Moduli spaces of high-dimensional discs

after Watanabe, Weiss, Kupers-Randal-Williams

GeMAT seminar
IMAR
4 June 2021

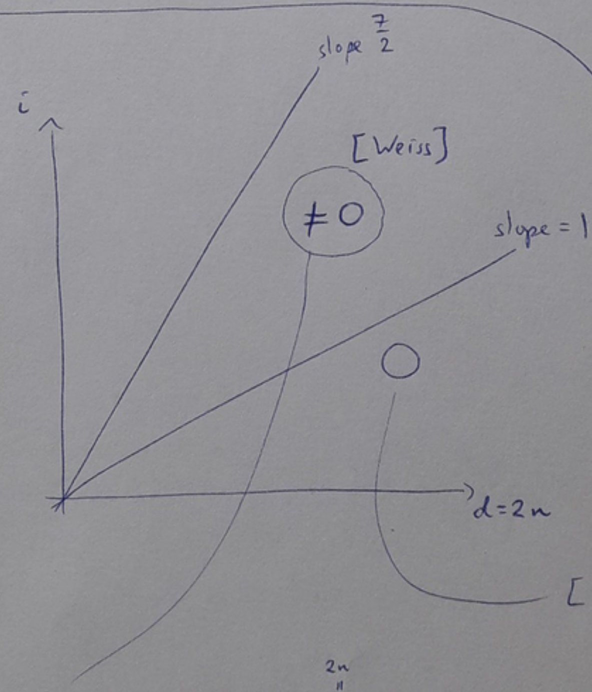
IV. Even-dimensional discs, Exotic Pontrjagin classes

Our general aim: understand $\pi_*(\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q}$.

Last two talks $\xrightarrow{d=2n+1}$ non-trivial elements detected by Kontsevich char. classes [Watanabe]

Today + next week $\xrightarrow{d=2n}$ non-trivial elements detected by "exotic Pontrjagin classes"

(Weiss char. classes)
[Weiss]
[Kupers-Randal-Williams]



$\pi_i(\mathrm{Diff}_\partial(D^{2n})) \otimes \mathbb{Q}$

[Farrell-Hsiang, Igusa, Randal-Williams]

[Weiss] $\exists$ non-0 element in bidegree $(\overset{2n}{\overset{\shortparallel}{d}}, i)$

whenever $i \equiv 2n-2 \pmod 4$.

(in this range)

Classifying spaces

G topological group

Thm [Milnor '56 explicit construction]
[Brown '62 categorical proof]

$\exists$ space BG such that

$$\left\{ \begin{array}{c} \text{principal } G\text{-bundles over } X \\ (\text{trivialised over } * \in X) \end{array} \right\} \Big/ \cong \quad \cong \quad [X, BG] = \mathrm{Maps}_*(X, BG) \big/ \text{htpy}$$

for all (paracompact) X, functorially in X.

Rmk
- BG is unique up to htpy equivalence
- $[BG, BG] \ni \mathrm{id} \longleftrightarrow \begin{array}{c} EG \\ \downarrow \\ BG \end{array}$ "universal" principal G-bundle.

Eg
- $BSO(n) \simeq Gr_n^+(\mathbb{R}^\infty)$
- $BG \simeq K(G, 1)$ if $\underbrace{G \text{ is discrete}}$

principal G-bundle $\equiv$ regular covering with $\mathrm{Deck} = G$.

Def F G-space

An ***F*-bundle** is a locally trivial fibration $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ with
- fibres $\cong F$
- transition maps given by $G \curvearrowright F$

(principal G-bundle $\longleftrightarrow$ $G \curvearrowright G = F$)
left mult^n

Eg

$\mathrm{Homeo}(F) \curvearrowright F$ ———— locally trivial fibrations

$GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$ ———— real vector bundles

$\mathrm{Diff}(M) \curvearrowright M$ ———— smooth M-bundles

Lemma By "replacing the fibre" we have

$$\{F\text{-bundles over } X\}/_{\cong} \quad \cong \quad \{\text{principal } G\text{-bundles over } X\}/_{\cong}$$

Hence both of these are classified by $[X, BG]$.

Coro For any G-space F,

$$\{\text{char. classes of } F\text{-bundles}\} \cong H^*(BG)$$

$$\text{``}\!\!=\!\!\text{''} \quad \left[\begin{pmatrix} F \to E \\ \downarrow \\ X \end{pmatrix} \longmapsto c(E) \in H^*(X) \right. \quad \left. \text{stable under pullback} \right]$$

Eg char. classes of

- real vector bundles (rank n) $\cong H^*(BGL_n(\mathbb{R}))$

  $\cong H^*(BO(n))$

  $O(n) \xhookrightarrow{\simeq} GL_n(\mathbb{R})$

- oriented real v. bundles (rank n) $\cong H^*(BSO(n))$
- smooth M-bundles $\cong H^*(B\mathrm{Diff}(M))$
- locally trivial fibrations with fibres $\cong F$ $\cong H^*(B\mathrm{Homeo}(F))$.

We will use the fact that the construction $B(-)$ can be applied also to <u>topological monoids</u> $\mathcal{M}$.

In general, $[X, B\mathcal{M}]$ does not have such an obvious geometric interpretation as $[X, BG] = \{\text{prin } G\text{-bundles} / X\}$.

But one important special case is:

<u>Eg</u> $\quad \mathcal{M} = hAut(F) := \{ f \in Map(F,F) \mid f \text{ has a hty inverse} \}$

$$[X, BhAut(F)] \cong \{\text{fibrations over } X \text{ with fibres} \simeq F\}/_{\cong}$$

not necessarily locally trivial, just satisfy some homotopy lifting property

e.g. 

So for a smooth manifold M we have:

$$Diff(M) \hookrightarrow Homeo(M) \hookrightarrow hAut(M)$$

$$\begin{tikzcd}
BDiff(M) \arrow[r] & BHomeo(M) \arrow[r] & BhAut(M) \\
& X \arrow[ul, dashed, "\substack{\text{smooth}\\ M\text{-bundles}}"] \arrow[u, dashed, "\substack{\text{locally trivial}\\ \text{fibrations}\\ \text{fibre} \cong M}"'] \arrow[ur, dashed, "\substack{\text{fibrations}\\ \text{fibre} \simeq M}"'] &
\end{tikzcd}$$

Pontrjagin classes, signature theorem

<u>Notation:</u> H^* means $H^*(-;\mathbb{Q})$ unless otherwise specified.

<u>Fact</u> (e.g. Milnor - Stasheff):

$$H^*(BO(2n)) \cong \mathbb{Q}[p_1, p_2, \dots, p_n] \qquad \text{polynomial algebra}$$

$$\deg(p_i) = 4i.$$

$$BO := \operatorname{colim}_n (BO(n))$$

$$\begin{tikzcd}
BO & H^*(BO) \arrow[d] & \cong & \mathbb{Q}[p_1, p_2, p_3, \dots] \arrow[d] \\
BO(2n) \arrow[u, hook] & H^*(BO(2n)) & \cong & \mathbb{Q}[p_1, \dots, p_n]
\end{tikzcd}$$

<u>Signature theorem (Hirzebruch):</u>

$\exists\ \mathcal{L}_n \in \mathbb{Q}[p_1, p_2, p_3, \dots]$ of homogeneous degree $4n$

such that if M is a smooth, closed, oriented $4n$-manifold

$$\text{then } \Big\langle \underbrace{\mathcal{L}_n\begin{pmatrix} TM \\ \downarrow \\ M \end{pmatrix}}_{\in H^{4n}(M)}, [M] \Big\rangle = \underbrace{\sigma(M)}_{\text{signature of the intersection form on } H^{2n}(M)}$$

<u>Eg</u> $\mathcal{L}_0 = 1$

$\mathcal{L}_1 = \frac{1}{3} p_1$

$\mathcal{L}_2 = \frac{7}{45} p_2 - \frac{1}{45} p_1^2$

....

Exotic Pontrjagin classes

$$BO := \operatorname{colim}_n (BO(n))$$

$$BTop := \operatorname{colim}_n (BHomeo(\mathbb{R}^{2n}))$$

$$\begin{tikzcd} BO(2n) \arrow[r] \arrow[d, hook] & BHomeo(\mathbb{R}^{2n}) \arrow[d, hook] \\ BO \arrow[r, "(*)"] & BTop \end{tikzcd}$$

Thm (Thom / Novikov / Kirby - Siebenmann / Sullivan)

$(*)$ induces $\cong$ on $H^*(-;\mathbb{Q})$

So we have:

$$\begin{tikzcd} \mathbb{Q}[p_1, p_2, \dots p_n] & H^*(BHomeo(\mathbb{R}^{2n})) \arrow[l] \\ \mathbb{Q}[p_1, p_2, \dots] \arrow[u, two heads] & \mathbb{Q}[p_1, p_2, \dots] \arrow[l, "\cong"'] \arrow[u] \end{tikzcd}$$

Cor $p_i \neq 0$ in $H^*(BHomeo(\mathbb{R}^{2n}))$ for $i \leq n$.

Def Exotic Pontrjagin classes are $p_i \in H^*(BHomeo(\mathbb{R}^{2n}))$ for $i > n$.

Q: Are there non-0 exotic Pontrjagin classes?

Conj. [Weiss '94] No.

Thm [Weiss '15] Yes!!

$p_i \neq 0 \in H^*(BHomeo(\mathbb{R}^{2n}))$ for all $i \leq \frac{9}{4}n$ - const.

Relation to $\mathrm{Diff}_\partial(D^d)$

- Slight extension of Weiss' theorem:

$$p_{n+k} \in H^{4n+4k}(B\mathrm{Homeo}(\mathbb{R}^{2n}))$$

For any space X there is a bilinear map

$$H^i(X) \times \pi_i(X)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

$$(\alpha, [S^i \xrightarrow{f} X]) \longmapsto \langle f^*(\alpha), [S^i] \rangle$$

Notation: $\pi_*(-)_{\mathbb{Q}} = \pi_*(-) \otimes \mathbb{Q}$

__Thm__ [Weiss '16] For all $k \leq \frac{5}{4}n - \text{const.}$, the linear map

$$\langle p_{n+k}, - \rangle : \pi_{4n+4k}(B\mathrm{Homeo}(\mathbb{R}^{2n}))_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

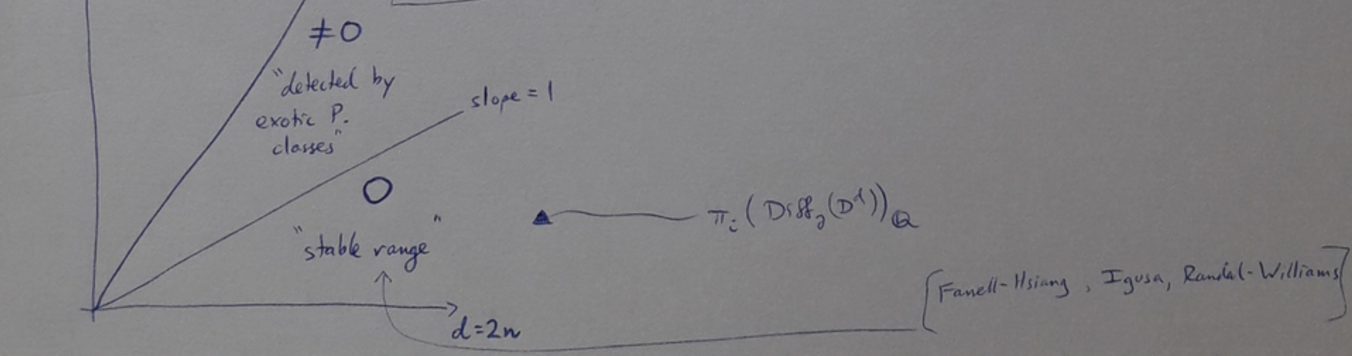
is non-zero.

__Coro__ $\pi_{4n+4k}(B\mathrm{Homeo}(\mathbb{R}^{2n}))_{\mathbb{Q}} \neq 0$ for these k.

- [Morlet] For $i > 0$, $\pi_i B\mathrm{Diff}_\partial(D^d) \cong \pi_{i+d}\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d)\right)$

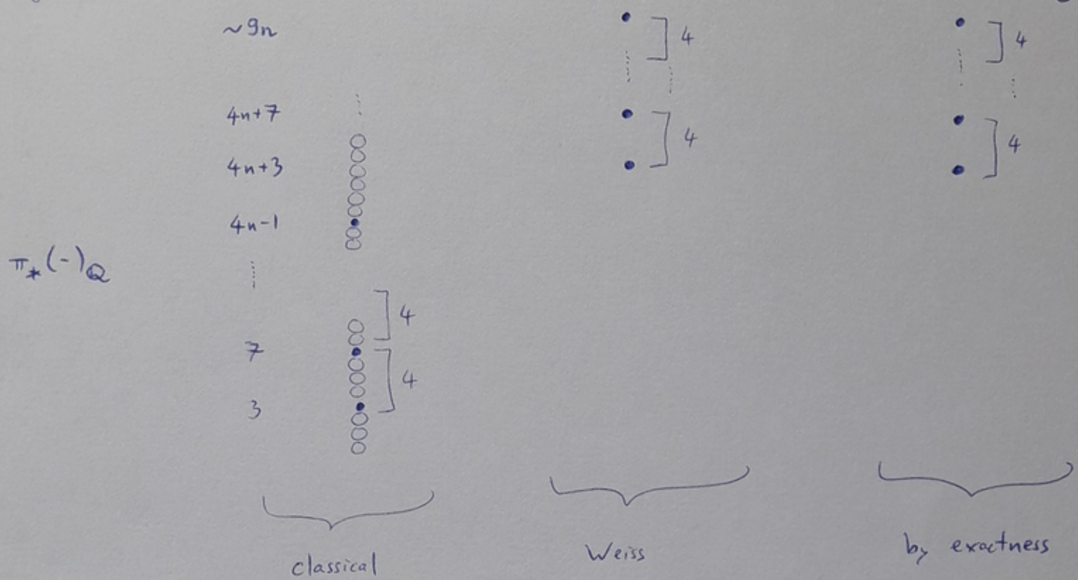
__Corollary__ [Weiss '16] For all $1 \leq k \leq \frac{5}{4}n - \text{const.}$,

$$\boxed{\pi_{2n+4k-2}(\mathrm{Diff}_\partial(D^{2n}))_{\mathbb{Q}} \neq 0}$$



proof of Corollary:

fibre bundle $\quad O(2n) \hookrightarrow \mathrm{Homeo}(\mathbb{R}^{2n}) \twoheadrightarrow \mathrm{Homeo}(\mathbb{R}^{2n})/O(2n)$



For $\quad 1 \leq k \leq \frac{5}{4}n - \text{const}$,

$$0 \neq \pi_{4n+4k-1}\left(\mathrm{Homeo}(\mathbb{R}^{2n}) / O(2n)\right) \underset{[\text{Morlet}]}{\cong} \pi_{2n+4k-1}\left(B\mathrm{Diff}_\partial(D^d)\right)$$

$$\cong \pi_{2n+4k-2}\left(\mathrm{Diff}_\partial(D^d)\right). \quad //$$

Aim: Explain Weiss' proof that

$$p_{n+k} \neq 0 \in H^{4n+4k}(B\mathrm{Homeo}(\mathbb{R}^{2n}))$$

for all $k \leq \frac{5}{4}n - \text{const}$.

κ - classes

Smooth:

$$\begin{tikzcd} \mathrm{Diff}(M) \arrow[r] & E\mathrm{Diff}(M) \arrow[d] \\ & B\mathrm{Diff}(M) \end{tikzcd}$$

universal principal $\mathrm{Diff}(M)$- bundle.

$\rightsquigarrow$ change fibres : $E_M = E\mathrm{Diff}(M) \times_{\mathrm{Diff}(M)} M$

$$\begin{tikzcd} \mathbb{R}^d \arrow[r] & T^v E_M \arrow[d] & \\ M \arrow[r] & E_M \arrow[r, "f"] \arrow[d, "\pi"] & BO(d) \\ & B\mathrm{Diff}(M) & \end{tikzcd}$$

vertical tangent bundle $\coprod_{x \in B\mathrm{Diff}(M)} T(\pi^{-1}(x))$

universal smooth M-bundle

$d = \dim(M)$.

f = map classifying the vector bundle $T^v E_M$.

$$\begin{tikzcd} H^*(BO(d)) \arrow[r, "f^*"] \arrow[rr, bend right, "\kappa"'] & H^*(E_M) \arrow[r, "\pi_!", "\int_{\text{fibre}}"'] & H^{*-d}(B\mathrm{Diff}(M)) \end{tikzcd}$$

Eg $d=2$, oriented version

$$\begin{tikzcd} H^*(BSO(2)) \arrow[r] & H^{*-2}(B\mathrm{Diff}(\Sigma)) \cong H^{*-2}(\mathcal{M}(\Sigma)) \end{tikzcd}$$

$H^*(BSO(2)) \cong \mathbb{Q}[e]$, $\deg(e) = 2$

$H^{*-2}(B\mathrm{Diff}(\Sigma)) \ni \kappa(e^i)$ degree $2i-2$

"Miller-Morita-Mumford classes"

non-smooth:

$$\begin{tikzcd} \mathbb{R}^d \arrow[r] & T^v E_M \arrow[d] & \\ M \arrow[r] & E_M \arrow[d, "\pi"'] \arrow[rr, "f"] & & B\mathrm{Homeo}(\mathbb{R}^d) \\ & B\mathrm{Homeo}(M) & \end{tikzcd}$$

universal loc. trivial fibration with fibre $\cong M$

vertical tangent microbundle $\coprod_{x \in B\mathrm{Homeo}(M)} \underbrace{T(\pi^{-1}(x))}_{\text{tangent microbundle}}$

- "microbundle" [Milnor]
- M topological manifold $\rightsquigarrow$ $\begin{matrix} TM \\ \downarrow \\ M \end{matrix}$ microbundle of rank $d = \dim(M)$
- <u>Thm</u> (Kister-Mazur) X paracompact

$$\begin{tikzcd} \left\{ \begin{matrix} \text{loc. trivial fibration over } X \\ \text{with fibre} \cong \mathbb{R}^d \end{matrix} \right\} \big/ \cong \arrow[r, "1:1"] \arrow[d, leftrightarrow, "1:1"'] & \left\{ \text{microbundles of rank } d \text{ over } X \right\} \big/ \cong \\ {[X, B\mathrm{Homeo}(\mathbb{R}^d)]} & \end{tikzcd}$$

$$\begin{tikzcd} H^*(B\mathrm{Homeo}(\mathbb{R}^d)) \arrow[r, "f^*"] \arrow[rr, bend right, "\kappa"'] & H^*(E_M) \arrow[r, "\pi_!"', "\int_{\text{fibre}}"] & H^{*-d}(B\mathrm{Homeo}(M)) \end{tikzcd}$$

The smooth & non-smooth constructions are compatible, i.e.

$$\begin{tikzcd} H^*(B\mathrm{Homeo}(\mathbb{R}^d)) \arrow[r, "\kappa"] \arrow[d] & H^{*-d}(B\mathrm{Homeo}(M)) \arrow[d] \\ H^*(BO(d)) \arrow[r, "\kappa"] & H^{*-d}(B\mathrm{Diff}(M)) \end{tikzcd}$$

commutes.

<u>Step 1</u>. Let $k \leq \frac{5}{4}n - \text{const.}$

$$W_g = \#_g (S^n \times S^n)$$

$(n=1,\ g=3)$

$$\underset{\substack{\| \\ H^*(B\mathrm{Top})}}{\mathbb{Q}[p_1, p_2, p_3, \dots]} \longrightarrow H^*(B\mathrm{Homeo}(\mathbb{R}^{2n})) \xrightarrow{\kappa} H^{*-2n}(B\mathrm{Homeo}(W_g))$$

<u>Notation</u>: $\left.\begin{array}{l} \mathcal{L}_{n+k} = \text{Hirzebruch } L\text{-polynomial} \\ P = \text{any monomial in } p_1, p_2, \dots \end{array}\right]$ in degree $4n+4k$

<u>Suppose we can find</u> $\left.\begin{array}{l} z \in H_{2n+4k}(B\mathrm{Homeo}(W_g)) \text{ such that} \\ \langle \kappa(\mathcal{L}_{n+k}), z \rangle \neq 0 \\ \langle \kappa(P), z \rangle = 0 \quad \text{whenever } P \text{ is } \underline{\text{decomposable}} \end{array}\right\} (*)$

<u>Then we would be done!</u>

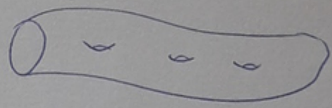
- $\mathcal{L}_{n+k} = \lambda . p_{n+k} + (\text{lin. comb}^{n} \text{ of decomposable monomials}), \quad \lambda \neq 0$

$\langle \kappa(-), z \rangle \rightsquigarrow$ • $0 \neq \lambda . \langle \kappa(p_{n+k}), z \rangle$

- $p_{n+k} \neq 0 \in H^{4n+4k}(B\mathrm{Homeo}(\mathbb{R}^{2n}))$. //

Step 2. First, construct something a bit different.

$$W_{g,1} = W_g \setminus \text{open disc}$$



$$\begin{array}{c} H^*(BO(2n)) \xrightarrow{\quad \kappa \quad} H^{*-2n}\left(BDiff_\partial(W_{g,1})\right) \\ \cong \\ \mathbb{Q}[p_1, \dots, p_n] \end{array}$$

$\boxed{\text{(!) From now on, we assume that } \underline{k \leqslant n - \text{const.}}}$

(The case $n - \text{const.} \leqslant k \leqslant \frac{5}{4}n - \text{const.}$ uses a similar trick.)

Fix integers $\frac{n+1}{4} \leqslant a, b \leqslant n-1 \quad : \quad a+b = n+k$

$$\left.\begin{array}{l} \underline{\underline{\text{There exists}}} \quad x \in H_{2n+4k}\left(BDiff_\partial(W_{g,1})\right) \quad \text{such that} \\ \langle \kappa(\mathcal{L}_{n+k}), x \rangle \neq 0 \\ \langle \kappa(P), x \rangle = 0 \quad \text{whenever } P \neq p_a p_b \end{array}\right\} (**)$$

Proof: We use the theorem of Galatius-Randal-Williams that

$$H^*(BDiff_\partial(W_{g,1})) \underset{g \gg *}{\cong} \mathbb{Q}\left[\kappa(c) \;\middle|\; \begin{array}{l} c \text{ monomial of degree} > 2n \\ \text{in } e, p_i \text{ for } \frac{n+1}{4} \leqslant i \leqslant n-1 \end{array}\right]$$

$$\kappa(p_a p_b) \in H^{2n+4k}\left(BDiff_\partial(W_{g,1})\right)$$

$$x := \kappa(p_a p_b)^* \in H_{2n+4k}\left(BDiff_\partial(W_{g,1})\right)$$

linear dual (UCThm)

$$\Rightarrow \quad \langle \kappa(p_a p_b), x \rangle = 1$$

$$\langle \kappa(P), x \rangle = 0 \qquad P \text{ monomial of degree } 4n+4k, \; P \neq p_a p_b$$

$$\mathcal{L}_{n+k} = \mu \cdot p_a p_b + (\text{lin. comb}^{\text{n}} \text{ of other monomials})$$

<u>Thm</u> (Berglund – Bergström '18)

- $\mu \neq 0$
- in fact if P is <u>any</u> monomial in $p_1, p_2, \dots$ of degree $4n+4k$, then $(\text{coeff of } P \text{ in } \mathcal{L}_{n+k}) \neq 0$.

Apply $\langle \kappa(-), x \rangle$

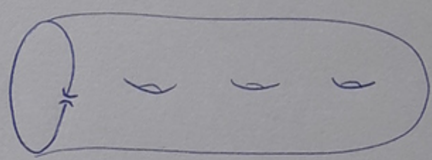
$$\rightsquigarrow \quad \langle \kappa(\mathcal{L}_{n+k}), x \rangle = \mu \cdot \langle \kappa(p_a p_b), x \rangle = \mu \neq 0. \quad //.$$

<u>Step 3</u>. (Idea)

$$W_{g,1}^* := W_{g,1} \setminus (\text{point on } \partial)$$

$$\partial W_{g,1} \cong S^{2n-1}$$

$$\partial W_{g,1}^* \cong \mathbb{R}^{2n-1}$$



Note that $(W_{g,1}^*)^+ \cong W_{g,1}$

one-point compactification.

$$\begin{tikzcd}
\mathrm{Diff}_\partial(W_{g,1}) \arrow[r, "\text{restrict}"'] & \mathrm{Diff}_\partial(W_{g,1}^*) \arrow[r, "\text{extend to 1-pt compact}^{\text{n}}"'] \arrow[rr, bend left] & \mathrm{Homeo}_\partial(W_{g,1}) \arrow[r, "\text{collapse } \partial"'] & \mathrm{Homeo}(W_g)
\end{tikzcd}$$

$$\begin{tikzcd}
W_{g,1} \arrow[r] \arrow[d, "/\partial"'] & W_{g,1} \arrow[d, "/\partial"] \\
W_g \arrow[r] & W_g
\end{tikzcd}$$

$\underline{H_*}$: $\quad x \longmapsto y \qquad \varphi_*(y) \longmapsto z$

$$\begin{tikzcd} BDiff_\partial(W_{g,1}) \arrow[r] & BDiff_\partial(W_{g,1}^*) \arrow[r] \arrow[loop below, "\varphi"] & BHomeo(W_g) \end{tikzcd}$$

endomorphism

$$\boxed{\begin{array}{l} \langle \kappa(\mathcal{L}_{n+k}), x\rangle \neq 0 \\ \langle \kappa(P), x\rangle = 0 \\ \quad \text{if } P \neq p_a p_b \end{array}} \qquad \boxed{\begin{array}{l} \langle \kappa(\mathcal{L}_{n+k}), z\rangle \neq 0 \\ \langle \kappa(P), z\rangle = 0 \\ \quad \text{if } P \neq p_{n+k} \end{array}}$$

<u>Got</u> $\qquad\qquad\qquad\qquad$ <u>Want</u>

It will be enough to construct φ so that φ^*

- preserves $\kappa(\mathcal{L}_{n+k})$
- if P is a <u>decomposable</u> monomial, then

$$\kappa(P) \longmapsto \begin{cases} 0 & P \text{ contains } p_b \\ \kappa(P) & \text{o/w} \end{cases}$$

<u>Idea</u>: $\boxed{\text{"decompose" } BDiff_\partial(W_{g,1}^*)}$ into 2 pieces, so that

- $\kappa(\mathcal{L}_{n+k})$ "lives" on one piece,
- $\kappa(P)$ [P decomposable monomial] "lives" on the other piece.

Then construct φ in pieces.......

<u>Manifold calculus</u> $\qquad$ originally : [Goodwillie - Weiss '99]

$\qquad\qquad\qquad\qquad$ reformulated : [Boavida de Brito - Weiss '18]

$\longrightarrow$ Next time .....