S^k \subseteq D^{2k+1}.$$

(3) $\rightsquigarrow$ framed $(W, \partial W)$-bundle over $(S^{k-1})^{2\ell}$

- pull back along each projection $(S^{k-1})^{2\ell} \xrightarrow{pr_i} S^{k-1}$
- take fibrewise $\sqcup$.

its ∂^{fib} is the <u>trivial</u> ∂W-bundle over $(S^{k-1})^{2\ell}$

glue this to the <u>trivial</u> $(D^{2k+1} \setminus int(W))$-bundle over $(S^{k-1})^{2\ell}$

$\rightsquigarrow$ framed (D^{2k+1}, ∂)-bundle over $(S^{k-1})^{2\ell}$. //

Footnote:

On page 4 it is not quite true that $\mathcal{A}_{n,m}^{\text{even}}$ is as written. There are <u>two</u> differences from the $\mathcal{A}_{n,m}^{\text{odd}}$ case.

- $$\mathcal{G}_{n,m}^{\text{odd}} = \mathbb{Q}\left\{\begin{array}{l}\text{fin. conn. graphs } \Gamma \text{ of valence} \geq 3 \\ V(\Gamma) \longleftrightarrow \{1,\dots,n\} \\ E(\Gamma) \longleftrightarrow \{1,\dots,m\} \\ \underline{\text{orientation}}\end{array}\right\} \Big/ \bar{\Gamma} \sim -\Gamma$$

$$\text{orientation} \longleftrightarrow \{\text{orientation of each edge of } \Gamma\} \Big/ \text{reversing orientations of an even \# of edges}$$

$$\mathcal{G}_{n,m}^{\text{even}} = \text{The same, except now } \underline{\text{orientation}}$$

$$\text{orientation} \longleftrightarrow \{\text{ordering of } E(\Gamma)\} \Big/ \text{even permutations}$$

$$\overset{\text{lemma}}{\cong} \mathbb{Q}\left\{\begin{array}{l}\text{fin. conn. graphs } \Gamma \text{ of valence} \geq 3 \\ V(\Gamma) \longleftrightarrow \{1,\dots,n\} \\ E(\Gamma) \longleftrightarrow \{1,\dots,m\}\end{array}\right\}$$

- $$\mathcal{I}_{n,m}^{\text{odd/even}} = \mathcal{G}_{n,m}^{\text{odd/even}} \Big/ \left(\begin{array}{c}\text{relations corresponding} \\ \text{to } d\gamma = 0\end{array}\right)$$

- $$\mathcal{A}_{n,m}^{\text{odd}} = \mathcal{I}_{n,m}^{\text{odd}} \Big/ \begin{array}{l}\Gamma' \sim -\Gamma \\ \Gamma'' \sim \Gamma\end{array}$$

$\Gamma' = \Gamma$ with two vertex labels swapped

$\Gamma'' = \Gamma$ with two edge labels swapped

$$\mathcal{A}_{n,m}^{\text{even}} = \mathcal{I}_{n,m}^{\text{even}} \Big/ \begin{array}{l}\Gamma' \sim \Gamma \\ \Gamma'' \sim -\Gamma\end{array}$$