Moduli spaces of high-dimensional discs

after Watanabe, Weiss, Kupers-Randal-Williams

GeMAT seminar
IMAR
7 May 2021

I. Introduction

Spaces of smooth structures

Fix
- a topological d-manifold M
- a smooth structure on a neighbourhood of ∂M (if $\partial M \neq \emptyset$)

"near"

The/an aim of differential topology is to understand

$$Sm(M) = \{\text{smooth structures on } M\}$$

- smooth structures: maximal C^∞ atlas
- on M: compatible with the given one near ∂M

as a topological space.

First Q: How do we topologise this?

→ $Sm(M)$ has an action of $Homeo(M)$ by pulling back smooth structures.

(!) slightly non-standard:
"homeomorphism" will mean "homeo. that is a diffeo. near the boundary"

so by the orbit-stabiliser theorem we have an isomorphism of $Homeo(M)$-sets

$$Sm(M) \cong \coprod_{\{\text{orbits } [\mathcal{A}] \text{ of smooth structures } \mathcal{A} \text{ on } M\}} \mathrm{Homeo}(M) / \mathrm{Diff}(M, \mathcal{A})$$

(exercise) 1:1

$$\{\text{smooth manifolds } X \text{ that are homeomorphic to } M\} / \text{diffeomorphism} = : \mathrm{Smooth}(M)$$

(differ. near ∂)

$$Sm(M) \cong \coprod_{[X] \in \mathrm{Smooth}(M)} \mathrm{Homeo}(X) / \mathrm{Diff}(X)$$

compact-open topology

A more flexible model

$$\underline{Sm}(M) := \left\{ \begin{matrix} Y \subset \mathbb{R}^\infty \\ M \xrightarrow{f} Y \end{matrix} \;\middle|\; \begin{matrix} Y \text{ smooth submanifold} \\ f \text{ homeomorphism} \end{matrix} \right\} \xleftarrow{\cong} \coprod_{[X] \in \mathrm{Smooth}(M)} \frac{\mathrm{Homeo}(M, X) \times \mathrm{Emb}(X, \mathbb{R}^\infty)}{\mathrm{Diff}(X)}$$

$$\begin{bmatrix} Y = e(X) \\ f = e \circ g \end{bmatrix} \longmapsto \left[M \xrightarrow{g} X \xrightarrow{e} \mathbb{R}^\infty \right]$$

$\mathrm{Emb}(-) = \{\text{smooth embeddings}\}$ with Whitney topology

Fact: $\underline{Sm}(M) \twoheadrightarrow Sm(M)$

$(Y, f) \longmapsto$ pull back smooth structure of Y along f

is a fibre bundle, and its fibres ($\mathrm{Emb}(X, \mathbb{R}^\infty)$) are contractible,

so it is a homotopy equivalence.

The moduli space

$\underline{Sm}(M)$ has a free action of $\mathrm{Homeo}(M)$:

The moduli space of smooth manifolds homeomorphic to M is

$$\mathcal{M}(M) := \underline{Sm}(M)/\mathrm{Homeo}(M) = \left\{ Y \subset \mathbb{R}^\infty \;\middle|\; \begin{array}{l} Y \text{ smooth submanifold} \\ \text{s.t. } \exists \text{ homeo. } M \to Y \end{array} \right\}$$

$$\cong \coprod_{[X] \in \mathrm{Smooth}_h(M)} \mathrm{Emb}(X, \mathbb{R}^\infty)/\mathrm{Diff}(X)$$

$$\cong \coprod_{[X] \in \mathrm{Smooth}_h(M)} B\mathrm{Diff}(X)$$

$B\mathrm{Diff}(X)$: classifying space for smooth X-bundles

We will be interested in the mod-∂ version:

First note that

$$Sm(M) \cong \coprod_{[X]} \mathrm{Homeo}(X)/\mathrm{Diff}(X) \cong \coprod_{[X]} \mathrm{Homeo}_\partial(X)/\mathrm{Diff}_\partial(X)$$

restricting to id. near the boundary

and similarly

$$\underline{Sm}(M) \cong \coprod_{[X]} \frac{\mathrm{Homeo}(X) \times \mathrm{Emb}(X, \mathbb{R}^\infty)}{\mathrm{Diff}(X)} \cong \coprod_{[X]} \frac{\mathrm{Homeo}_\partial(X) \times \mathrm{Emb}(X, \mathbb{R}^\infty)}{\mathrm{Diff}_\partial(X)}$$

$$\begin{array}{ccc} \mathrm{Diff}_\partial(X) & \hookrightarrow & \mathrm{Homeo}_\partial(X) \\ \downarrow & & \downarrow \\ \mathrm{Diff}(X) & \hookrightarrow & \mathrm{Homeo}(X) \end{array}$$

is a pushout of topological groups

Then we define:

$$\mathcal{M}_\partial(M) := \underline{Sm}(M) \,/\, Homeo_\partial(M) = \left\{ \begin{array}{l} Y \subseteq \mathbb{R}^\infty \text{ smooth submanifold} \\ \text{choice of a "germ" near } \partial M \text{ of} \\ \quad \text{a homeomorphism } M \to Y \end{array} \right\}$$

$$\cong \coprod_{[X] \in Smooth(M)} Emb(X, \mathbb{R}^\infty) \,/\, Diff_\partial(X)$$

$$\cong \coprod_{[X] \in Smooth(M)} BDiff_\partial(X)$$

$BDiff_\partial(X)$: classifying space for smooth X-bundles whose fibrewise boundary is the trivial ∂X-bundle

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Summary

$$Homeo_\partial(M)$$
$$\downarrow$$
$$\underline{Sm}(M) \xrightarrow{\simeq} Sm(M) \cong \coprod_{[X]} Homeo_\partial(X) \,/\, Diff_\partial(X)$$
$$\downarrow$$
$$\mathcal{M}_\partial(M) \cong \coprod_{[X]} BDiff_\partial(X)$$

$\pi_0(\mathcal{M}_\partial(M)) = Smooth(M) = \{\text{set of inequivalent smooth structures on } M\}$

$H^*(\mathcal{M}_\partial(M)) =$ characteristic classes of smooth bundles with fibre M (with some smooth structure)

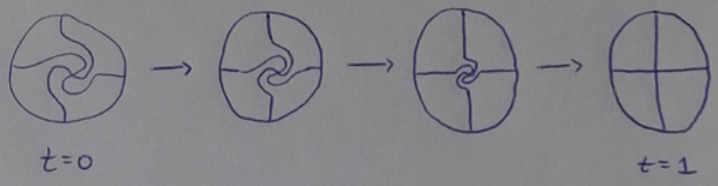
$\pi_*(\mathcal{M}_\partial(M)) =$ isom. classes of smooth M-bundles over spheres (with some smooth str.)

Plan for today:

- What is known for $M = D^d, S^d, \mathbb{R}^d$
- "Landscape" of calculations of $Diff_\partial(D^d)$, $d \geq 5$

What is known for D^d?

First note that $\mathrm{Homeo}_\partial(D^d) \simeq *$ by Alexander's trick



Hence $\mathcal{S}m(D^d) \simeq \mathcal{M}_\partial(D^d)$

Fact: For $d \neq 4$, D^d has a unique smooth structure up to diffeomorphism.

$d=2$: Radó
$d=3$: Moise
$d \geq 6$: $\Leftarrow$ h-cobordism theorem (Smale)
$d=5$: Milnor

So $\mathcal{M}_\partial(D^d)$ is connected,
$\mathcal{M}_\partial(D^d) \cong B\mathrm{Diff}_\partial(D^d)$

For $d \leq 3$, $\mathrm{Diff}_\partial(D^d) \simeq *$ ($d=2$: Smale '59; $d=3$: Hatcher '83)

But $\mathrm{Diff}_\partial(D^4) \not\simeq *$ (Watanabe '18)

For $d \geq 5$, $\mathrm{Diff}_\partial(D^d)$ → the main topic of these talks.....

But first we briefly look at $M = S^d$
$M = \mathbb{R}^d$

What is known for S^d?

$$Sm(S^d) = \coprod_{[\Sigma] \in \Theta_d} \mathrm{Homeo}(\Sigma)/\mathrm{Diff}(\Sigma) \qquad \mathcal{M}(S^d) = \coprod_{[\Sigma] \in \Theta_d} B\mathrm{Diff}(\Sigma)$$

So we need to understand the set Θ_d

& the topological groups $\mathrm{Diff}(\Sigma)$

$\mathrm{Homeo}(\Sigma) \cong \mathrm{Homeo}(S^d)$

- Θ_d is an <u>abelian group</u> under connected sum, if $d \neq 4$

  (Θ_4 is an abelian monoid)

  $\Theta_1 = \Theta_2 = \Theta_3 = *$ ——— since all manifolds of dim ≤ 3 have a unique smooth structure up to diffeo.

  [Radó, $d=2$; Moise, $d=3$]

  Θ_d is <u>finite</u> if $d \neq 4$ [Kervaire-Milnor]

  and fits into a short exact sequence

  $$1 \to \boxed{\text{cyclic}} \to \Theta_d \to \boxed{\begin{array}{c}\text{subquotient} \\ \text{of } \pi_d(\mathbb{S})\end{array}} \to 1$$

- $\mathrm{Diff}(S^d) \simeq \mathrm{Diff}_\partial(D^d) \times \underbrace{O(d+1)}_{\text{"well-understood"}}$ [Antonelli-Burghelea-Kahn]

  So this boils down to understanding $\mathrm{Diff}_\partial(D^d)$.....

- $\mathrm{Diff}(\Sigma) \simeq ?$

- $\mathrm{Homeo}(S^d)$:

Fact: For all smooth manifolds M of dim ≤ 3,

$$\mathrm{Diff}(M) \hookrightarrow \mathrm{Homeo}(M)$$

is a homotopy equivalence.

$d=3$: [Cerf '59], assuming the Smale conjecture [Hatcher '83]

So for $d \leq 3$ we have:

$$O(d+1) \xrightarrow{\simeq} \mathrm{Diff}(S^d) \xrightarrow{\simeq} \mathrm{Homeo}(S^d)$$

[Smale, Hatcher]

But this certainly fails in higher dimensions, e.g.

$|\pi_0 \mathrm{Diff}(S^6)| = 56$

$|\pi_0 \mathrm{Homeo}(S^6)| = 2$

(*) **Fact:** $\boxed{\pi_0 \mathrm{Homeo}(S^d) \cong \mathbb{Z}/2}$ ⟷

In contrast:

$\pi_0 \mathrm{Diff}^+(S^d) \cong \Theta_{d+1}$

$[\varphi] \longmapsto D^{d+1} \cup_{\varphi} D^{d+1}$

[Cerf / Smale]

Let's prove this using the stable homeomorphism conjecture

$$\mathrm{Homeo}^s(M) = \left\{ \varphi \in \mathrm{Homeo}(M) \;\middle|\; \begin{array}{l} \varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k \\ \varphi_i|_{U_i} = \mathrm{id} \quad \text{for some} \\ \text{non-}\emptyset\text{, open } U_i \subsetneq M \end{array} \right\}$$

Thm ($d=3$: Fisher '60, $d \geq 5$: Kirby '69, $d=4$: Quinn '82)

$$\mathrm{Homeo}^s(\mathbb{R}^d) = \mathrm{Homeo}^+(\mathbb{R}^d)$$

Hence we have:

$\pi_0 \mathcal{M}(S^d) \cong \Theta_d$

$\pi_1\left(\mathcal{M}(S^d), S^d\right) \geqslant \Theta_{d+1}$ (index-2)

Proof of (*)

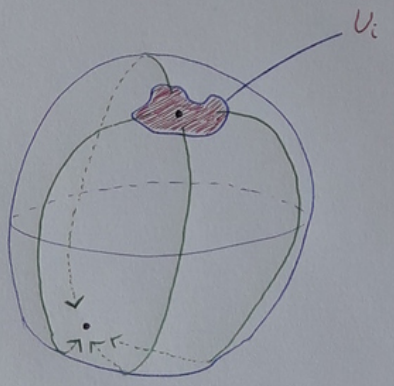
Let $\varphi \in \mathrm{Homeo}^+(S^d)$

- choose a stable homeo. φ_0 such that $\varphi_0^{-1} \circ \varphi$ fixes the north pole. so $\varphi_0^{-1} \circ \varphi$ restricts to $\in \mathrm{Homeo}^+(\mathbb{R}^d)$

- SHC $\Rightarrow$ $\underbrace{\varphi_0^{-1}}_{\text{stable}} \circ \varphi = \underbrace{\varphi_1 \circ \cdots \circ \varphi_k}_{\text{stable}} \in \mathrm{Homeo}^+(\mathbb{R}^d)$

- one-point-compactify $\rightsquigarrow$ $\varphi = \varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_k \in \mathrm{Homeo}^+(S^d)$

- $\varphi_i|_{U_i} = \mathrm{id}$



Alexander's trick $\Rightarrow \varphi_i \simeq \mathrm{id}$

hence $\varphi \simeq \mathrm{id}$. $\square$

What is known for $\mathbb{R}^d$?

- For $d \neq 4$, $\mathbb{R}^d$ has a unique smooth structure up to diffeomorphism. [Stallings, Munkres, Hirsch]

- By the opposite of the Alexander trick ("stretching"), $O(d) \xrightarrow{\simeq} \mathrm{Diff}(\mathbb{R}^d)$.

So $\mathrm{Sm}(\mathbb{R}^d) \cong \mathrm{Homeo}(\mathbb{R}^d)/O(d)$ $\qquad$ $\mathcal{M}(\mathbb{R}^d) \cong BO(d)$ ("well-understood")

This also reduces to studying $\mathrm{Diff}_\partial(D^d)$, via...

Smoothing theory

- In general ($d \neq 4$):

$$\mathrm{Sm}(M) \cong \coprod_{[X]} \mathrm{Homeo}_\partial(X)/\mathrm{Diff}_\partial(X)$$

fibre over $p \in X$ is $\mathrm{Sm}(T_p X)$

$$\mathrm{Homeo}_\partial(X)/\mathrm{Diff}_\partial(X) \xrightarrow{\text{remember the smooth str. infinitesimally at each point}} \Gamma_\partial\left(\begin{array}{c}\mathrm{Sm}(TX)\\ \downarrow \\ X\end{array}\right)$$

Fact: (Burghelea-Lashof '74, Hirsch-Mazur '74, Kirby-Siebenmann '77)

This is a homotopy equivalence onto a union of path-components.

("h-principle")

- Apply to $M = D^d$:

$$\mathrm{Homeo}_\partial(D^d)/\mathrm{Diff}_\partial(D^d) \xrightarrow{\sim} \mathrm{Map}_\partial(D^d, \mathrm{Sm}(\mathbb{R}^d))$$

$$= \mathrm{Map}_*\left(S^d, \mathrm{Homeo}(\mathbb{R}^d)/O(d)\right)$$

$$= \Omega^d\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d)\right)$$

(Alexander trick) — is $B\mathrm{Diff}_\partial(D^d)$

(notation)

Cor:

$$\boxed{\mathrm{Diff}_\partial(D^d) \simeq \Omega^{d+1}\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d)\right)}$$

"Morlet's lemma".

Summary :

$d \leq 3$

	$\mathbb{R}^d$	D^d	S^d
$\mathrm{Smooth}(M)$	$*$	$*$	$*$
$\mathrm{Diff}_\partial(M)$	$O(d)$	$*$	$O(d+1)$
$\mathrm{Homeo}_\partial(M)$	same π_i as $O(d)$ for $i \geq d+1$	$*$	$O(d+1)$

$d \geq 5$

	$\mathbb{R}^d$	D^d	S^d
$\mathrm{Smooth}(M)$	$*$	$*$	Θ_d
$\mathrm{Diff}_\partial(M)$	$O(d)$	$\square$	$\square$
$\mathrm{Homeo}_\partial(M)$	$\square$	$*$	?

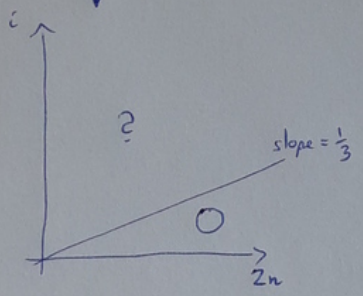
Morlet

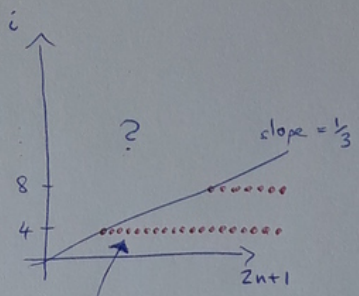
$\mathrm{Diff}_\partial(D^d) \times O(d+1) \simeq \mathrm{Diff}(S^d)$

"Landscape" of calculations of $\pi_i(\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q}$

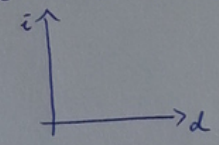
Low degrees

"Classical approach" (surgery theory + pseudoisotopy theory + algebraic K-theory) [Farrell-Hsiang '78, Igusa '84]





Graphs of $\pi_i \mathrm{BDiff}_\partial(D^d) \otimes \mathbb{Q}$



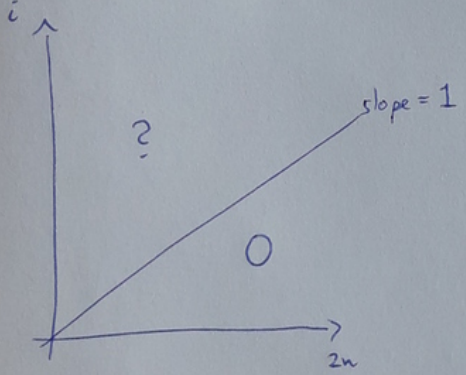
$d = 2n$
$d = 2n+1$

$\bullet = \mathbb{Q}$

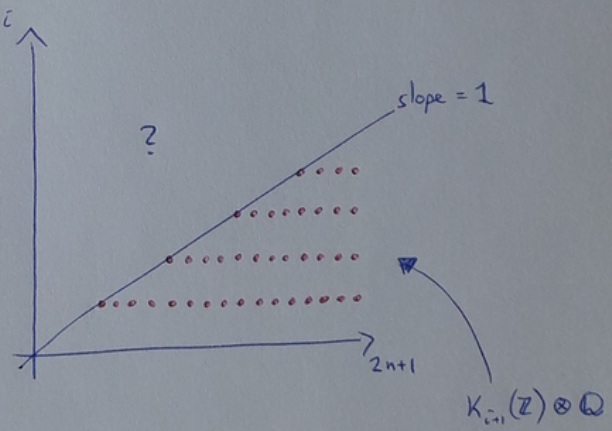
$$K_{i+1}(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i \equiv 0 \pmod 4 \\ 0 & \text{o/w.} \end{cases}$$

[Borel '74]

More recently (high-dimensional topology)



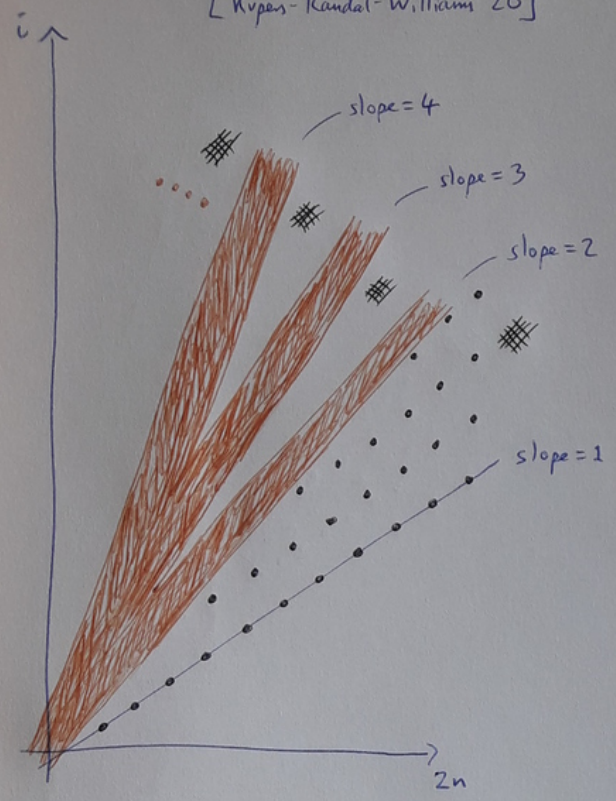
[Randal-Williams '17]



[Krannich '20]

High degrees

[Kupers-Randal-Williams '20]



$$\text{hatched} = \begin{cases} \mathbb{Q} & i \equiv 2n-1 \pmod 4 \\ 0 & \text{o/w} \end{cases}$$

generated by "exotic Pontrjagin classes"

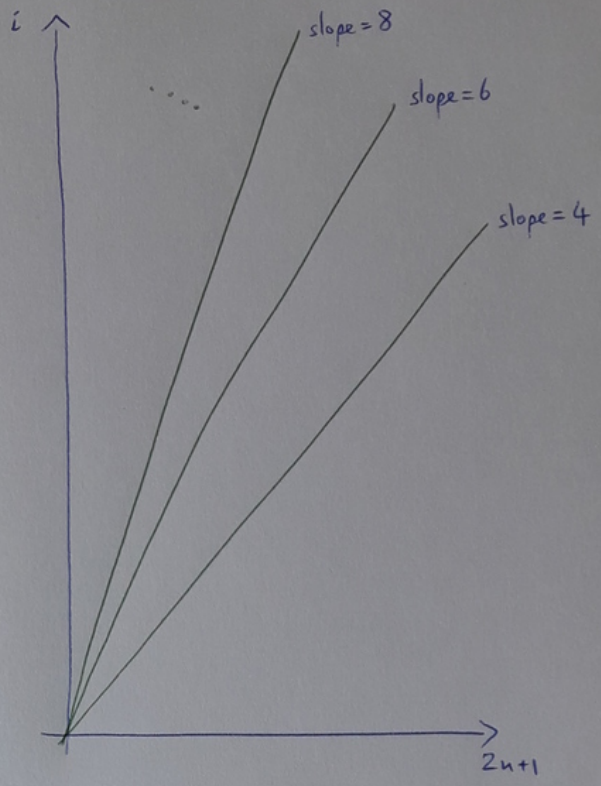
discovered by [Weiss '15]

"bands" = ?

[bands of even slope : exotic P. classes $\neq 0$ but there could be more

bands of odd slope : ???]

[Watanabe '09]



Each entry on a green line of slope $2k$

$\downarrow$

$\mathcal{A}_k$ — graph homology in degree k

Plan for next 4 talks:

~2 talks on [Watanabe '09] → graph homology classes

~2 talks on [Weiss '15] + [Kupers-Randal-Williams '20] → exotic Pontrjagin classes.

Moduli spaces of high-dimensional discs

after Watanabe, Weiss, Krpasa - Randal-Williams

GeMAT seminar
IMAR
13 May 2021

II. Odd-dimensional discs, just above the stable range

0. Recollections

Aim: Study the homotopy type of $\mathrm{Diff}_\partial(D^d) = \{ D^d \xrightarrow{\varphi} D^d \mid \varphi$ diffeomorphism fixing a neighbourhood of $\partial D^d \}$

$d \leq 3$: $\mathrm{Diff}_\partial(D^d) \simeq *$ [Smale '59, Hatcher '83]

$d \geq 7$: $\mathrm{Diff}_\partial(D^d) \not\simeq$ any finite CW-complex [ABK]'72 = [Antonelli - Burghelea - Kahn]

In particular: $\boxed{\pi_*(\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q}}$ $\boxed{d = 2k+1}$

"Stable range": $\pi_i(\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & i \equiv 3 \pmod 4 \\ 0 & \text{o/w} \end{cases}$

in the range $i \leq \frac{d-43}{6}$ [Farrell - Hsiang '78]

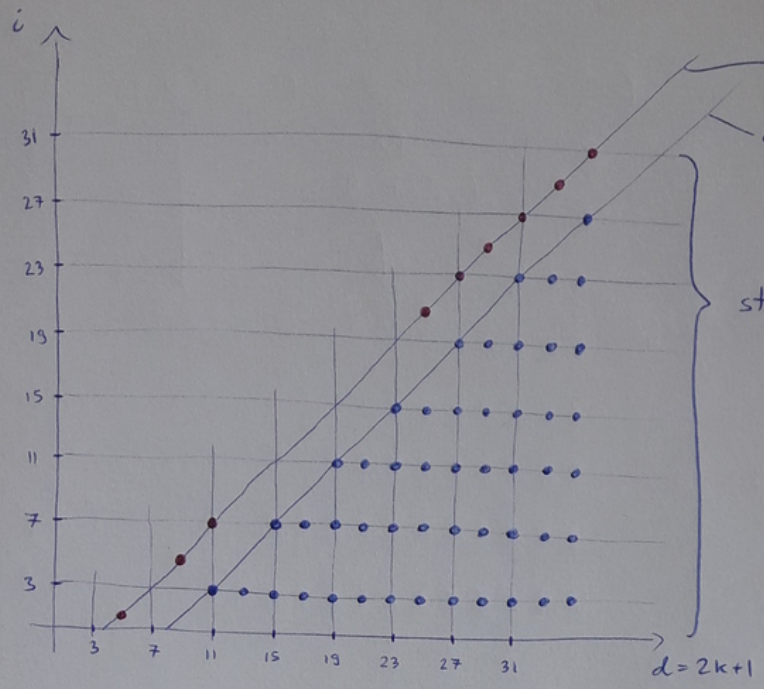
$i \leq \min\left(\frac{d-2}{3}, \frac{d-7}{2}\right)$ [Igusa '84]

$i \leq d-8$ [Krannich '20]

Just above the stable range:

[Watanabe '09] detects new elements of $\pi_i(\mathrm{Diff}_\partial(D^{2k+1})) \otimes \mathbb{Q}$

with $\underline{i = d-4}$.



• = • = $\mathbb{Q}$

stable — detected by Watanabe

(In the stable range, the elements • generate all of $\pi_*(-) \otimes \mathbb{Q}$.)

by building on work of [Antonelli-Burghelea-Kahn '72]

(*)
- [ABK] — construct elements of $\pi_*(\mathrm{Diff}_\partial(D^d))$ by "plumbing"
  - detect <u>non-triviality</u> via invariants with values in finite cyclic groups.
- [W] — detects <u>∞-order</u> by lifting these invariants (up to scalar) to $\mathbb{Z}$.

<u>Today</u>: Understand (*).

<u>Next time</u>: Look at [Watanabe '09 Part II]

↳ elements on lines of higher slope.

Outline

1. The Growoll filtration
2. Plumbing $\Rightarrow$ exotic spheres
3. Refinement of plumbing
4. The subgroup $bP_{n+1} \subseteq \Theta_n$
5. Detection results (torsion)
6. Infinite-type results
7. Watanabe's construction $\hat{\Sigma}_2$
8. Detection results (non-torsion)

1. The Gromoll filtration

Def $\Theta_n = \{$smooth, closed, connected, oriented manifolds homeo to $S^n\}$ / diffeomorphism.

Lemma ($n \neq 4$) Θ_n is an abelian group wrt $\#$

$\exists$ homomorphism $\pi_0(\mathrm{Diff}^+(S^{n-1})) \longrightarrow \Theta_n$

$\varphi \longmapsto D^n \cup_\varphi D^n$

Thm (Cerf, Smale) This is an isomorphism if $n \neq 4$.

Def (Novikov) $\Sigma : \pi_i(\mathrm{Diff}_\partial(D^n)) \longrightarrow \pi_{i-j}(\mathrm{Diff}_\partial(D^{n+j}))$

Idea of construction:

$\pi_i(\mathrm{Diff}_\partial(D^n))$ $\qquad$ $\pi_{i-j}(\mathrm{Diff}_\partial(D^{n+j}))$

$\varphi : I^i \longrightarrow \mathrm{Diff}(I^n)$ $\left[\begin{array}{l} \varphi(x) = id \ , \ x \in \partial I^i \\ \varphi(x)|_{\partial I^n} = id \end{array}\right]$

$\tilde{\varphi}$ diffeo of $I^i \times I^n$

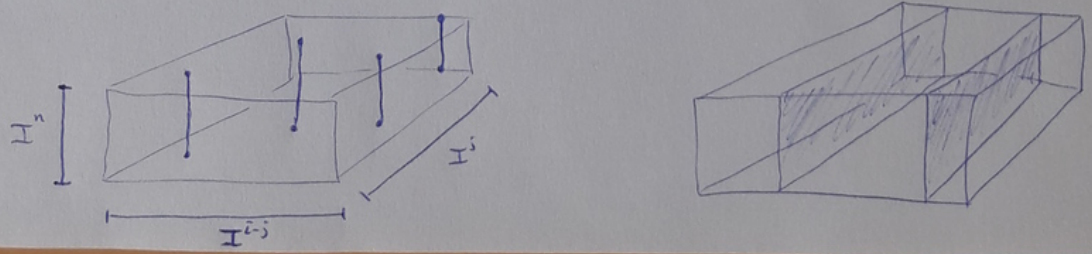
- id near the ∂
- preserves the partition $\{\{t\} \times I^n\}_{t \in I^i}$

$\longrightarrow$

$\tilde{\varphi}$ diffeo of $I^{i-j} \times I^{n+j}$

- id near the ∂
- preserves the partition $\{\{s\} \times I^{n+j}\}_{s \in I^{i-j}}$

Picture for $i=2$, $n=1$, $j=1$



Def (Gromoll)

$$\theta_n(i) := \text{image}\left(\pi_i \mathrm{Diff}_\partial(D^{n-1-i}) \xrightarrow{\Psi} \pi_0 \mathrm{Diff}_\partial(D^{n-1}) \;\|\; \pi_0 \mathrm{Diff}^+(S^{n-1}) \xrightarrow{\cong} \theta_n \right)$$

$$\theta_n = \theta_n(0) \supseteq \theta_n(1) \supseteq \cdots\cdots\cdots \supseteq \theta_n(n-2) = 0$$

Gromoll filtration.

In fact $\theta_n(n-4) = 0$ [Hatcher]

$\theta_n(0) = \theta_n(1)$ [Cerf]

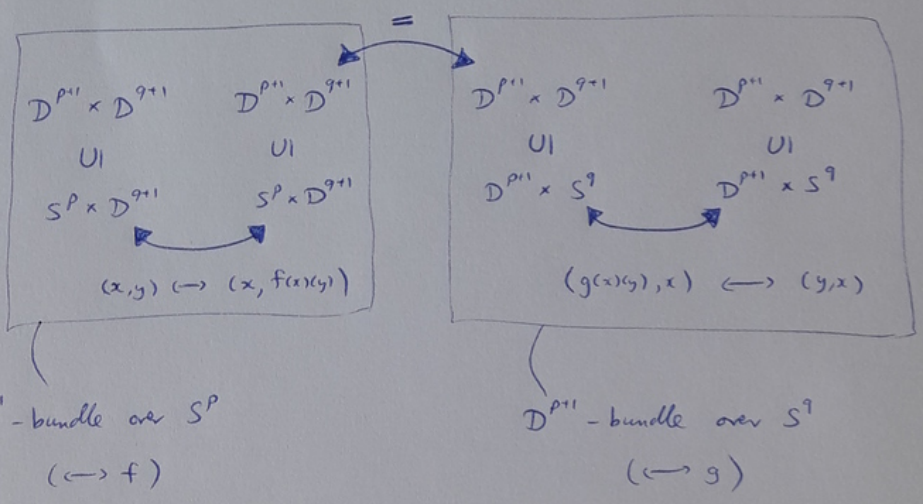
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2. Plumbing ⇒ exotic spheres

Def (Milnor) $f: S^p \to SO(q+1)$

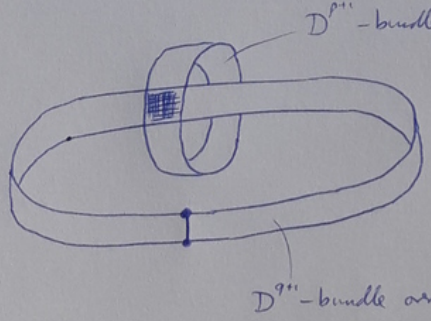
$g: S^q \to SO(p+1)$

$M(f,g) :=$



Picture:

(schematic)



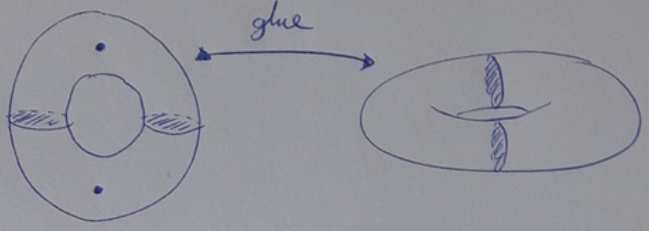
$= M(f,g)$

Its boundary looks like:

$$\begin{array}{ccc} D^{p+1} \times S^q & & D^{q+1} \times S^p \\ \cup & & \cup \\ S^p \times S^q & \longleftrightarrow & S^q \times S^p \end{array}$$

$$(x,y) \longmapsto \left(f(x)(y) \,,\, g\big(f(x)(y)\big)^{-1}(x) \right)$$

Another picture: $(p=q=1)$



$= \partial M(f,g)$.

Lemma: $M(f,g)$ depends only on the homotopy classes of $f : S^p \to SO(q+1)$, $g : S^q \to SO(p+1)$

Lemma (Milnor) if $\text{image}(f) \subseteq SO(q)$ or $\text{image}(g) \subseteq SO(p)$ then $\partial M(f,g)$ is homeo to S^{p+q+1}.

Idea: Construct an explicit Morse function on $\partial M(f,g)$ with exactly 2 critical points.

- $D^{p+1} \times S^q \longrightarrow \mathbb{R}$
  $(x,y) \longmapsto \text{height}(y) \cdot (1+|x|^2)^{-1/2}$ } 2 critical pts.
- extend to $S^p \times D^{q+1}$ with no extra critical points. //

Cor. $\exists$ linear map

$$\pi_p SO(q) \otimes \pi_q SO(p) \xrightarrow{\text{PL.}} \theta_{p+q+1}$$

"plumbing"

3. Refinement of plumbing => element of Gromoll filtration

Proposition [ABK] PL. may be lifted to:

$$\begin{array}{ccc} \pi_p SO(q-a) \otimes \pi_q SO(p-b) & \xrightarrow{\widetilde{PL.}} & \pi_{a+b+1} Diff_\partial(D^{p+q-a-b-1}) \\ \downarrow & & \downarrow \Psi \\ \pi_p SO(q) \otimes \pi_q SO(p) & \xrightarrow{PL.} & \theta_{p+q+1} \supseteq \theta_{p+q+1}(a+b+1) \\ & & \| \\ & & \pi_0 Diff_\partial(D^{p+q}) \end{array}$$

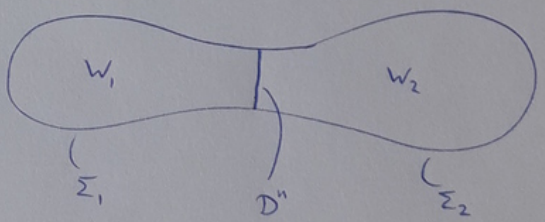
4. The subgroup $bP_{n+1} \subset \theta_n$

Def $bP_{n+1} = \{ \Sigma \in \theta_n \mid \Sigma = \partial W^{n+1} \text{ with } TW \text{ trivial} \}$

Lemma This is a subgroup

Idea: $\Sigma_1 = \partial W_1$, $\Sigma_2 = \partial W_2$ $\rightsquigarrow$ $\Sigma_1 \# \Sigma_2 = \partial(W_1 \natural W_2)$ (∂-connected sum)

give trivialisations of TW_1 and TW_2 along $W_1 \cap W_2 \cong D^n$



//

$n = 4k-1$

let $\sigma_k := \gcd\left\{ |\mathrm{sign}(W)| : \begin{array}{l} W \text{ } 4k\text{-manifold} \\ TW \text{ trivial} \\ \mathrm{sign}(W) \neq 0 \end{array} \quad \partial W = S^{4k-1} \right\}$

$\tau_k := \gcd\left\{ \text{---}''\text{---} \quad \partial W \text{ homeo to } S^{4k-1} \right\}$

Lemma $\Sigma \in bP_{4k}$ $\quad \Sigma = \partial W_1 = \partial W_2$ $\quad (TW_i \text{ trivial})$

Then $\mathrm{sign}(W_i) \in \tau_k \mathbb{Z}$

$\mathrm{sign}(W_1) - \mathrm{sign}(W_2) \in \sigma_k \mathbb{Z}$

Def (Milnor)

$$\begin{array}{ccc} bP_{4k} & \xrightarrow{\lambda} & \mathbb{Z}/b_k \\ \cup & & \cup \\ \Sigma = \partial W & \longmapsto & \frac{1}{\tau_k}\mathrm{sign}(W) \end{array}$$

$b_k = \dfrac{\sigma_k}{\tau_k}$

Thm (Kervaire-Milnor) This is an isomorphism.

$\tau_k = 8$

$\sigma_k = 2^{2k+1}\left(2^{2k-1}-1\right) \cdot a_k \cdot \mathrm{Numerator}\left(\dfrac{B_k}{4k}\right)$

$a_k = \begin{cases} 1 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$

B_k = Bernoulli number

5. Detection results (torsion)

$$\begin{array}{ccccc} \pi_p SO(q-a) \otimes \pi_q SO(p-b) & \xrightarrow{\widetilde{PL}} & \pi_{2k-3} \mathrm{Diff}_\partial(D^{2k+1}) & & \\ \downarrow i_* & & \downarrow \Psi & & \\ \pi_p SO(q) \otimes \pi_q SO(p) & \xrightarrow{PL} & \Theta_{4k-1} & \supseteq & bP_{4k} \\ & & & & \cong \downarrow \lambda' \\ & & & & \mathbb{Z}/b_k \end{array}$$

$k = r+s$

$p = 4r-1$
$q = 4s-1$
$a = 4s-2r-2$
$b = 4r-2s-2$

[ABK]: Find elements of $\pi_p SO(q-a) \otimes \pi_q SO(p-b)$ that map to bP_{4k} & whose λ'-invariant is $8b_r b_s \in \mathbb{Z}/b_k$

whenever $r, s \geq 6$ and $\frac{1}{2}r < s < 2r$

Construction
- quite involved
- key step "pulling back" elements along i_* depends on studying the LES of $SO(n) \hookrightarrow SO(2n) \twoheadrightarrow V_{2n,n}$ (Stiefel manifold) in the "metastable range". [Barratt - Mahowald '64]

Number theory $\Rightarrow$ $\forall k \geq 12$ $\exists r, s \geq 6$: $k = r+s$, $\frac{1}{2}r < s < 2r$, $8b_r b_s \not\equiv 0 \pmod{b_k}$

Cor: $\forall k \geq 12$, $\boxed{\pi_{2k-3} \mathrm{Diff}_\partial(D^{2k+1}) \neq 0}$

[Burghelea - Lashof '74] find torsion in $\pi_* \mathrm{Diff}_\partial(D^d)$ of specific prime order.

E.g. they prove that $\pi_{2p-2} \mathrm{Diff}_\partial(D^{2p^2-2p-4})$ contains p-torsion (p odd prime).

6. Infinite-type results

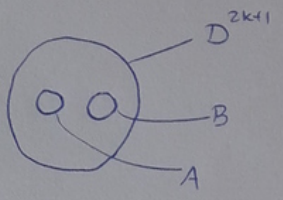
Thm [ABK] $\forall k \geq 12$

$\mathrm{Diff}_\partial(D^{2k+1})$ is not $\simeq$ any finite CW-complex

In fact they prove this for $\mathrm{Diff}_\partial(D^d)$, $d \geq 7$

proof: $G = \mathrm{Diff}_\partial(D^{2k+1})_0$ — identity component

claim: G is homotopy-abelian



$$\begin{array}{ccccc} \mathrm{Diff}_\partial(A)_0 \times \mathrm{Diff}_\partial(B)_0 & \xrightarrow{\simeq} & G \times G & \xrightarrow{\circ} & G \\ \| & & \| & & \\ \mathrm{Diff}_\partial(A)_0 \times \mathrm{Diff}_\partial(B)_0 & \xrightarrow{\simeq} & G \times G & \xrightarrow[\text{swap}]{} G \times G & \xrightarrow{\circ} G \end{array}$$

//

Thm [Hubbuck '69] if G is a (connected / htpy-abelian) H-space & $G \simeq$ finite CW-complex

then $G \simeq (S^1)^n$.

But $(S^1)^n$ has vanishing π_*, $* \geq 2$.

whereas $\pi_{2k-3}(G) \neq 0$.

□

7. Watanabe's construction $\hat{z}_2$

M homology d-sphere $\qquad d = 2k+1$

$M^\circ = M \setminus \mathrm{int}(D^d)$

$$\left(\begin{array}{ll} E & \text{smooth } M^\circ\text{-bundle} \\ \downarrow \pi & \text{with trivial } \partial^{fib} \\ B & + \text{"vertical framing of } TE\text{"} \end{array}\right) \longmapsto \xi_2 \longmapsto H^{2k-2}(B)$$

Idea:

$$\begin{array}{ccccc} \bar{C}_2(M^\circ) & \supseteq & \partial\bar{C}_2(M^\circ) & \longrightarrow & S^{2k} \\ \downarrow & & \downarrow & \nearrow p & \\ E\bar{C}_2(\pi) & \supseteq & \partial^{fib}E\bar{C}_2(\pi) & & \end{array}$$

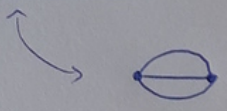
"propagator"

$E\bar{C}_2(\pi) \to B$: associated bundle whose fibre is the Fulton–MacPherson compactification of $C_2(M^\circ)$

(1) pull back gen. of $H^{2k}(S^{2k})$ along p

(2) extend (uniquely) to $\omega \in H^{2k}\left(E\bar{C}_2(\pi)\right)$

(3) take <u>triple cup product</u> $\omega^3 \in H^{6k}\left(E\bar{C}_2(\pi)\right)$



(4) "integrate along the fibre" $\longrightarrow H^{2k-2}(B)$.

<u>Caveat</u>: Do steps (1)–(3) carefully at the level of <u>cochains</u>

$\rightsquigarrow$ element of $H^{6k}\left(E\bar{C}_2(\pi), \partial^{fib}E\bar{C}_2(\pi)\right) \ni \tilde{\omega}^3$

$$\downarrow$$

$$H^{6k}\left(E\bar{C}_2(\pi)\right) \ni \omega^3$$

Then step (4) works.

<u>Thm</u> (Kontsevich) This is well-defined.

Def (Watanabe)

$$\pi_{2k-2} BDiff_\partial(D^{2k+1}) \xrightarrow{\hat{z}_2} \mathbb{Q}$$

$$\| \text{IS}$$

$$\left\{ \begin{array}{ll} E & \text{smooth } D^{2k+1}\text{-bundle} \\ \pi \downarrow & \\ S^{2k-2} & \text{with trivial } \partial^{fib} \end{array} \right\}$$

Idea: (1) obstructions to $\exists$ of vertical framing of TE

live in $\underbrace{\pi_{4k-2} SO(2k+1)}_{\text{finite}}$ [Milnor-Moore]

$q_k :=$ smallest $d \in \mathbb{Z}_{>0}$ killing all elements of this group.

Then $q_k \cdot E$ always admits a vertical framing.

(2) Choose one. $\longrightarrow$ $\zeta_2(q_k \cdot E) \in H^{2k-2}(S^{2k-2})$

$$\hat{z}_2(E) := \langle \zeta_2(q_k \cdot E), [S^{2k-2}] \rangle + (\text{framing correction}) \in \mathbb{Q}$$

$$\cap \qquad\qquad \cap$$

$$\mathbb{Z} \qquad\qquad \mathbb{Q}$$

Thm (Watanabe) This is a well-defined homomorphism.

↑ main difficult theorem!

8. Detection results (non-torsion)

Thm (Watanabe) $k \geq 3$

$$\begin{array}{ccccccc}
\pi_{2k-3}\,\mathrm{Diff}_\partial(D^{2k+1}) \cong \pi_{2k-2}\,B\mathrm{Diff}_\partial(D^{2k+1}) & & \xrightarrow{\hat{z}_2} & & \mathbb{Q} & & \\
\downarrow \Psi & & \cup & & \cup & & \\
\Theta_{4k-1} \supseteq bP_{4k} & \xrightarrow{\lambda'} & \mathbb{Z}/b_k & \xrightarrow{\cdot c_k q_k} & \mathbb{Z}/b_k & & \\
 & & \cup & & & & \\
 & & 8b_r b_s & & & & \\
\Psi^{-1}(bP_{4k}) & \longrightarrow & \frac{1}{b_k}\mathbb{Z} & \xrightarrow{\cdot b_k} & \mathbb{Z} & &
\end{array}$$

b_k, c_k ⟶ Bernoulli #s

q_k ⟵ torsion in $\pi_{4k-2}\,SO(2k+1)$

Recall that $[A,k] \longrightarrow x \in \Psi^{-1}(bP_{4k})$

with $\lambda'(\Psi(x)) = 8b_r b_s \in \mathbb{Z}/b_k$

$k = r+s$
$r, s \geq 6$
$\frac{1}{2}r < s < 2r$

Cor If $8b_r b_s c_k q_k \not\equiv 0 \pmod{b_k}$ (*)

then $\pi_{2k-3}\,\mathrm{Diff}_\partial(D^{2k+1})$ contains an element of ∞ order.

i.e. $\pi_{2k-3}\,\mathrm{Diff}_\partial(D^{2k+1}) \otimes \mathbb{Q} \neq 0$.

Watanabe checks (*) for $k = 4, 5, 12, \ldots, 41, 45, 54, \ldots, 2212$.

also any $k = 2m$ where $4m-1$ is prime & $8m-1$ is a regular prime.

Note:

- In the stable range, non-zero $\pi_*(-) \otimes \mathbb{Q}$ occur only for $* \equiv 3 \pmod 4$.
- Whenever (*) holds for even k, we get non-zero $\pi_*(-) \otimes \mathbb{Q}$ for $* \equiv 1 \pmod 4$.

Moduli spaces of high-dimensional discs

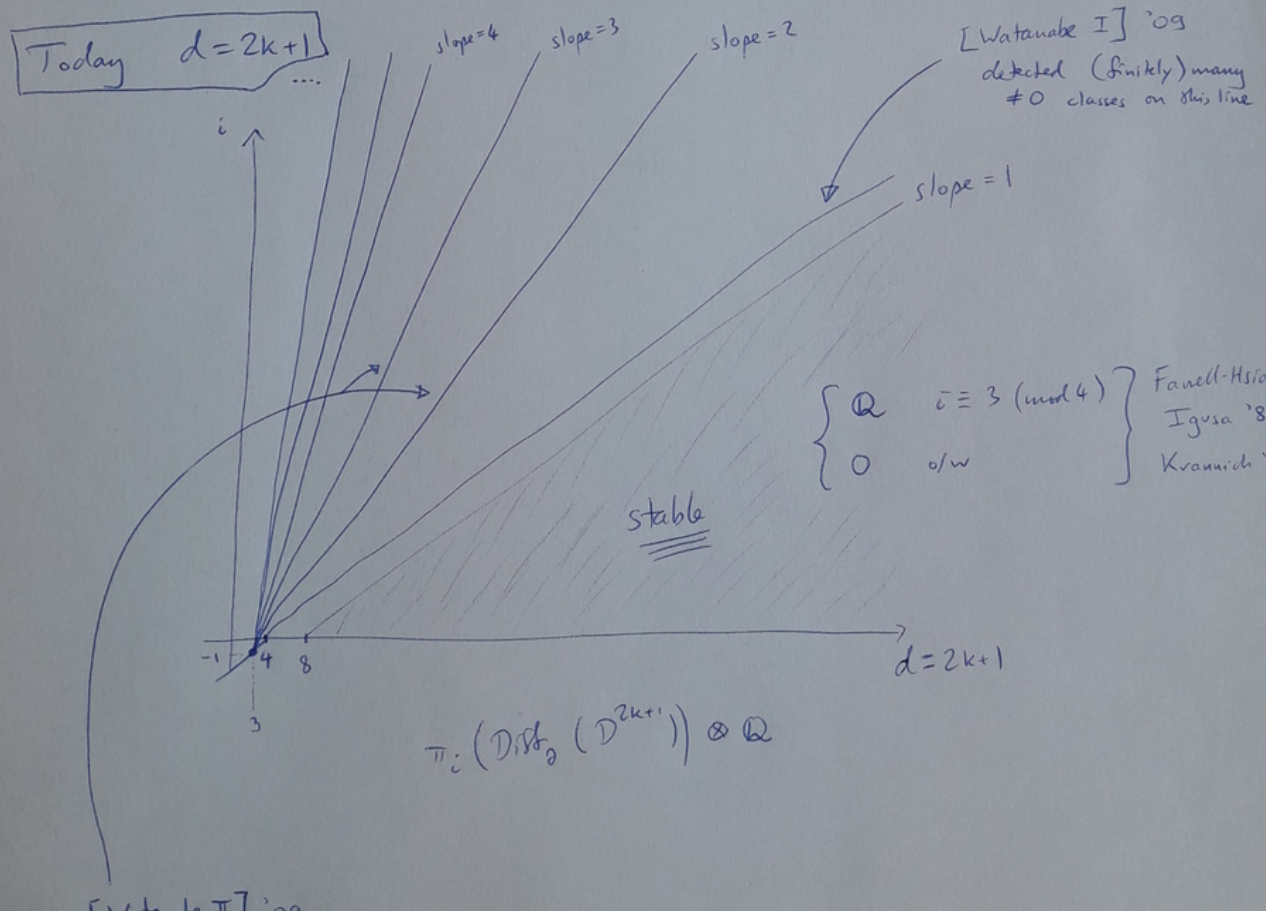
after Watanabe, Weiss, Kupers-Randal-Williams

GeMAT seminar
IMAR
28 May 2021

III. Odd-dimensional discs, far above the stable range

Aim: Study the rational homotopy groups of

$$\mathrm{Diff}_\partial(D^d) = \left\{ D^d \xrightarrow{\varphi} D^d \text{ diffeomorphism fixing pointwise a neighbourhood of } \partial D^d \right\}$$



[Watanabe II] :

Every rational homotopy group lying on one of the lines of slope $n \gg 2$ above has dimension $\geq \dim(A_{2n,3n})$

when k is odd, i.e. $d \equiv 3 \pmod 4$

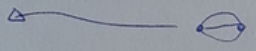
$\{$trivalent graphs$\}$ / relations

↑ purely combinatorial (in principle)

Lower bounds on $\dim(A_{2n,3n})$ come from

- weight systems — Lie algebras; Rozansky-Witten
- brute force computations

n	$\dim(A_{2n,3n})$
1	1
2	1
3	1
4	2
5	2
6	3
7	4
8	5
9	6
10	8
11	9



[Bar-Natan]
[Kneissler]

Plan

- Graph complexes
- Kontsevich integral I_π
- Watanabe's main theorem
- Idea of construction of I_π — detects non-trivial D^d-bundles
- Idea of construction of Ψ_{2n} — builds non-trivial D^d-bundles

Graph complexes

Def $\mathcal{G}_{n,m} := \mathbb{Q}\Big\{$ finite, connected graphs Γ with valence ≥ 3

$V(\Gamma) \longleftrightarrow \{1,\dots,n\}$

$E(\Gamma) \longleftrightarrow \{1,\dots,m\}$

orientation $\Big\}\Big/\sim$

$or(\Gamma) = \{$ orientation of each edge of $\Gamma\}$ / reversing orientations of an even # of edges

$\bar{\Gamma} \sim -\Gamma$

where $\bar{\Gamma}$ means Γ with opposite orientation

Note: trivalent $\longleftrightarrow$ $n = 2\ell$, $m = 3\ell$.

Def $\mathcal{G}_{n,m} \xrightarrow{d} \mathcal{G}_{n-1,m-1}$

$\Gamma \longmapsto \sum_{e \in E(\Gamma)} \Gamma/e$ ~ induced labelling & orientation

Lemma $d^2 = 0$

Observation: $\gamma \in \mathcal{G}_{n,m}$

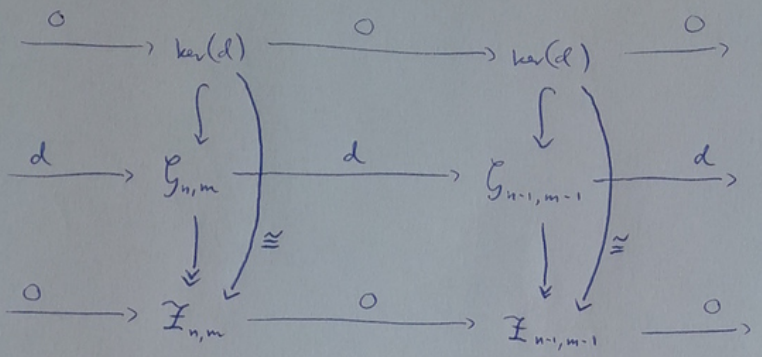
$d\gamma = 0$ $\longleftrightarrow$ system of linear equations in the coeffs of γ

$R_{n,m}$

collection of relations on $\mathcal{G}_{n,m}$

Def $\mathcal{I}_{n,m} := \mathcal{G}_{n,m} / \mathcal{R}_{n,m}$

Key property : this allows us to view $\ker(d)$ as a quotient of $\mathcal{G}_{n,m}$:



Rmk $\mathcal{I}_{2\ell,3\ell} = \{ \text{IHX relations} \}$

Def $\Gamma' = \Gamma$ with two vertex labels swapped
$\Gamma'' = \Gamma$ with two edge labels swapped

$$\mathcal{A}^{odd}_{n,m} := \mathcal{I}_{n,m} \Big/ \begin{matrix} \Gamma' \sim -\Gamma \\ \Gamma'' \sim \Gamma \end{matrix}$$

} target for Kontsevich char. classes of odd-dim. disc bundles

$$\mathcal{A}^{even}_{n,m} := \mathcal{I}_{n,m} \Big/ \begin{matrix} \Gamma' \sim \Gamma \\ \Gamma'' \sim -\Gamma \end{matrix}$$

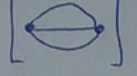
} target for Kontsevich char. classes of even-dim. disc bundles

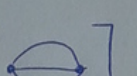
Today : $\mathcal{A}_{n,m} = \mathcal{A}^{odd}_{n,m}$

Comparison :

ℓ	1	2	3	4	5	6	7	8	9
$\dim\left(\mathcal{A}^{odd}_{2\ell,3\ell}\right)$	1	1	1	2	2	3	4	5	6
$\dim\left(\mathcal{A}^{even}_{2\ell,3\ell}\right)$	0	1	0	0	1	0	0	0	1

In particular,

$$\left[\ \text{  }\ \right] \neq 0 \in \mathcal{A}_{2,3}^{\text{odd}}$$

$$\left[\ \text{  }\ \right] = 0 \in \mathcal{A}_{2,3}^{\text{even}}$$

- $\Gamma =$  has an auto. swapping two edges, so $\Gamma = -\Gamma$ in $\mathcal{A}_{2,3}^{\text{even}}$

- $\Gamma =$  has an auto. swapping two vertices, but this <u>also</u> reverses orientations of 3 edges, so $\Gamma = (-1)^4 \Gamma$ in $\mathcal{A}_{2,3}^{\text{odd}}$.

<u>Why $\mathcal{A}_{n,m}$?</u> $\rightsquigarrow$ "universal" cycles!

$(\mathcal{G}_{n,m}, d)$ does not contain a <u>canonical</u> cycle.

$(\mathcal{A}_{n,m} \otimes \mathcal{G}_{n,m}, 1 \otimes d)$ has a canonical (<u>"universal"</u>) cycle:

$$\tilde{\gamma}_{n,m} = \frac{1}{n!\,m!} \sum_{\Gamma} [\Gamma] \otimes \Gamma$$

Γ: oriented labelled graphs

Kontsevich integral I_π

Black box (Kontsevich)

Associated to any framed (D^d, ∂)-bundle $\begin{array}{c} E \\ \downarrow \pi \\ B \end{array}$ over a smooth manifold,

there is a chain map

$$\begin{array}{ccc} \xi_{n,m} & \xrightarrow{I_\pi} & \Omega_{dR}^{m(d-1)-nd}(B) \\ \downarrow d & & \downarrow d \\ \xi_{n-1,m-1} & \xrightarrow{I_\pi} & \Omega_{dR}^{m(d-1)-nd+1}(B) \end{array}$$

(1) I_π depends on a choice of "propagator", but if $\gamma \in \xi_{n,m}$ is a cycle, then $[I_\pi(\gamma)] \in H^{m(d-1)-nd}(B;\mathbb{R})$ does not.

(2) $[I_\pi(\gamma)]$ is a characteristic class of framed (D^d, ∂)-bundles (it behaves well under pullback)

(3) If $\dim(B) = m(d-1)-nd$, B closed & oriented, then $\int_B [I_\pi(\gamma)]$ depends on the oriented bundle $\begin{array}{c} E \\ \downarrow \pi \\ B \end{array}$ up to cobordism of bundles.

Corollary Each cycle $\gamma \in \mathcal{G}_{n,m}$ defines a homomorphism

$$FDisc^{d}_{m(d-1)-nd} \longrightarrow \mathbb{R}$$

$$\Big\| \quad \Big/ \text{cobordism}$$

$$\left\{ \begin{array}{l} \text{framed } (D^d, \partial)\text{-bundles over closed} \\ \text{oriented } (m(d-1)-nd)\text{-manifolds} \end{array} \right\}$$

operation $= \amalg = \#$

$$\begin{bmatrix} E \\ \downarrow \pi \\ B \end{bmatrix} \longmapsto \int_B \left[I_\pi(\gamma) \right]$$

Obs Properties (1)–(3) hold also if I_π is tensored with $\mathcal{A}_{n,m}$.

$$\rightsquigarrow \quad \left[\mathcal{A}_{n,m} \otimes I_\pi(\gamma) \right] \in H^{m(d-1)-nd}\left(B ; \mathcal{A}_{n,m} \otimes \mathbb{R} \right)$$

$\rightsquigarrow$ and integrating, we get:

Corollary' Each cycle $\gamma \in \mathcal{A}_{n,m} \otimes \mathcal{G}_{n,m}$ defines a homomorphism

$$FDisc^{d}_{m(d-1)-nd} \longrightarrow \mathcal{A}_{n,m} \otimes \mathbb{R} \qquad (*)$$

Def The universal Kontsevich characteristic class $\kappa_{n,m}^{KCC}$ is the homomorphism $(*)$ associated to the "universal" cycle $\tilde{\gamma}_{n,m} = \frac{1}{n!\,m!} \sum_{\Gamma} [\Gamma] \otimes \Gamma$

$$\in \mathcal{A}_{n,m} \otimes \mathcal{G}_{n,m}.$$

Watanabe's main theorem

How are framed (D^d, ∂)-bundles related to $\pi_*(\mathrm{Diff}_\partial(D^d))$?

Def $B\mathrm{Diff}_\partial(D^d) :=$ the space classifying smooth (D^d, ∂)-bundles

↳ • it comes equipped with a smooth (D^d, ∂)-bundle

• for all (paracompact) B,

$$\text{pullback} : \underbrace{\mathrm{Maps}(B, B\mathrm{Diff}_\partial(D^d))}_{\text{htpy}} \longrightarrow \underbrace{\left\{\begin{array}{l}\text{smooth } (D^d,\partial)\text{-} \\ \text{-bundles over } B\end{array}\right\}}_{\cong}$$

is a bijection.

(This exists & is unique)

[Milnor] — up to htpy equivalence — Lemma

Lemma $\pi_i(\mathrm{Diff}_\partial(D^d)) \cong \pi_{i+1}(B\mathrm{Diff}_\partial(D^d))$

Def $\widetilde{B\mathrm{Diff}}_\partial(D^d) :=$ the space classifying smooth (D^d, ∂)-bundles that are framed

↖ trivialisation of the vertical tangent bundle

$$\begin{array}{ccc} \mathbb{R}^d \cong T_e E_{\pi(e)} & \longrightarrow & T^v E \\ & & \downarrow \\ D^d \cong E_b & \longrightarrow & E \ni e \\ & & \downarrow \pi \\ & & B \ni b \end{array}$$

$$\begin{array}{ccc} T^v E & \cong & \mathbb{R}^d \times E \\ \downarrow & \swarrow & \\ E & & \end{array}$$

$\exists$ canonical map (up to $\simeq$)

$$B\widetilde{Diff}_\partial(D^d) \longrightarrow BDiff_\partial(D^d)$$

Prop (Watanabe) The induced map on $\pi_i(-)\otimes\mathbb{Q}$ is an isomorphism

if · d odd
· i even
· $i > 2k-2$.

$\exists$ canonical map

$$\pi_i\left(B\widetilde{Diff}_\partial(D^d)\right) \xrightarrow{H} \left\{\begin{array}{l}\text{framed } (D^d,\partial)\text{-bundles}\\ \text{over closed oriented } i\text{-mflds}\end{array}\right\}\Big/\text{cob.}$$

$$\Big\| \qquad\qquad \Big\| = FDisc_i^d$$

$$\left\{\begin{array}{l}\text{framed } (D^d,\partial)\text{-bundles}\\ \text{over } S^i\end{array}\right\}\Big/\cong$$

Now set $(n,m) = (2\ell, 3\ell)$, $d = 2k+1$

$$\longrightarrow\quad m(d-1) - nd = 2\ell(k-1)$$

Theorem (Watanabe) For all $\ell \geq 1$, k odd,

$\exists$ $\mathbb{Q}$-linear map $\zeta_{2\ell,3\ell} \xrightarrow{\Psi_{2\ell}} FDisc^{2k+1}_{2\ell(k-1)} \otimes \mathbb{Q}$

such that

$$\begin{array}{ccccc}
 & & \pi_{2\ell(k-1)}\left(B\widetilde{Diff}_\partial(D^{2k+1})\right)\otimes\mathbb{Q} & & \\
 & & \downarrow H\otimes\mathbb{Q} & & \\
\zeta_{2\ell,3\ell} & \xrightarrow{\Psi_{2\ell}} & FDisc^{2k+1}_{2\ell(k+1)}\otimes\mathbb{Q} & \xrightarrow{kcc_{2\ell,3\ell}} & A_{2\ell,3\ell}\otimes\mathbb{R} \\
\downarrow & & & \nearrow \text{mult}^{\text{n}} \text{ by a certain +ve integer} & \\
A_{2\ell,3\ell} & \hookrightarrow & A_{2\ell,3\ell}\otimes\mathbb{R} & &
\end{array}$$

commutes and image$(\Psi_{2\ell}) \subseteq$ image$(H\otimes\mathbb{Q})$.

Corollary (Watanabe) For all $\ell \geq 2$, k odd,

$$\dim_{\mathbb{Q}} \left(\pi_{2\ell(k-1)-1}\left(\mathrm{Diff}_\partial(D^{2k+1}) \right) \otimes \mathbb{Q} \right) \geq \dim_{\mathbb{Q}} \left(A_{2\ell,3\ell} \right)$$

proof
- Tensor the big diagram with $\mathbb{R}$.
- Each $x \in A_{2\ell,3\ell} \otimes \mathbb{R}$ lifts to $y \in \mathrm{image}(\Psi_{2\ell}) \otimes \mathbb{R}$
- Each $y \in \mathrm{image}(\Psi_{2\ell}) \otimes \mathbb{R}$ lifts to $z \in \pi_{2\ell(k-1)}\left(B\widetilde{\mathrm{Diff}}_\partial(D^{2k+1}) \right) \otimes \mathbb{R}$
- Hence

$$\dim_{\mathbb{R}} \left(\pi_{2\ell(k-1)}\left(B\widetilde{\mathrm{Diff}}_\partial(D^{2k+1}) \right) \otimes \mathbb{R} \right) \geq \dim_{\mathbb{R}} \left(A_{2\ell,3\ell} \otimes \mathbb{R} \right)$$

$$\Rightarrow \quad \dim_{\mathbb{Q}} \left(\pi_{2\ell(k-1)}\left(B\widetilde{\mathrm{Diff}}_\partial(D^{2k+1}) \right) \otimes \mathbb{Q} \right) \geq \dim_{\mathbb{Q}} \left(A_{2\ell,3\ell} \right)$$

Since $\left(\begin{array}{l} d = 2k+1 \quad \text{odd} \\ i = 2\ell(k-1) \quad \text{even} \\ \ell \geq 2 \Rightarrow i > 2k-2 \end{array} \right)$ we have an isomorphism

$$\pi_{2\ell(k-1)}\left(B\widetilde{\mathrm{Diff}}_\partial(D^{2k+1}) \right) \otimes \mathbb{Q} \cong \pi_{2\ell(k-1)}\left(B\mathrm{Diff}_\partial(D^{2k+1}) \right) \otimes \mathbb{Q}$$

$$\cong \pi_{2\ell(k-1)-1}\left(\mathrm{Diff}_\partial(D^{2k+1}) \right) \otimes \mathbb{Q}$$

Rmk: [Watanabe II]: lines of slope $\ell \geq 2$ ⟶ all odd k. [when $A_{2\ell,3\ell} \neq 0$]

[Watanabe I]: $\ell = 1$ ⟶ only finitely many values of k (even & odd)

$d = 2k+1$

$k = \frac{1}{2}(d-1)$

Idea of construction of I_π.

Input $\begin{array}{c} E \\ \pi \downarrow \\ B \end{array}$ smooth (D^d, ∂)-bundle + framing of $T^v E$ $\quad (d \text{ odd})$

Output $\zeta_{n,m} \xrightarrow{I_\pi} \Omega_{dR}^{m(d-1)-nd}(B)$

$\omega \in \Gamma \longmapsto$ differential form on B

① Associated bundle $E\bar{C}_n(\pi) \downarrow B$ whose fibre over $b \in B$ is

$$\bar{C}_n(E_b) \cong \bar{C}_n(D^d)$$

↖ Fulton-MacPherson compactn of the ordered config space of n pts in E_b

$\dim = nd$.

② $E\bar{C}_2(\pi) \supseteq \partial^{fib} E\bar{C}_2(\pi) \xrightarrow{p} S^{d-1}$

$E\bar{C}_2(\pi) \downarrow B$ ↙ $\partial^{fib} E\bar{C}_2(\pi)$

p defined using the topology of $\partial\bar{C}_2(D^d)$.

$$\bar{C}_2(D^d) = Bl_\Delta\left(Bl_{S^d \vee S^d}\left(Bl_{\{(\infty,\infty)\}}\left(S^d \times S^d\right)\right)\right)$$

$$\partial\bar{C}_2(D^d) \cong S^d \times S^{d-1} \xrightarrow{pr_2} S^{d-1}$$

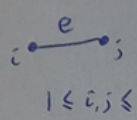
Choose $\omega \in \Omega_{dR}^{d-1}\left(E\bar{C}_2(\pi)\right)$:

$$\omega|_{\partial^{fib}} = p^*(\text{vol})$$

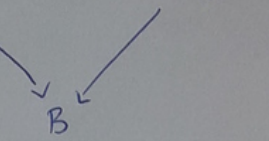
"propagator"

(!) The framing of $T^v E$ is needed to make this definition on each fibre glue together well.

③ $e \in E(\Gamma) \longrightarrow$ $E\bar{C}_n(\pi) \xrightarrow{f_e} E\bar{C}_2(\pi)$



$1 \leq i, j \leq n$



forget all config. points except i^{th} + j^{th}

$$\bigwedge_{e \in E(\Gamma)} f_e^*(\omega) \quad \in \quad \Omega_{dR}^{m(d-1)}\left(E\bar{C}_n(\pi)\right)$$

④ Integrate along the fibre : $\int_{\text{fibre}} \bigwedge_{e \in E(\Gamma)} f_e^*(\omega) \quad \in \Omega_{dR}^{m(d-1)-nd}(B).$

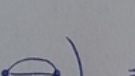
$\| \cdot I_\pi(\Gamma)$

Simplest example:

$n = 2$
$m = 3$

$\Gamma =$ 

3 edges , each $f_e = id$

$$I_\pi\left(\text{  }\right) = \int_{\bar{C}_2(D^d)} \omega \wedge \omega \wedge \omega \quad \in \quad \Omega_{dR}^{d-3}(B)$$

$\longrightarrow$ This is the special case that Watanabe used in Part I (cf. last talk) to detect classes in $\pi_{2k-3}\left(\text{Diff}_\partial(D^{2k+1})\right) \otimes \mathbb{Q}$.

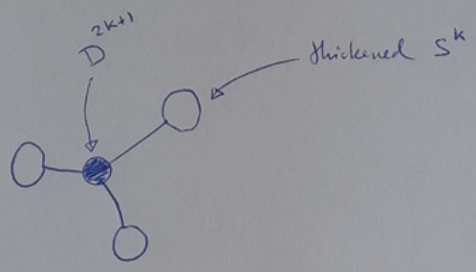
Idea of construction of $\Psi_{2\ell}$.

Input $\Gamma \in \zeta_{2\ell,3\ell}$

Output Framed (D^{2k+1}, ∂)-bundle over a closed, oriented $2\ell(k-1)$-manifold.

$B = (S^{k-1})^{2\ell}$.

① $V =$

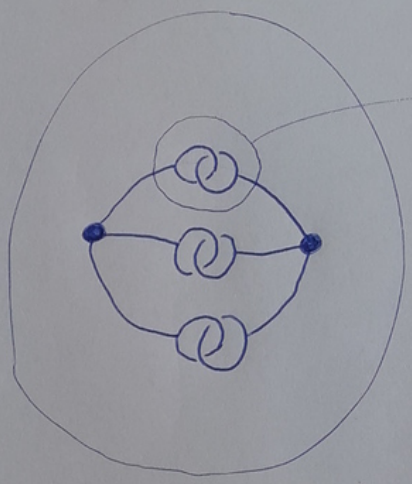


$= (0\text{-handle}) \cup (\text{three } k\text{-handles})$

$(2k+1)$-dim. handlebody.

Take 2ℓ copies, named $V_1, \dots, V_{2\ell}$

Embed according to Γ into $\text{int}(D^{2k+1})$:



+1 Hopf link $S^k \sqcup S^k \subseteq D^{2k+1}$.

$W := V_1 \sqcup \dots \sqcup V_{2\ell} \subseteq \text{int}(D^{2k+1})$.

② Construct a non-trivial framed $(V, \partial V)$-bundle over S^{k-1}.

Black box — via the "Borromean rings"

$$S^{2k-1} \amalg S^k \amalg S^k \subseteq D^{2k+1}$$

③ $\rightsquigarrow$ framed $(W, \partial W)$-bundle over $(S^{k-1})^{2\ell}$

- pull back along each projection $(S^{k-1})^{2\ell} \xrightarrow{pr_i} S^{k-1}$
- take fibrewise $\amalg$.

its ∂^{fib} is the <u>trivial</u> ∂W-bundle over $(S^{k-1})^{2\ell}$

glue this to the <u>trivial</u> $(D^{2k+1} \setminus \mathrm{int}(W))$-bundle over $(S^{k-1})^{2\ell}$

$\rightsquigarrow$ framed (D^{2k+1}, ∂)-bundle over $(S^{k-1})^{2\ell}$. //

Footnote:

On page 4 it is not quite true that $\mathcal{A}^{even}_{n,m}$ is as written. There are <u>two</u> differences from the $\mathcal{A}^{odd}_{n,m}$ case.

- $\mathcal{G}^{odd}_{n,m} = \mathbb{Q}\left\{ \begin{array}{l} \text{fin. conn. graphs } \Gamma \text{ of valence } \geq 3 \\ V(\Gamma) \longleftrightarrow \{1,\dots,n\} \\ E(\Gamma) \longleftrightarrow \{1,\dots,m\} \end{array} \right\} \Big/ \bar{\Gamma} \sim -\Gamma$

  orientation

  $\{\text{orientation of each edge of } \Gamma\}$ / reversing orientations of an even # of edges

$\mathcal{G}^{even}_{n,m}$ = The same, except now orientation

$\{\text{ordering of } E(\Gamma)\}$ / even permutations

$\overset{\text{lemma}}{\cong} \mathbb{Q}\left\{ \begin{array}{l} \text{fin. conn. graphs } \Gamma \text{ of valence } \geq 3 \\ V(\Gamma) \longleftrightarrow \{1,\dots,n\} \\ E(\Gamma) \longleftrightarrow \{1,\dots,m\} \end{array} \right\}$

- $\mathcal{I}^{odd/even}_{n,m} = \mathcal{G}^{odd/even}_{n,m} \Big/ \left(\begin{array}{c}\text{relations corresponding} \\ \text{to } d\gamma = 0\end{array}\right)$

- $\mathcal{A}^{odd}_{n,m} = \mathcal{I}^{odd}_{n,m} \Big/ \begin{array}{l} \Gamma' \sim -\Gamma \\ \Gamma'' \sim \Gamma \end{array}$

Γ' = Γ with two vertex labels swapped

Γ'' = Γ with two edge labels swapped

$$\mathcal{A}^{even}_{n,m} = \mathcal{I}^{even}_{n,m} \Big/ \begin{array}{l} \Gamma' \sim \Gamma \\ \Gamma'' \sim -\Gamma \end{array}$$

Moduli spaces of high-dimensional discs

after Watanabe, Weiss, Kupers-Randal-Williams

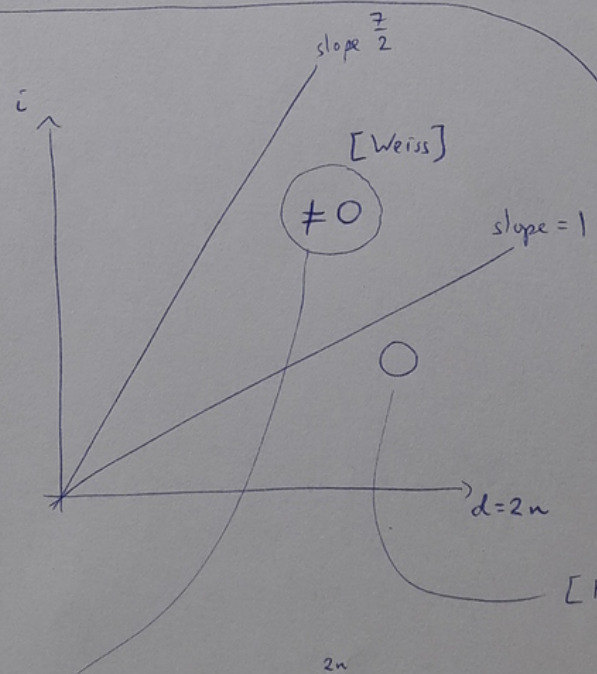
GeMAT seminar
IMAR
4 June 2021

IV. Even-dimensional discs, Exotic Pontrjagin classes

Our general aim: understand $\pi_*(\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q}$.

Last two talks $\xrightarrow{d=2n+1}$ non-trivial elements detected by Kontsevich char. classes [Watanabe]

Today + next week $\xrightarrow{d=2n}$ non-trivial elements detected by "exotic Pontrjagin classes" (Weiss char. classes) [Weiss] [Kupers-Randal-Williams]



$\pi_i(\mathrm{Diff}_\partial(D^{2n})) \otimes \mathbb{Q}$

[Weiss] $\exists$ non-0 element in bidegree $\overset{2n}{\overset{\shortparallel}{(d,i)}}$ whenever $i \equiv 2n-2 \pmod 4$.

(in this range)

Classifying spaces

G topological group

Thm [Milnor '56 explicit construction]
[Brown '62 categorical proof]

$\exists$ space BG such that

$$\left\{\begin{array}{c}\text{principal } G\text{-bundles over } X \\ (\text{trivialised over } * \in X)\end{array}\right\}\Big/\cong \quad \cong \quad [X, BG] = \mathrm{Maps}_*(X, BG)/\mathrm{hty}$$

for all (paracompact) X, functorially in X.

Rmk • BG is unique up to hty equivalence

• $[BG, BG] \ni \mathrm{id} \longleftrightarrow \begin{array}{c} EG \\ \downarrow \\ BG \end{array}$ "universal" principal G-bundle.

Eg • $BSO(n) \simeq Gr_n^+(\mathbb{R}^\infty)$

• $BG \simeq K(G, 1)$ if G is discrete

principal G-bundle $\equiv$ regular covering with Deck $= G$.

Def F G-space

An **F-bundle** is a **locally trivial fibration** $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ with fibres $\cong F$, transition maps given by $G \curvearrowright F$

(principal G-bundle $\longleftrightarrow$ $G \curvearrowright G = F$)

(left mult^n)

Eg $\mathrm{Homeo}(F) \curvearrowright F$ ——— locally trivial fibrations

$GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$ ——— real vector bundles

$\mathrm{Diff}(M) \curvearrowright M$ ——— smooth M-bundles

Lemma By "replacing the fibre" we have

$$\{F\text{-bundles over } X\}/\cong \quad \cong \quad \{\text{principal } G\text{-bundles over } X\}/\cong$$

Hence both of these are classified by $[X, BG]$.

Coro For any G-space F,

$$\{\text{char. classes of } F\text{-bundles}\} \quad \cong \quad H^*(BG)$$

$$\vdots$$

$$\| \qquad \left[\begin{pmatrix} F \to E \\ \downarrow \\ X \end{pmatrix} \longmapsto c(E) \in H^*(X) \right]$$

stable under pullback

Eg char. classes of

- real vector bundles (rank n) $\cong H^*(BGL_n(\mathbb{R}))$
  $\cong H^*(BO(n))$
  $O(n) \xrightarrow{\simeq} GL_n(\mathbb{R})$
- oriented real v. bundles (rank n) $\cong H^*(BSO(n))$
- smooth M-bundles $\cong H^*(B\mathrm{Diff}(M))$
- locally trivial fibrations with fibres $\cong F$ $\cong H^*(B\mathrm{Homeo}(F))$.

We will use the fact that the construction $B(-)$ can be applied also to <u>topological monoids</u> $\mathcal{M}$.

In general, $[X, B\mathcal{M}]$ does not have such an obvious geometric interpretation as $[X, BG] = \{\text{prin } G\text{-bundles } / X\}$.

But one important special case is:

<u>Eg</u> $\mathcal{M} = hAut(F) := \{f \in Map(F,F) \mid f \text{ has a hty inverse}\}$

$$[X, BhAut(F)] \cong \{\text{fibrations over } X \text{ with fibres} \simeq F\}/\cong$$

not necessarily locally trivial, just satisfy some homotopy lifting property

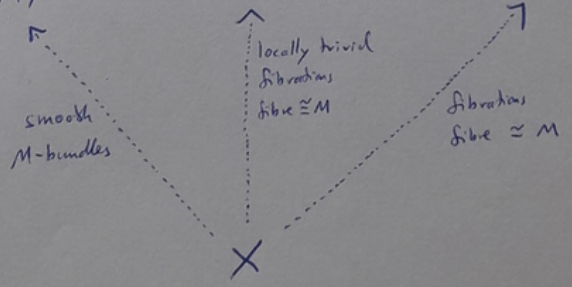
e.g.



So for a smooth manifold M we have:

$$Diff(M) \hookrightarrow Homeo(M) \hookrightarrow hAut(M)$$

$$BDiff(M) \longrightarrow BHomeo(M) \longrightarrow BhAut(M)$$



Pontrjagin classes, signature theorem

Notation: H^* means $H^*(-;\mathbb{Q})$ unless otherwise specified.

Fact (e.g. Milnor-Stasheff):

$$H^*(BO(2n)) \cong \mathbb{Q}[p_1, p_2, \dots, p_n] \quad \text{polynomial algebra}$$

$$\deg(p_i) = 4i.$$

$$BO := \operatorname{colim}_n (BO(n))$$

$$\begin{array}{ccccc} BO & & H^*(BO) & \cong & \mathbb{Q}[p_1, p_2, p_3, \dots] \\ \uparrow & & \downarrow & & \downarrow \\ BO(2n) & & H^*(BO(2n)) & \cong & \mathbb{Q}[p_1, \dots, p_n] \end{array}$$

Signature theorem (Hirzebruch):

$\exists\ L_n \in \mathbb{Q}[p_1, p_2, p_3, \dots]$ of homogeneous degree $4n$

such that if M is a smooth, closed, oriented $4n$-manifold

then $\left\langle \underbrace{L_n\left(\overset{TM}{\downarrow}{}_M\right)}_{\in H^{4n}(M)}, [M] \right\rangle = \sigma(M)$

$\sigma(M)$ = signature of the intersection form on $H^{2n}(M)$

Eg

$L_0 = 1$

$L_1 = \frac{1}{3} p_1$

$L_2 = \frac{7}{45} p_2 - \frac{1}{45} p_1^2$

...

Exotic Pontrjagin classes

$$BO := \operatorname{colim}_n (BO(n))$$

$$BTop := \operatorname{colim}_n \left(B\,\mathrm{Homeo}(\mathbb{R}^{2n})\right)$$

$$\begin{array}{ccc} BO(2n) & \longrightarrow & B\,\mathrm{Homeo}(\mathbb{R}^{2n}) \\ \downarrow & & \downarrow \\ BO & \xrightarrow{(*)} & BTop \end{array}$$

Thm (Thom / Novikov / Kirby – Siebenmann / Sullivan)

$(*)$ induces $\cong$ on $H^*(-;\mathbb{Q})$

So we have:

$$\begin{array}{ccc} \mathbb{Q}[p_1, p_2, \dots p_n] & \longleftarrow & H^*(B\,\mathrm{Homeo}(\mathbb{R}^{2n})) \\ \uparrow\uparrow & & \uparrow \\ \mathbb{Q}[p_1, p_2, \dots] & \xleftarrow{\cong} & \mathbb{Q}[p_1, p_2, \dots] \end{array}$$

Cor $p_i \neq 0$ in $H^*(B\,\mathrm{Homeo}(\mathbb{R}^{2n}))$ for $i \leq n$.

Def *Exotic Pontrjagin classes* are $p_i \in H^*(B\,\mathrm{Homeo}(\mathbb{R}^{2n}))$ for $i > n$.

Q: Are there non-0 exotic Pontrjagin classes?

Conj. [Weiss '94] No.

Thm [Weiss '15] Yes!!

$p_i \neq 0 \in H^*(B\,\mathrm{Homeo}(\mathbb{R}^{2n}))$ for all $i \leq \frac{9}{4}n - \text{const.}$

Relation to $\mathrm{Diff}_\partial(D^d)$

- Slight extension of Weiss' theorem:

$$p_{n+k} \in H^{4n+4k}\left(B\mathrm{Homeo}(\mathbb{R}^{2n})\right)$$

For any space X there is a bilinear map

$$H^i(X) \times \pi_i(X)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

$$\left(\alpha, [S^i \xrightarrow{f} X]\right) \longmapsto \left\langle f^*(\alpha), [S^i]\right\rangle$$

Notation:
$\pi_*(-)_{\mathbb{Q}}$
$=$
$\pi_*(-)\otimes\mathbb{Q}$

Thm [Weiss '16] For all $k \leq \frac{5}{4}n - \text{const.}$, the linear map

$$\langle p_{n+k}, - \rangle : \pi_{4n+4k}\left(B\mathrm{Homeo}(\mathbb{R}^{2n})\right)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

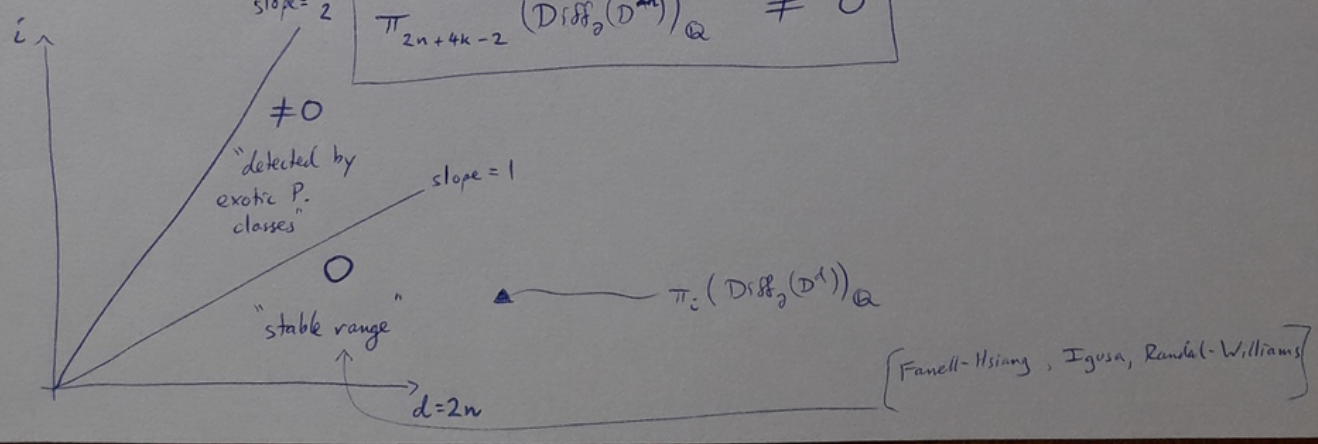
is non-zero.

Cor $\pi_{4n+4k}\left(B\mathrm{Homeo}(\mathbb{R}^{2n})\right)_{\mathbb{Q}} \neq 0$ for these k.

- [Morlet] For $i>0$, $\pi_i B\mathrm{Diff}_\partial(D^d) \cong \pi_{i+d}\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d)\right)$

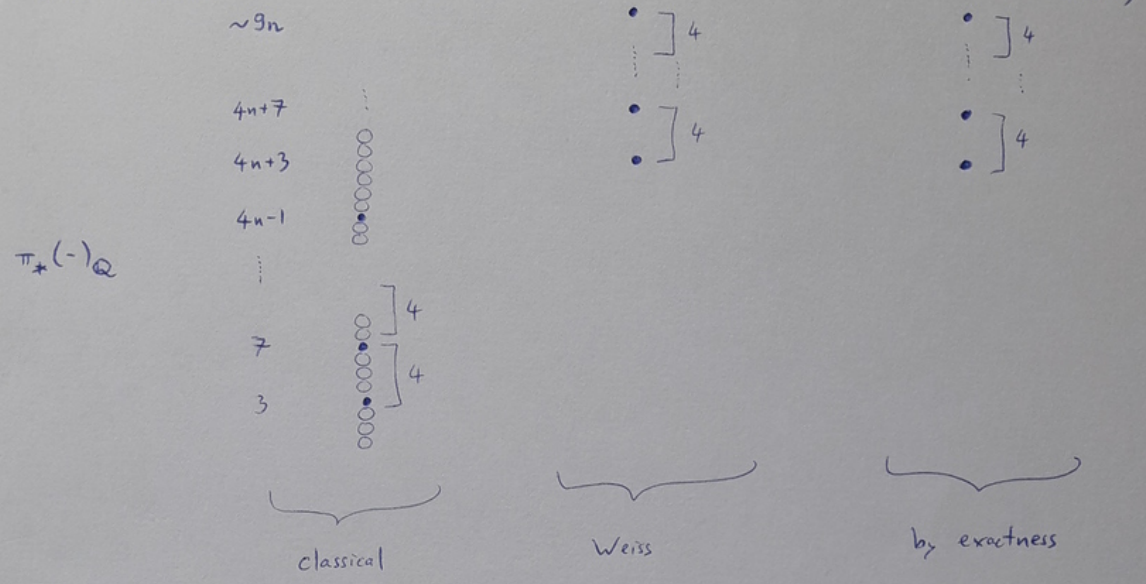
Corollary [Weiss '16] For all $1 \leq k \leq \frac{5}{4}n - \text{const.}$,

$$\pi_{2n+4k-2}\left(\mathrm{Diff}_\partial(D^{2n})\right)_{\mathbb{Q}} \neq 0$$



proof of Corollary:

fibre bundle $O(2n) \hookrightarrow \mathrm{Homeo}(\mathbb{R}^{2n}) \twoheadrightarrow \mathrm{Homeo}(\mathbb{R}^{2n})/O(2n)$



For $1 \leq k \leq \frac{5}{4}n$ - const,

$$0 \neq \pi_{4n+4k-1}\left(\frac{\mathrm{Homeo}(\mathbb{R}^{2n})}{O(2n)}\right) \cong \pi_{2n+4k-1}\left(BDiff_\partial(D^d)\right)$$

[Morlet]

$$\cong \pi_{2n+4k-2}\left(Diff_\partial(D^d)\right). \quad //$$

Aim: Explain Weiss' proof that

$$p_{n+k} \neq 0 \in H^{4n+4k}(B\mathrm{Homeo}(\mathbb{R}^{2n}))$$

for all $k \leq \frac{5}{4}n$ - const.

κ-classes

Smooth:

$$\begin{array}{ccc} \mathrm{Diff}(M) & \longrightarrow & E\mathrm{Diff}(M) \\ & & \downarrow \\ & & B\mathrm{Diff}(M) \end{array}$$

universal principal $\mathrm{Diff}(M)$-bundle.

$\rightsquigarrow$ change fibres: $E_M = E\mathrm{Diff}(M) \underset{\mathrm{Diff}(M)}{\times} M$

$$\begin{array}{ccccc} \mathbb{R}^d & \longrightarrow & T^v E_M & & \\ & & \downarrow & & \\ M & \longrightarrow & E_M & \xrightarrow{f} & BO(d) \\ & & \pi \downarrow & & \\ & & B\mathrm{Diff}(M) & & \end{array}$$

vertical tangent bundle $\coprod_{x \in B\mathrm{Diff}(M)} T(\pi^{-1}(x))$

universal smooth M-bundle

$d = \dim(M)$.

f = map classifying the vector bundle $T^v E_M$.

$$H^*(BO(d)) \xrightarrow{f^*} H^*(E_M) \xrightarrow{\pi_!} H^{*-d}(B\mathrm{Diff}(M))$$

$\pi_!$: $\int$ fibre

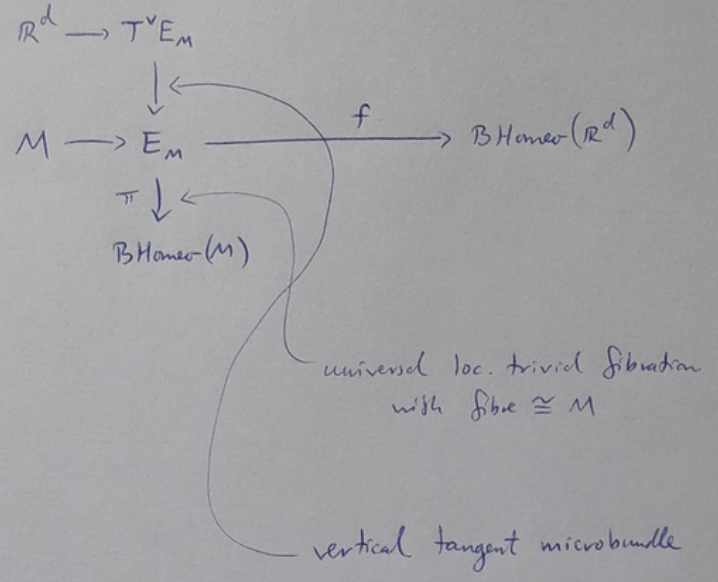
κ

Eg $d=2$, oriented version

$$\begin{array}{ccccc} H^*(BSO(2)) & \longrightarrow & H^{*-2}(B\mathrm{Diff}(\Sigma)) & \cong & H^{*-2}(\mathcal{M}(\Sigma)) \\ \| & & \cup & & \\ \mathbb{Q}[e] & & \kappa(e^i) & \text{degree } 2i-2 & \\ \deg(e)=2 & & & & \end{array}$$

"Miller-Morita-Mumford classes"

non-smooth:



- "microbundle" [Milnor]
- M topological manifold $\rightsquigarrow$ $TM \downarrow M$ microbundle of rank $d = \dim(M)$
- Thm (Kister-Mazur) X paracompact

$$\left\{\text{loc. trivial fibration over } X \text{ with fibre} \cong \mathbb{R}^d\right\}/\cong \xrightarrow{1:1} \left\{\text{microbundles of rank } d \text{ over } X\right\}/\cong$$

$$\left\{\text{loc. trivial fibration over } X \text{ with fibre} \cong \mathbb{R}^d\right\}/\cong \;\overset{1:1}{\longleftrightarrow}\; [X, B\mathrm{Homeo}(\mathbb{R}^d)]$$

$$H^*(B\mathrm{Homeo}(\mathbb{R}^d)) \xrightarrow{f^*} H^*(E_M) \xrightarrow[\text{fibre}]{\pi_!} H^{*-d}(B\mathrm{Homeo}(M))$$

κ

The smooth & non-smooth constructions are compatible, i.e.

$$\begin{array}{ccc} H^*(B\mathrm{Homeo}(\mathbb{R}^d)) & \xrightarrow{\kappa} & H^{*-d}(B\mathrm{Homeo}(M)) \\ \downarrow & & \downarrow \\ H^*(BO(d)) & \xrightarrow{\kappa} & H^{*-d}(B\mathrm{Diff}(M)) \end{array}$$

commutes.

Step 1. Let $k \leq \frac{5}{4}n - \text{const.}$

$W_g = \#_g (S^n \times S^n)$



$(n=1, g=3)$

$$\mathbb{Q}[p_1, p_2, p_3, \dots] \longrightarrow H^*(B\mathrm{Homeo}(\mathbb{R}^{2n})) \xrightarrow{\kappa} H^{*-2n}(B\mathrm{Homeo}(W_g))$$

$\| \; H^*(BTop)$

Notation: L_{n+k} = Hirzebruch L-polynomial; P = any monomial in $p_1, p_2, \dots$ — in degree $4n+4k$

Suppose we can find $z \in H_{2n+4k}(B\mathrm{Homeo}(W_g))$ such that (*)

$$\langle \kappa(L_{n+k}), z \rangle \neq 0$$

$$\langle \kappa(P), z \rangle = 0 \quad \text{whenever } P \text{ is } \underline{\text{decomposable}}$$

Then we would be done!

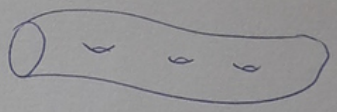
- $L_{n+k} = \lambda \cdot p_{n+k} + (\text{lin. comb}^{\text{n}} \text{ of decomposable monomials}), \quad \lambda \neq 0$

$\langle \kappa(-), z \rangle \rightsquigarrow$

- $0 \neq \lambda \cdot \langle \kappa(p_{n+k}), z \rangle$
- $p_{n+k} \neq 0 \in H^{4n+4k}(B\mathrm{Homeo}(\mathbb{R}^{2n}))$. //

Step 2. First, construct something a bit different.

$W_{g,1} = W_g \setminus$ open disc



$$H^*(BO(2n)) \xrightarrow{\kappa} H^{*-2n}\left(BDiff_\partial(W_{g,1})\right)$$

$\|$

$\mathbb{Q}[p_1, \dots, p_n]$

(!) From now on, we assume that $k \leq n\text{-const.}$

(The case $n\text{-const.} \leq k \leq \frac{5}{4}n\text{-const.}$ uses a similar trick.)

Fix integers $\frac{n+1}{4} \leq a, b \leq n-1$: $a+b = n+k$

There exists $x \in H_{2n+4k}\left(BDiff_\partial(W_{g,1})\right)$ such that

$$\langle \kappa(\xi_{n+k}), x \rangle \neq 0$$

$$\langle \kappa(P), x \rangle = 0 \quad \text{whenever } P \neq p_a p_b$$

(**)

Proof: We use the theorem of Galatius-Randal-Williams that

$$H^*(BDiff_\partial(W_{g,1})) \cong \mathbb{Q}\left[\kappa(c) \;\middle|\; \begin{array}{l} c \text{ monomial of degree } > 2n \\ \text{in } e, p_i \text{ for } \frac{n+1}{4} \leq i \leq n-1 \end{array}\right]$$

$g \gg *$

$$\kappa(p_a p_b) \in H^{2n+4k}\left(BDiff_\partial(W_{g,1})\right)$$

$$x := \kappa(p_a p_b)^* \in H_{2n+4k}\left(BDiff_\partial(W_{g,1})\right)$$

* linear dual (UCThm)

$$\Rightarrow \quad \langle \kappa(p_a p_b), x \rangle = 1$$

$$\langle \kappa(P), x \rangle = 0$$

P monomial of degree $4n+4k$, $P \neq p_a p_b$

$$L_{n+k} = \mu \cdot p_a p_b + (\text{lin. comb}^{\text{n}} \text{ of other monomials})$$

Thm (Berglund - Bergström '18)

- $\mu \neq 0$
- in fact if P is any monomial in $p_1, p_2, \ldots$ of degree $4n+4k$, then $(\text{coeff of } P \text{ in } L_{n+k}) \neq 0$.

Apply $\langle \kappa(-), x \rangle$

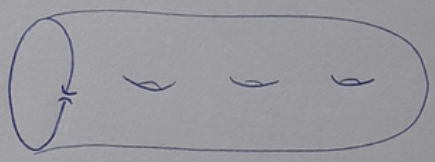
$$\rightsquigarrow \quad \langle \kappa(L_{n+k}), x \rangle = \mu \cdot \langle \kappa(p_a p_b), x \rangle = \mu \neq 0. \quad //.$$

Step 3 (Idea)

$W_{g,1}^* := W_{g,1} \setminus (\text{point on } \partial)$

$\partial W_{g,1} \cong S^{2n-1}$

$\partial W_{g,1}^* \cong \mathbb{R}^{2n-1}$



Note that $(W_{g,1}^*)^+ \cong W_{g,1}$

one-point compactification.

$$\text{Diff}_\partial(W_{g,1}) \xrightarrow[\text{restrict}]{} \text{Diff}_\partial(W_{g,1}^*) \xrightarrow[\text{extend to 1-pt compact}^{\text{n}}]{} \text{Homeo}_\partial(W_{g,1}) \xrightarrow[\text{collapse } \partial]{} \text{Homeo}(W_g)$$

$$\begin{array}{ccc} W_{g,1} & \longrightarrow & W_{g,1} \\ \downarrow & & \downarrow \\ W_g & \longrightarrow & W_g \end{array}$$

H_*: $x \longmapsto y$ $\qquad \varphi_*(y) \longmapsto z$

$$BDiff_\partial(W_{g,1}) \longrightarrow BDiff_\partial(W^*_{g,1}) \longrightarrow B\,Homeo(W_g)$$

φ

endomorphism

Got	Want
$\langle \kappa(f_{n+k}), x \rangle \neq 0$	$\langle \kappa(f_{n+k}), z \rangle \neq 0$
$\langle \kappa(P), x \rangle = 0$	$\langle \kappa(P), z \rangle = 0$
if $P \neq P_a P_b$	if $P \neq P_{n+k}$

It will be enough to construct φ so that φ^*

- preserves $\kappa(f_{n+k})$
- if P is a <u>decomposable</u> monomial, then

$$\kappa(P) \longmapsto \begin{cases} 0 & P \text{ contains } P_b \\ \kappa(P) & \text{o/w} \end{cases}$$

Idea: "decompose" $BDiff_\partial(W^*_{g,1})$ into 2 pieces, so that

- $\kappa(f_{n+k})$ "lives" on one piece,
- $\kappa(P)$ [P decomposable monomial] "lives" on the other piece.

Then construct φ in pieces.....

<u>Manifold calculus</u>

originally: [Goodwillie - Weiss '99]

reformulated: [Boavida de Brito - Weiss '18]

→ Next time.....

Moduli spaces of high-dimensional discs

after Watanabe, Weiss, Kupers-Randal-Williams

GeMAT seminar
IMAR
11 June 2021

V. Manifold calculus & exotic Pontrjagin classes

- Recap of last talk
  - $\pi_* \mathrm{Diff}_\partial(D^d)$ vs. $\pi_* \mathrm{BHomeo}(\mathbb{R}^d)$
  - exotic Pontrjagin classes
  - strategy of proof of Weiss' theorem
- Pulling back $\kappa(f_{\mathrm{nat}})$ and $\kappa(\text{decomposable})$
- Manifold calculus
- End of the proof of Weiss' theorem
- The work of Kupers-Randal-Williams

Recap of last talk

$$\pi_* \mathrm{Diff}_\partial(D^d) \cong \pi_{*+d+1}\left(\frac{\mathrm{Homeo}(\mathbb{R}^d)}{O(d)}\right) \qquad \text{[Morlet]}$$

$$\cdots \to \underbrace{\pi_{*+1} BO(d)}_{\text{vanishes for } * \gtrsim 2d} \to \pi_{*+1} B\mathrm{Homeo}(\mathbb{R}^d) \to \underbrace{\pi_*\left(\frac{\mathrm{Homeo}(\mathbb{R}^d)}{O(d)}\right)}_{\text{isomorphic in degrees } * \gtrsim 2d} \to \underbrace{\pi_* BO(d)}_{\text{vanishes for } * \gtrsim 2d} \to \cdots\cdots$$

$\otimes\mathbb{Q}$:

Putting this together:

In degrees $* \gtrsim d$, ("above the stable range") we have $\pi_* \mathrm{Diff}_\partial(D^d)_{\mathbb{Q}} \cong \pi_{*+d+2} B\mathrm{Homeo}(\mathbb{R}^d)_{\mathbb{Q}}$

Now set $\boxed{d = 2n}$

$$\begin{array}{ccccc}
\mathbb{Q}[p_1,\dots,p_n] & \cong & H^*(BO(2n)) & \longleftarrow & H^*(B\mathrm{Homeo}(\mathbb{R}^{2n})) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{Q}[p_1,p_2,p_3,\dots] & \cong & H^*(BO) & \longleftarrow & H^*(B\mathrm{Top}) \cong \mathbb{Q}[p_1,p_2,p_3,\dots]
\end{array}$$

$H^*(B\mathrm{Homeo}(\mathbb{R}^{2n}))$: $p_1, \dots, p_n$ (0 ≠) | $p_{n+1}, p_{n+2}, \dots$ (?) "exotic"

$\cong$: Thom / Novikov / Kirby–Siebenmann / Sullivan

Theorem [Weiss '15] (1) $p_{n+k} \neq 0 \in H^{4n+4k}(B\mathrm{Homeo}(\mathbb{R}^{2n}))$ for all $k \leq \frac{5}{4}n$

(2) Moreover, the linear map $\pi_{4n+4k} B\mathrm{Homeo}(\mathbb{R}^{2n})_{\mathbb{Q}} \longrightarrow \mathbb{Q}$

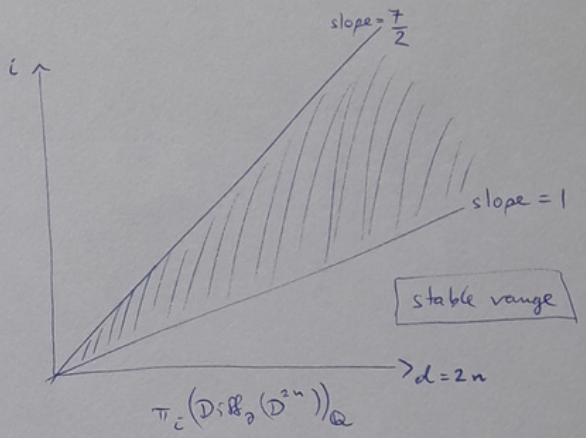
$$[f : S^{4n+4k} \to B\mathrm{Homeo}(\mathbb{R}^{2n})] \longmapsto \int_{S^{4n+4k}} f^*(p_{n+k})$$

is non-zero.

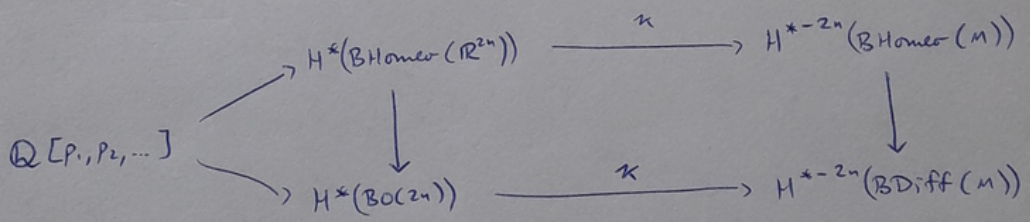
Note: (2) $\Rightarrow$ $\pi_{2n+4k-2}\left(\mathrm{Diff}_\partial(D^{2n})\right)_{\mathbb{Q}} \cong \pi_{4n+4k}\left(B\mathrm{Homeo}(\mathbb{R}^{2n})\right)_{\mathbb{Q}} \neq 0$

for $k \leq \frac{5}{4}n$

i.e. $2n+4k-2 \leq 7n$



Aim: Understand the strategy to prove that $p_{n+k} \neq 0 \in H^*(B\mathrm{Homeo}(\mathbb{R}^{2n}))$ in the slightly smaller range $k \leq n$.

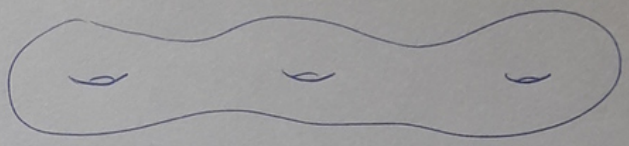


$L_d \in \mathbb{Q}[p_1, p_2, \dots]$ Hirzebruch L-polynomial of homogeneous degree $= 4d$

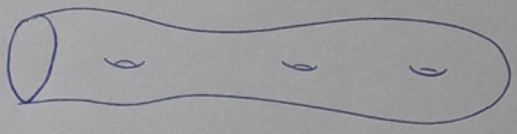
[Hirzebruch] • If $\dim(M) = 4d$ (closed, oriented) $\quad \langle L_d(TM), [M] \rangle =$ signature of M

[Berglund-Bergström '18] • if P is any monomial of degree $4d$ then the coefficient of P in L_d is $\neq 0$.

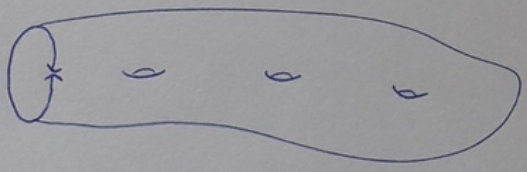
$W_g = \#_g (S^n \times S^n)$



$W_{g,1} = W_g \setminus \mathrm{int}(D^{2n})$



$W_{g,1}^* = W_{g,1} \setminus (\text{one point on } \partial)$



Fix $\frac{n+1}{4} \leq a, b \leq n \quad : \quad a+b = n+k$ — possible since $k \leq n$

$$BDiff_\partial(W_{g,1}) \longrightarrow BDiff_\partial(W_{g,1}^*) \longrightarrow BHomeo(W_g)$$

restrict

1-point compactify, collapse ∂

$x \in H_*$

$\langle \kappa(f_{n+k}), x \rangle \neq 0$

$\langle \kappa(P), x \rangle = 0$ unless $P = p_a p_b$

$x = \kappa(p_a p_b)^*$

$z \in H_*$

$\langle \kappa(f_{n+k}), z \rangle \neq 0$

$\langle \kappa(P), z \rangle = 0$ unless $P = p_{n+k}$

?

write $f_{n+k} = \lambda \cdot p_{n+k} + (\text{lin. comb. of decomposable monomials})$

$0 \neq \langle \kappa(f_{n+k}), z \rangle = \lambda \cdot \langle \kappa(p_{n+k}), z \rangle$

$p_{n+k} \neq 0$

Galatius-Randal-Williams

for $g \geq 2*+3$,

$H^* \cong \mathbb{Q}[\kappa(c) \mid c \text{ certain monomials in } e, p_i \quad \frac{n+1}{4} \leq c \leq n]$

higher-dimensional Madsen-Weiss theorem needs $2n \geq 6$

$$x \longmapsto y \longmapsto \varphi_*(y) \longmapsto z$$

$$BDiff_\partial(W_{g,1}) \longrightarrow BDiff_\partial(W_{g,1}^*) \longrightarrow BHomeo(W_g)$$

$$\downarrow \simeq \quad \text{(Lemma)}$$

$$BEmb_\partial(W_{g,1}^*, W_{g,1}^*)$$

φ

$$\varphi^*: \quad \kappa(f_{n+k}) \longmapsto \kappa(f_{n+k})$$

$$\kappa(P) \longmapsto \kappa(P) \qquad P \text{ decomposable monomial } \underline{\text{not}} \text{ containing } P_b$$

$$\kappa(P) \longmapsto 0 \qquad P \text{ decomposable monomial containing } P_b$$

$$\langle \kappa(f_{n+k}), z \rangle = \langle \kappa(f_{n+k}), \varphi_*(y) \rangle = \langle \varphi^*\kappa(f_{n+k}), y \rangle = \langle \kappa(f_{n+k}), y \rangle = \langle \kappa(f_{n+k}), x \rangle \neq 0$$

if P decomposable monomial not containing P_b, then:

$$\langle \kappa(P), z \rangle = \langle \kappa(P), \varphi_*(y) \rangle = \langle \varphi^*\kappa(P), y \rangle = \langle \kappa(P), y \rangle = \langle \kappa(P), x \rangle = 0$$

(since $P \neq P_a P_b$)

if P decomposable monomial containing P_b, then:

$$\langle \kappa(P), z \rangle = \langle \kappa(P), \varphi_*(y) \rangle = \langle \varphi^*\kappa(P), y \rangle = \langle 0, y \rangle = 0 .$$

Next: How to construct $BEmb_\partial(W_{g,1}^*, W_{g,1}^*)$.

φ

Where $\kappa(\ell_{n+k})$ comes from

Lemma: $BDiff_\partial(W_{g,1}^*) \xrightarrow{\text{1-pt-compactify}} BHomeo_\partial(W_{g,1}) \longrightarrow BhAut_\partial(W_{g,1})$

on H^*: $\kappa(\ell_{n+k}) \longleftarrow \bar{\kappa}(\ell_{n+k})$

Idea of proof — Use the signature theorem.

For any M^{2n+4k} ($TM \cong$ trivial) and $E^{4n+4k} \xrightarrow{\pi} M \xrightarrow{f} BDiff_\partial(W_{g,1}^*)$

$$\begin{aligned} \langle \kappa(\ell_{n+k}), f_*[M] \rangle &= \langle \ell_{n+k}(T^\vee E), [E] \rangle && \text{(naturality \& construction of } \kappa) \\ &= \langle \ell_{n+k}(TE), [E] \rangle && (TE \cong T^\vee E \oplus \pi^* TM) \\ &= \sigma(E) && \text{(Hirzebruch signature thm)} \end{aligned}$$

Define $\bar{\kappa}(\ell_{n+k}) \in H^*(BhAut_\partial(W_{g,1})) \cong Hom\left(H_*(BhAut_\partial(W_{g,1})), \mathbb{R}\right)$

by $f_*[M] \longmapsto \sigma(E)$

where $E \to M^{2n+4k} \xrightarrow{f} BhAut_\partial(W_{g,1})$

Now only a fibration with fibres $\simeq W_{g,1}$

(E is not generally a manifold!)

E still has Poincaré duality of formal dimension $4n+4k$

so $H^{2n+2k}(E)$ has a symmetric bilinear form

so $\sigma(E)$ still makes sense.

Where $\kappa(P)$ comes from (P decomposable monomial):

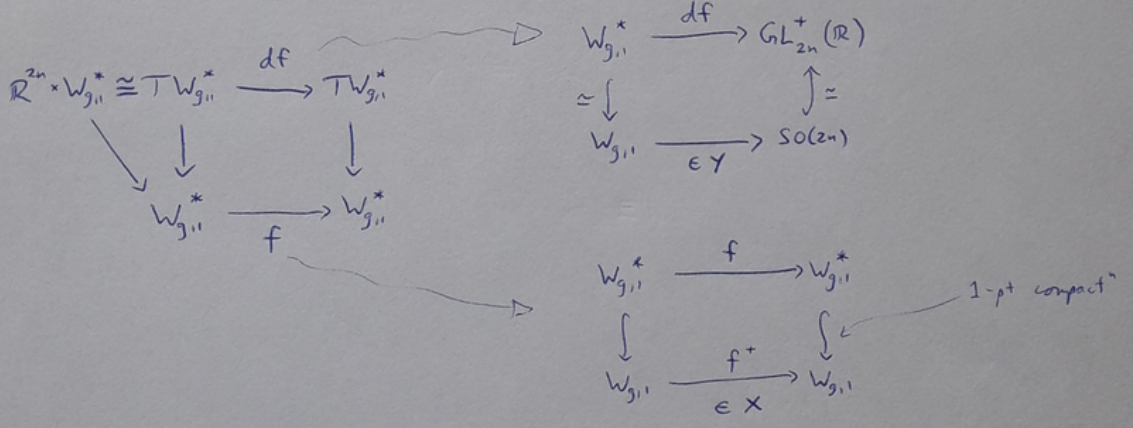
Topological monoids $X = hAut_\partial(W_{g,1})$ composition

$Y = Map_*(W_{g,1}, SO(2n))$ pointwise mult^n in $SO(2n)$

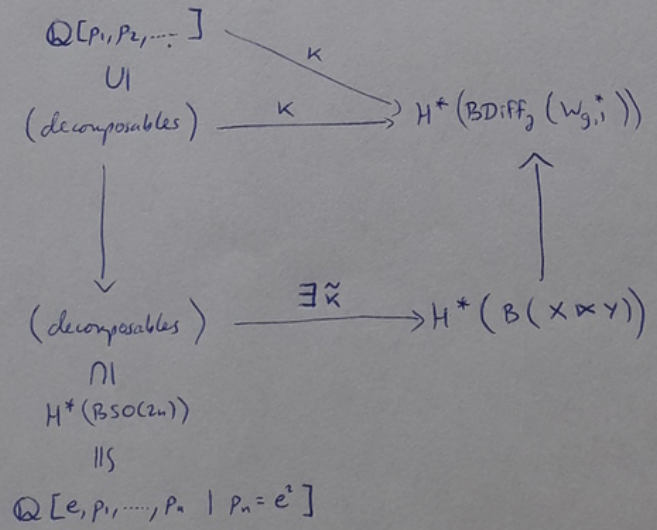
(actually a group)

X acts on Y $\longrightarrow$ semi-direct product $X \ltimes Y$

"Linearisation": $Diff_\partial(W_{g,1}^*) \longrightarrow X \ltimes Y$



Lemma



$$\kappa(f_{\kappa\circ\kappa}) \longleftarrow\!\shortmid\; \tilde{\kappa}(f_{\kappa\circ\kappa})$$

$$\begin{array}{ccc} \kappa(P) & BDiff_\partial(W_{g,1}^1) & \longrightarrow BhAut_\partial(W_{g,1}) = BX \\ \uparrow & \downarrow & \\ \tilde{\kappa}(P) & B(X \ltimes Y) & \end{array}$$

$X = hAut_\partial(W_{g,1})$

$Y = Map_*(W_{g,1}, SO(2n))$

P decomposable

Idea: Build the endomorphism φ of $BDiff_\partial(W_{g,1}^1)$

via · id on $BhAut_\partial(W_{g,1}) = BX$

· something non-trivial on $B(X \ltimes Y)$.

Manifold calculus

1st formulation [Goodwillie-Weiss '99]

$$\begin{array}{ccc} & \vdots & \\ & \downarrow f_4 & \\ & T_3 Emb_\partial(M,N) \longleftarrow\!\shortmid\; fib(f_3) = L_3 Emb_\partial(M,N) & \\ & \downarrow f_3 & \\ & T_2 Emb_\partial(M,N) \longleftarrow\!\shortmid\; fib(f_2) = L_2 Emb_\partial(M,N) & \\ & \downarrow f_2 & \\ Emb_\partial(M,N) & \longrightarrow T_1 Emb_\partial(M,N) & \end{array}$$

Analogy $T_i \mathrm{Emb}_d(-,N)$ is the "i^{th} Taylor approximation" of the functor $\mathrm{Emb}_d(-,N) : \left\{ \begin{matrix} \text{smooth } d\text{-manifolds} \\ \text{open embeddings} \end{matrix} \right\}^{op} \longrightarrow \text{Spaces}$

Prop [Weiss] $T_1 \mathrm{Emb}_d(M,N) \simeq \mathrm{Imm}_d(M,N) \overset{h\text{-principle}}{\simeq} f\mathrm{Imm}_d(M,N)$

(immersions ; (formal immersions

$$\begin{array}{ccc} TM & \xrightarrow{\varphi} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

φ linear & injective on fibres

Thm [Weiss] $L_i \mathrm{Emb}_d(M,N) \simeq \Gamma_d \left(\begin{array}{c} Z_k \\ \downarrow \\ \tilde{C}_k(M) \end{array} \right)$ — a certain fibration depending on N.

$\pi_*(-)$ computable

Rmk Everything works similarly for any "good" functor $\left\{ \begin{matrix} \text{smooth } d\text{-manifolds} \\ \text{open embeddings} \end{matrix} \right\}^{op} \longrightarrow \text{Spaces}$

Thm [Goodwillie-Klein-Weiss '15] ("Convergence")

If $\dim(N) - \underbrace{\mathrm{hdim}}_{\text{handle dimension}}(M) \geq 3$

then $\mathrm{Emb}_d(M,N) \longrightarrow \operatorname{holim}_{i \to \infty} T_i \mathrm{Emb}_d(M,N) =: T_\infty \mathrm{Emb}_d(M,N)$

is a homotopy equivalence.

2nd formulation [Boavida de Brito - Weiss '18]

Instead of looking at $L_i \mathrm{Emb}_\partial(M,N) := \mathrm{hofib}\left(T_i \to T_{i-1}\right)$,

try to understand $\boxed{\mathrm{hofib}\left(T_i \to T_1\right)}$ for all i.

Thm [Boavida-de-Brito – Weiss]

There is a homotopy pullback square

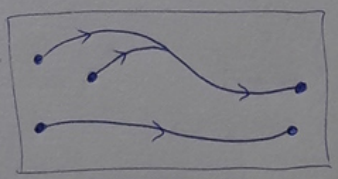
$$\begin{array}{ccc} T_i \mathrm{Emb}_\partial(M,N) & \longrightarrow & \mathrm{Map}_\partial\left(\mathrm{con}(M;i),\ \mathrm{con}(N;i)\right) \\ \downarrow & & \downarrow (*) \\ T_1 \mathrm{Emb}_\partial(M,N) & \longrightarrow & \mathrm{Map}_\partial\left(\mathrm{con}^{\mathrm{loc}}(M;i),\ \mathrm{con}^{\mathrm{loc}}(N;i)\right) \end{array}$$

so $\boxed{\mathrm{hofib}\left(T_i \to T_1\right) \simeq \mathrm{hofib}((*))}$

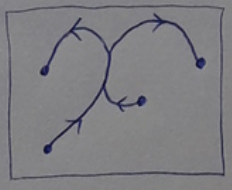
$\mathrm{con}(M;i)$ = "configuration category" of M

= [objects : ordered configurations of $\leq i$ points in M

morphisms : paths of configurations where points may collide, but NOT anticollide

Eg  ✓

$\left(\mathrm{con}^{\mathrm{loc}}(M;i)\right.$ = "local" version of $\left.\mathrm{con}(M;i)\right)$

 ×

Now set $M = N = W^*_{g,1}$

$\dim = 2n$

$\underline{\text{hdim}} = n$

handle dimension rel. ∂

$$\text{hdim}(M) = \min_{\substack{\mathcal{H} \text{ handle decomp' of } M \text{ rel. } \partial}} \left\{ \max \{ \text{index}(H) \mid H \text{ handle in } \mathcal{H} \} \right\}$$

For $M = W^*_{g,1}$: start with $\partial W^*_{g,1} \cong \mathbb{R}^{2n}$, attach $2g$ n-handles — $\dim - \text{hdim} = n$

For $M = W_{g,1}$: start with $\partial W_{g,1} \cong S^{2n}$, attach $2g$ n-handles, attach one $2n$-handle — $\dim - \text{hdim} = 0$

So we may apply manifold calculus to $\text{Emb}_\partial(W^*_{g,1}, W^*_{g,1})$ if $n \geq 3$.

$T_i \text{Emb}_\partial(W^*_{g,1}, W^*_{g,1}) \to \text{Map}_\partial(\text{con}(W^*_{g,1}; i), \text{con}(W^*_{g,1}; i))$ — isom. on π_* for $* \leq (n-2)i - n + 2$ [Goodwillie–Klein–Weiss]

$\text{Emb}_\partial(W^*_{g,1}, W^*_{g,1})$ is $\text{Diff}_\partial(W^*_{g,1})$

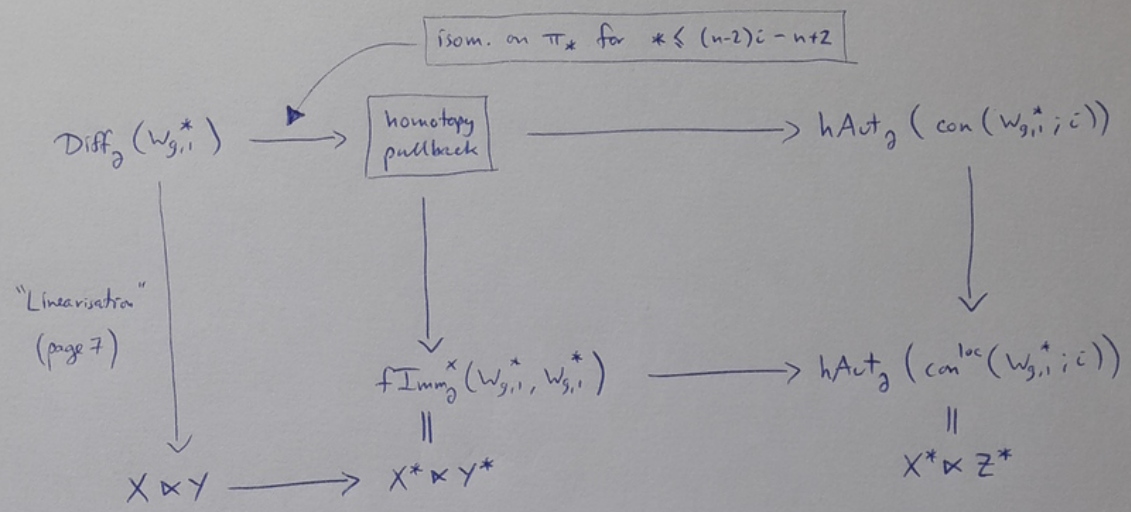
htpy pullback

$T_1 \text{Emb}_\partial(W^*_{g,1}, W^*_{g,1}) \to \text{Map}_\partial(\text{con}^{\text{loc}}(W^*_{g,1}; i), \text{con}^{\text{loc}}(W^*_{g,1}; i))$

$T_1 \text{Emb}_\partial(W^*_{g,1}, W^*_{g,1})$ is $\text{fImm}_\partial(W^*_{g,1}, W^*_{g,1})$

These are all topological monoids (or at least A_∞-spaces).

Restrict to homotopy-invertible path-components:



$X = h\mathrm{Aut}_\partial(W_{g,1})$

$X^* = h\mathrm{Aut}_\partial(W_{g,1}^*)$

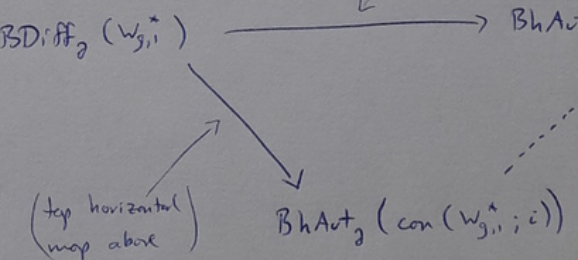
$Y = \mathrm{Map}_*(W_{g,1}, SO(2n))$

$Y^* = \mathrm{Map}_*(W_{g,1}^*, SO(2n))$

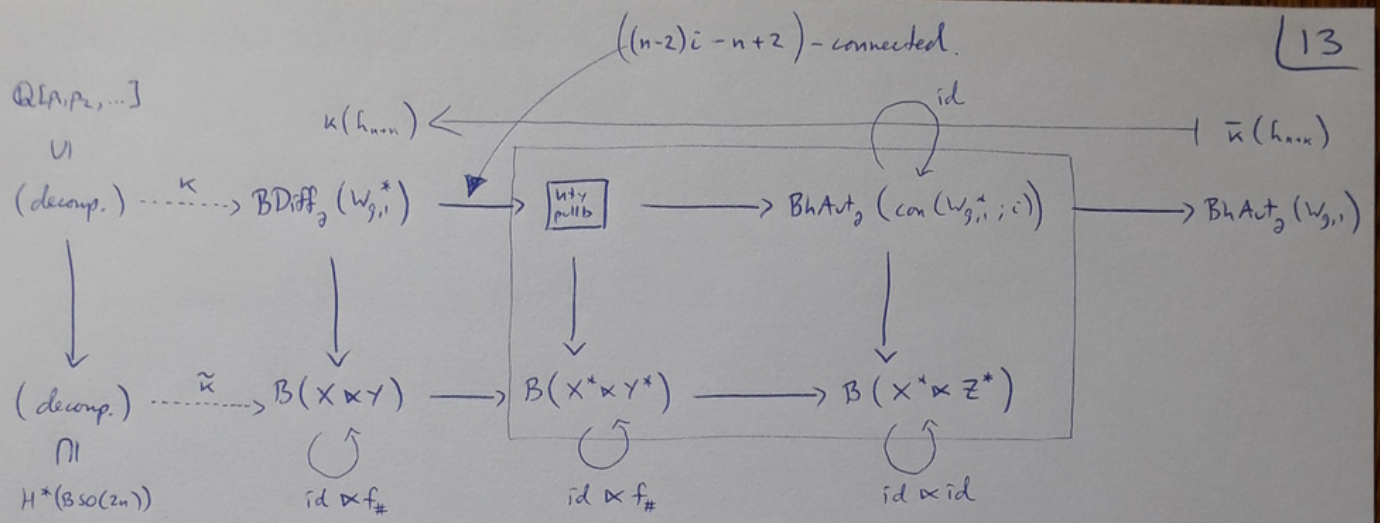
$Z^* = \mathrm{Map}_*(W_{g,1}^*, h\mathrm{Aut}(\mathrm{con}(\mathbb{R}^{2n}; i)))$

Thm [S. Tillmann-Weiss; Weiss]

The map $B\mathrm{Diff}_\partial(W_{g,1}^*) \to B h\mathrm{Aut}_\partial(W_{g,1})$ (page 6) factors like:



(after killing the homotopy groups of $B h\mathrm{Aut}_\partial(W_{g,1})$ in degrees $\geq (n-2)i - n$).



$X^* = hAut_\partial(W_{g,1}^*)$

$Y^* = Map_*(W_{g,1}^*, SO(2n))$

$Z^* = Map_*(W_{g,1}^*, hAut(con(\mathbb{R}^{2n}; i)))$

- Localise the whole diagram at $\mathbb{Q}$.
- Fact: $BSO(2n)_{\mathbb{Q}} \simeq K(\mathbb{Q}, 2n) \times \prod_{i=1}^{n-1} K(\mathbb{Q}, 4i)$ — (e), (p_i)

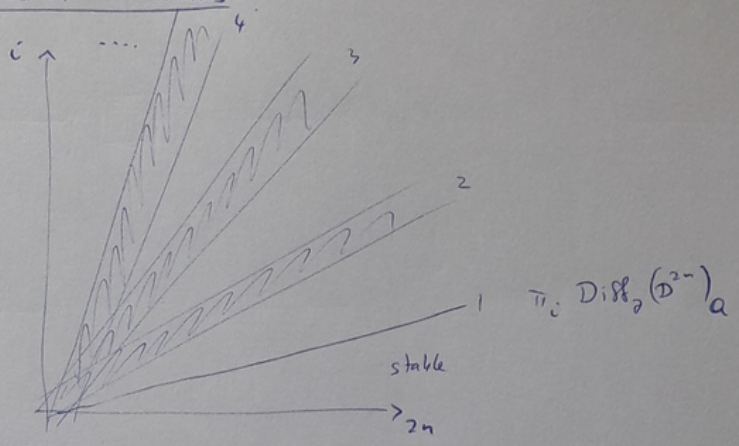
$\frac{n+1}{4} \leq a, b \leq n.$

- Choose $f: BSO(2n)_{\mathbb{Q}} \longrightarrow BSO(2n)_{\mathbb{Q}}$ — (p_b)
  - nullhomotopic on the $K(\mathbb{Q}, 4b)$ factor
  - identity on all other factors
- Then $f^*: e \mapsto e$, $p_j \mapsto p_j$ $(j \neq b)$, $p_b \mapsto 0$
- Technical theorem [Weiss]

$$\begin{array}{ccc} BSO(2n)_{\mathbb{Q}} & \longrightarrow & BhAut(con(\mathbb{R}^{2n}; i))_{\mathbb{Q}} \\ f \downarrow & \nearrow & \\ BSO(2n)_{\mathbb{Q}} & & \end{array}$$

commutes in degrees $\leq 8n$ (i.e. if we kill hty groups above degree $8n$). □

The work of Kupers - Randal - Williams

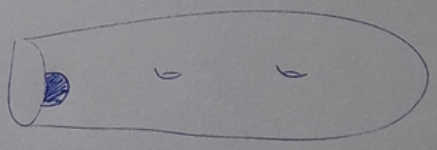


Thm [KRW20] Above the stable range & outside of these "bands", $\pi_* \mathrm{Diff}_\partial(D^{2n})_Q$ is given precisely by the (duals of) exotic Pontryagin classes.

Idea:

Weiss fibre sequence

$$\mathrm{Diff}_\partial(D^{2n}) \longrightarrow \mathrm{Diff}_\partial(W_{g,1}) \longrightarrow \mathrm{Emb}_\partial(W_{g,1}^*, W_{g,1}^*)$$



(1) Calculate $\pi_* L_i \mathrm{Emb}_\partial(W_{g,1}^*, W_{g,1}^*)$ explicitly

Emb. Calc $\Rightarrow$ $\pi_* \mathrm{Emb}_\partial(W_{g,1}^\epsilon, W_{g,1}^\epsilon)$ is concentrated in the "bands"

(2) [Galatius - Randal-Williams]: $H^*(B\mathrm{Diff}_\partial(W_{g,1}))$ well-understood.

X htpy associative H-space $\longrightarrow$ [Cartan-Serre]

$$\pi_*(X)_{\mathbb{Q}} \cong PH_*(X)$$

[Milnor-Moore]

$$H_*(X) \cong \mathcal{U}(PH_*(X)) \cong \mathcal{U}(\pi_*(X)_{\mathbb{Q}})$$

X nilpotent $\longrightarrow$ unstable Adams SS

$$(\text{gen. } H_*) \Rightarrow \pi_*$$

But $BDiff_2(W_{g,1})$ is not nilpotent!

$\pi_1(-) \cong \pi_0 Diff_2(W_{g,1}) \xrightarrow[\text{action on } H_n]{} \text{f.i. subgroup of } \begin{cases} Sp_{2g}(\mathbb{Z}) & n \text{ odd} \\ O_{gg}(\mathbb{Z}) & n \text{ even} \end{cases}$

Let $Tor_2(W_{g,1}) := \ker\left(Diff_2(W_{g,1}) \longrightarrow Aut(H_n(-))\right)$

Then $BTor_2(W_{g,1}) \longrightarrow BDiff_2(W_{g,1})$ is a covering space, so $\pi_{\geq 2}(-)$ agree.

Thm [KRW]
- $BTor_2(W_{g,1})$ is nilpotent
- analogue of [GRW] for $H^*(BTor_2(W_{g,1}))$ (hom stab' + stable H_*)
  - concentrated in "bands"
  - $\hookrightarrow$ exotic P. classes.

unstable Adams SS $\Rightarrow$ $\square$.