Moduli spaces of high-dimensional discs

after Watanabe, Weiss, Kupers–Randal-Williams

GeMAT seminar
IMAR
11 June 2021

V. Manifold calculus & exotic Pontrjagin classes

- Recap of last talk
  - $\pi_* \mathrm{Diff}_\partial(D^d)$ vs. $\pi_* B\mathrm{Homeo}(\mathbb{R}^d)$
  - exotic Pontrjagin classes
  - strategy of proof of Weiss' theorem
- Pulling back $\kappa(\mathcal{L}_{n+k})$ and $\kappa(\text{decomposable})$
- Manifold calculus
- End of the proof of Weiss' theorem
- The work of Kupers–Randal-Williams

Recap of last talk:

$$\boxed{\pi_* \mathrm{Diff}_\partial(D^d) \cong \pi_{*+d+1}\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d) \right)}$$ [Morlet]

$$\cdots \to \underbrace{\pi_{*+1} BO(d)}_{\text{vanishes for } * \geq 2d} \to \pi_{*+1} B\mathrm{Homeo}(\mathbb{R}^d) \to \pi_*\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d) \right) \to \underbrace{\pi_* BO(d)}_{\text{vanishes for } * \geq 2d} \to \cdots$$

$\otimes\mathbb{Q}$:

isomorphism in degrees $* \geq 2d$

Putting this together:

$\underbrace{\text{In degrees } * \geq d}_{\text{"above the stable range"}}$, we have $\boxed{\pi_* \mathrm{Diff}_\partial(D^d)_\mathbb{Q} \cong \pi_{*+d+2} B\mathrm{Homeo}(\mathbb{R}^d)_\mathbb{Q}}$

Now set $\boxed{d = 2n}$

$$\begin{array}{ccc} \mathbb{Q}[p_1,\dots,p_n] \cong H^*(BO(2n)) & \longleftarrow & H^*(B\mathrm{Homeo}(\mathbb{R}^{2n})) \ni \overbrace{p_1,\dots,p_n}^{\neq 0}, \overbrace{p_{n+1}, p_{n+2}, \dots}^{?} \\ \uparrow & & \uparrow \\ \mathbb{Q}[p_1,p_2,p_3,\dots] \cong H^*(BO) & \xleftarrow{\cong} & H^*(B\mathrm{Top}) \cong \mathbb{Q}[p_1,p_2,p_3,\dots] \end{array}$$

"exotic"

Thom / Novikov / Kirby-Siebenmann / Sullivan

<u>Theorem</u> [Weiss '15] (1) $p_{n+k} \neq 0 \in H^{4n+4k}(B\mathrm{Homeo}(\mathbb{R}^{2n}))$ $\boxed{\text{for all } k \lesssim \tfrac{5}{4}n}$

(2) Moreover, the linear map $\pi_{4n+4k} B\mathrm{Homeo}(\mathbb{R}^{2n})_\mathbb{Q} \longrightarrow \mathbb{Q}$

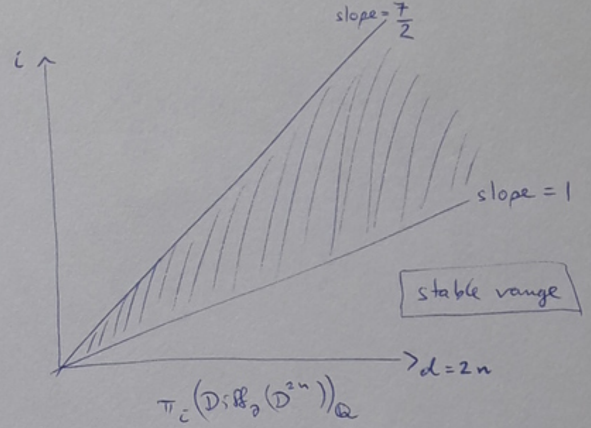
$$[f : S^{4n+4k} \to B\mathrm{Homeo}(\mathbb{R}^{2n})] \longmapsto \int_{S^{4n+4k}} f^*(p_{n+k})$$

is <u>non-zero</u>.

<u>Note:</u> (2) $\Rightarrow$ $\pi_{2n+4k-2}(\mathrm{Diff}_\partial(D^{2n}))_{\mathbb{Q}} \cong \pi_{4n+4k}(B\mathrm{Homeo}(\mathbb{R}^{2n}))_{\mathbb{Q}} \neq 0$

for $k \lesssim \frac{5}{4} n$

i.e. $2n + 4k - 2 \lesssim 7n$



<u>Aim:</u> Understand the strategy to prove that $p_{n+k} \neq 0 \in H^*(B\mathrm{Homeo}(\mathbb{R}^{2n}))$ in the slightly smaller range $k \lesssim n$.

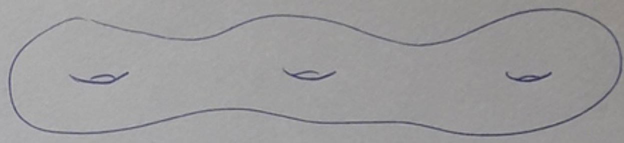
$$\begin{tikzcd}
 & H^*(B\mathrm{Homeo}(\mathbb{R}^{2n})) \arrow[r, "\kappa"] \arrow[d] & H^{*-2n}(B\mathrm{Homeo}(M)) \arrow[d] \\
\mathbb{Q}[p_1, p_2, \dots] \arrow[ur] \arrow[dr] & & \\
 & H^*(BO(2n)) \arrow[r, "\kappa"] & H^{*-2n}(B\mathrm{Diff}(M))
\end{tikzcd}$$

$\mathcal{L}_d \in \mathbb{Q}[p_1, p_2, \dots]$ Hirzebruch L-polynomial of homogeneous degree $= 4d$

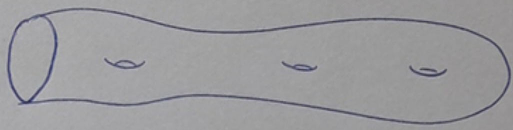
[Hirzebruch] • If $\dim(M) = 4d$ (closed, oriented) $\langle \mathcal{L}_d(TM), [M] \rangle =$ signature of M

[Berglund-Bergström '18] • If P is any <u>monomial</u> of degree $4d$ then the coefficient of P in $\mathcal{L}_d$ is $\neq 0$.

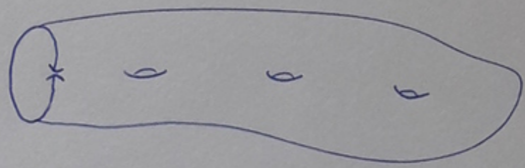
$W_g = \#_g (S^n \times S^n)$



$W_{g,1} = W_g \setminus \mathrm{int}(D^{2n})$



$W_{g,1}^* = W_{g,1} \setminus (\text{one point on } \partial)$



$\boxed{\text{Fix } \frac{n+1}{4} \leq a,b \leq n \;:\; a+b = n+k}$ possible since $k \leq n$

$$\boxed{B\mathrm{Diff}_\partial(W_{g,1}) \xrightarrow{\text{restrict}} B\mathrm{Diff}_\partial(W_{g,1}^*) \xrightarrow{\text{1-point compactify, collapse } \partial} B\mathrm{Homeo}(W_g)}$$

$x \in H_*$

$\langle \kappa(\mathcal{L}_{n+k}), x \rangle \neq 0$
$\langle \kappa(P), x \rangle = 0$ unless $P = p_a p_b$

$\dashrightarrow$?

$z \in H_*$

$\langle \kappa(\mathcal{L}_{n+k}), z \rangle \neq 0$
$\langle \kappa(P), z \rangle = 0$ unless $P = p_{n+k}$

$\Uparrow$ $x = \kappa(p_a p_b)^*$

Galatius–Randal-Williams

for $g \geq 2* + 3$,

$H^* \cong \mathbb{Q}[\kappa(c) \mid c \text{ certain monomials in } e, p_i \;\; \frac{n+1}{4} \leq i \leq n]$

higher-dimensional Madsen-Weiss theorem
needs $2n \geq 6$

$\Downarrow$

write $\mathcal{L}_{n+k} = \lambda \cdot p_{n+k} + \left(\text{lin. comb. of decomposable monomials}\right)$

$0 \neq \langle \kappa(\mathcal{L}_{n+k}), z \rangle = \lambda \cdot \langle \kappa(p_{n+k}), z \rangle$

$\Downarrow$

$p_{n+k} \neq 0$

$$x \longmapsto y \longmapsto \varphi_*(y) \longmapsto z$$

$$\begin{tikzcd}
B\mathrm{Diff}_\partial(W_{g,1}) \arrow[r] & B\mathrm{Diff}_\partial(W_{g,1}^*) \arrow[r] \arrow[d, "\simeq"] & B\mathrm{Homeo}(W_g) \\
& B\mathrm{Emb}_\partial(W_{g,1}^*, W_{g,1}^*) \arrow[loop below, "\varphi"] &
\end{tikzcd}$$

Lemma

φ^* : $\kappa(\mathcal{L}_{n+k}) \longmapsto \kappa(\mathcal{L}_{n+k})$

$\kappa(P) \longmapsto \kappa(P)$ P decomposable monomial <u>not</u> containing p_b

$\kappa(P) \longmapsto 0$ P decomposable monomial containing p_b

$$\langle \kappa(\mathcal{L}_{n+k}), z \rangle = \langle \kappa(\mathcal{L}_{n+k}), \varphi_*(y) \rangle = \langle \varphi^*\kappa(\mathcal{L}_{n+k}), y \rangle = \langle \kappa(\mathcal{L}_{n+k}), y \rangle = \langle \kappa(\mathcal{L}_{n+k}), x \rangle \neq 0$$

if P decomposable monomial <u>not</u> containing p_b, then:

$$\langle \kappa(P), z \rangle = \langle \kappa(P), \varphi_*(y) \rangle = \langle \varphi^*\kappa(P), y \rangle = \langle \kappa(P), y \rangle = \langle \kappa(P), x \rangle = 0$$

since $P \neq p_a p_b$

if P decomposable monomial containing p_b, then:

$$\langle \kappa(P), z \rangle = \langle \kappa(P), \varphi_*(y) \rangle = \langle \varphi^*\kappa(P), y \rangle = \langle 0, y \rangle = 0 .$$

<u>Next:</u> How to construct $\begin{tikzcd} B\mathrm{Emb}_\partial(W_{g,1}^*, W_{g,1}^*) \arrow[loop below, "\varphi"] \end{tikzcd}$.

Where $\kappa(\mathcal{L}_{n+k})$ comes from

Lemma:
$$BDiff_\partial(W_{g,1}^*) \xrightarrow[\text{1-pt-compactify}]{\sim} BHomeo_\partial(W_{g,1}) \longrightarrow BhAut_\partial(W_{g,1})$$

$\boxed{\text{on } H^*:}$ $$\kappa(\mathcal{L}_{n+k}) \longleftarrow\!\!\shortmid \bar{\kappa}(\mathcal{L}_{n+k})$$

Idea of proof — Use the signature theorem.

For any M^{2n+4k} ($TM \cong_s$ trivial) and
$$\begin{array}{l} E^{4n+4k} \\ \pi\downarrow \\ M \xrightarrow{f} BDiff_\partial(W_{g,1}^*) \end{array}$$

$$\begin{aligned}
\langle \kappa(\mathcal{L}_{n+k}), f_*[M] \rangle &= \langle \mathcal{L}_{n+k}(T^vE), [E] \rangle && \text{(naturality \& construction of } \kappa) \\
&= \langle \mathcal{L}_{n+k}(TE), [E] \rangle && (TE \cong T^vE \oplus \pi^*TM) \\
&= \sigma(E) && \text{(Hirzebruch signature thm)}
\end{aligned}$$

Define $\bar{\kappa}(\mathcal{L}_{n+k}) \in H^*(BhAut_\partial(W_{g,1})) \cong Hom\Big(H_*(BhAut_\partial(W_{g,1})), \mathbb{Q}\Big)$

by $$f_*[M] \longmapsto \sigma(E)$$

where
$$\begin{array}{l} E \\ \downarrow \\ M^{2n+4k} \xrightarrow{f} BhAut_\partial(W_{g,1}) \end{array}$$

Now only a fibration with fibres $\simeq W_{g,1}$
(E is not generally a manifold!)

E still has Poincaré duality of formal dimension $4n+4k$
so $H^{2n+2k}(E)$ has a symmetric bilinear form
so $\sigma(E)$ still makes sense.

//

Where $\kappa(P)$ comes from (P decomposable monomial) :

Topological monoids $X = hAut_\partial(W_{g,1})$ composition

$Y = Map_*(W_{g,1}, SO(2n))$ pointwise mult^n in $SO(2n)$

(actually a group)

X acts on Y $\rightsquigarrow$ semi-direct product $X \ltimes Y$

"Linearisation" : $Diff_\partial(W_{g,1}^*) \longrightarrow X \ltimes Y$

$$\begin{tikzcd}
\mathbb{R}^{2n} \times W_{g,1}^* \cong TW_{g,1}^* \arrow[r, "df"] \arrow[d] \arrow[dr] & TW_{g,1}^* \arrow[d] \\
W_{g,1}^* \arrow[r, "f"'] & W_{g,1}^*
\end{tikzcd}$$

$$\begin{tikzcd}
W_{g,1}^* \arrow[r, "df"] \arrow[d, "\simeq"'] & GL_{2n}^+(\mathbb{R}) \\
W_{g,1} \arrow[r, "\in Y"'] & SO(2n) \arrow[u, "\simeq"']
\end{tikzcd}$$

$$\begin{tikzcd}
W_{g,1}^* \arrow[r, "f"] \arrow[d, hook] & W_{g,1}^* \arrow[d, hook] \\
W_{g,1} \arrow[r, "f^+"', "\in X"] & W_{g,1}
\end{tikzcd}$$

1-pt compact^n

Lemma

$$\begin{tikzcd}
\mathbb{Q}[p_1, p_2, \ldots] \arrow[drr, "\kappa"] & & \\
\cup\!\!\!| & & \\[-2em]
(\text{decomposables}) \arrow[rr, "\kappa"] \arrow[d] & & H^*(BDiff_\partial(W_{g,1}^*)) \\
(\text{decomposables}) \arrow[rr, "\exists \tilde{\kappa}"] & & H^*(B(X \ltimes Y)) \arrow[u]
\end{tikzcd}$$

$$\begin{gathered}
(\text{decomposables}) \\
\cap\!\!\!| \\
H^*(BSO(2n)) \\
\cong \\
\mathbb{Q}[e, p_1, \ldots, p_n \mid p_n = e^2]
\end{gathered}$$

$$\kappa(\mathcal{L}_{n+k}) \longleftarrow\!\shortmid \tilde{\kappa}(\mathcal{L}_{n+k})$$

$$\begin{tikzcd}
\kappa(P) & B\mathrm{Diff}_\partial(W_{g,1}^*) \arrow[r] \arrow[d] & Bh\mathrm{Aut}_\partial(W_{g,1}) = BX \\
\tilde{\kappa}(P) \arrow[u, mapsto] & B(X \ltimes Y) &
\end{tikzcd}$$

$X = h\mathrm{Aut}_\partial(W_{g,1})$

$Y = \mathrm{Map}_*(W_{g,1}, SO(2n))$

P decomposable

Idea: Build the endomorphism φ of $B\mathrm{Diff}_\partial(W_{g,1}^*)$

via
- id on $Bh\mathrm{Aut}_\partial(W_{g,1}) = BX$
- something non-trivial on $B(X \ltimes Y)$.

Manifold calculus

1^{st} formulation [Goodwillie–Weiss '99]

$$\begin{tikzcd}
& \vdots \arrow[d, "f_4"] & \\
& T_3\mathrm{Emb}_\partial(M,N) \arrow[d, "f_3"] & \mathrm{fib}(f_3) = L_3\mathrm{Emb}_\partial(M,N) \arrow[l, hook'] \\
& T_2\mathrm{Emb}_\partial(M,N) \arrow[d, "f_2"] & \mathrm{fib}(f_2) = L_2\mathrm{Emb}_\partial(M,N) \arrow[l, hook'] \\
\mathrm{Emb}_\partial(M,N) \arrow[r] \arrow[ur] \arrow[uur] & T_1\mathrm{Emb}_\partial(M,N) &
\end{tikzcd}$$

Analogy $T_i \mathrm{Emb}_\partial(-,N)$ is the "i^{th} Taylor approximation" of

the functor $\mathrm{Emb}_\partial(-,N) : \left\{\begin{array}{c}\text{smooth } d\text{-manifolds} \\ \text{open embeddings}\end{array}\right\}^{op} \longrightarrow \text{Spaces}$

Prop [Weiss] $T_1 \mathrm{Emb}_\partial(M,N) \simeq \underbrace{\mathrm{Imm}_\partial(M,N)}_{\text{immersions}} \overset{h\text{-principle}}{\simeq} \underbrace{f\mathrm{Imm}_\partial(M,N)}_{\text{formal immersions}}$

$$\begin{array}{ccc} TM & \xrightarrow{\varphi} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

φ linear & injective on fibres

Thm [Weiss] $L_i \mathrm{Emb}_\partial(M,N) \simeq \Gamma_\partial\left(\begin{array}{c} Z_k \\ \downarrow \\ \tilde{C}_k(M)\end{array}\right)$ a certain fibration depending on N.

$\pi_*(-)$ computable

Rmk Everything works similarly for any "good" functor $\left\{\begin{array}{c}\text{smooth } d\text{-manifolds} \\ \text{open embeddings}\end{array}\right\}^{op} \longrightarrow \text{Spaces}$

Thm [Goodwillie-Klein-Weiss ≤'15] ("Convergence")

If $\dim(N) - \underbrace{\mathrm{hdim}}_{\boxed{\text{handle dimension}}}(M) \geq 3$

then $\mathrm{Emb}_\partial(M,N) \longrightarrow \operatorname{holim}_{i\to\infty} T_i \mathrm{Emb}_\partial(M,N) =: T_\infty \mathrm{Emb}_\partial(M,N)$

is a homotopy equivalence.

2^{nd} formulation [Boavida de Brito - Weiss '18]

Instead of looking at $L_i \operatorname{Emb}_\partial(M,N) := \operatorname{hofib}\left(T_i \to T_{i-1}\right)$,

try to understand $\boxed{\operatorname{hofib}\left(T_i \to T_1\right)}$ for all i.

Thm [Boavida-de-Brito - Weiss]

There is a homotopy pullback square

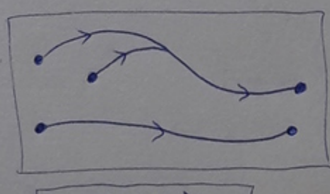
$$\begin{tikzcd}
T_i \operatorname{Emb}_\partial(M,N) \arrow[r] \arrow[d] & \operatorname{Map}_\partial\left(\operatorname{con}(M;i), \operatorname{con}(N;i)\right) \arrow[d, "(*)"] \\
T_1 \operatorname{Emb}_\partial(M,N) \arrow[r] & \operatorname{Map}_\partial\left(\operatorname{con}^{\mathrm{loc}}(M;i), \operatorname{con}^{\mathrm{loc}}(N;i)\right)
\end{tikzcd}$$

so $\boxed{\operatorname{hofib}\left(T_i \to T_1\right) \simeq \operatorname{hofib}((*))}$

$\operatorname{con}(M;i)$ = "configuration category" of M

= objects : ordered configurations of $\leq i$ points in M
  morphisms : paths of configurations where points may collide, but NOT anticollide.

Eg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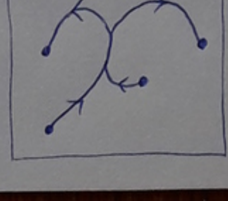
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($\operatorname{con}^{\mathrm{loc}}(M;i)$
= "local" version of $\operatorname{con}(M;i)$)

Now set $M = N = W_{g,1}^*$

$\dim = 2n$

$\underbrace{\mathrm{hdim}}_{\text{handle dimension rel. } \partial} = n$

$$\mathrm{hdim}(M) = \min_{\boxed{\substack{\mathcal{H} \text{ handle decomp}^n \\ \text{of } M \text{ rel. } \partial}}} \Big\{ \max \{ \mathrm{index}(H) \mid H \text{ handle in } \mathcal{H} \} \Big\}$$

For $M = W_{g,1}^*$: $\left[\begin{array}{l} \text{start with } \partial W_{g,1}^* \cong \mathbb{R}^{2n} \\ \text{attach } 2g \ n\text{-handles} \end{array}\right]$ $\dim - \mathrm{hdim} = n$

For $M = W_{g,1}$: $\left[\begin{array}{l} \text{start with } \partial W_{g,1} \cong S^{2n} \\ \text{attach } 2g \ n\text{-handles} \\ \text{attach one } 2n\text{-handle} \end{array}\right]$ $\dim - \mathrm{hdim} = 0$

So we may apply manifold calculus to $\mathrm{Emb}_\partial(W_{g,1}^*, W_{g,1}^*)$ if $n \geq 3$.

$$\begin{tikzcd}
\mathrm{Emb}_\partial(W_{g,1}^*, W_{g,1}^*) \arrow[r, "\boxed{\text{isom. on } \pi_* \text{ for } * \leq (n-2)i - n+2} \ \text{[Goodwillie-Klein-Weiss]}"] \arrow[d, phantom, "\cong" rotate=90] & T_i \mathrm{Emb}_\partial(W_{g,1}^*, W_{g,1}^*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{\text{hty pullback}}"] & \mathrm{Map}_\partial(\mathrm{con}(W_{g,1}^*; i), \mathrm{con}(W_{g,1}^*; i)) \arrow[d] \\
\mathrm{Diff}_\partial(W_{g,1}^*) & T_1 \mathrm{Emb}_\partial(W_{g,1}^*, W_{g,1}^*) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Map}_\partial(\mathrm{con}^{loc}(W_{g,1}^*; i), \mathrm{con}^{loc}(W_{g,1}^*; i)) \\
& f\mathrm{Imm}_\partial(W_{g,1}^*, W_{g,1}^*) &
\end{tikzcd}$$

These are all topological monoids (or at least A_∞-spaces).

Restrict to homotopy-invertible path-components:

$$\begin{tikzcd}
\mathrm{Diff}_\partial(W_{g,1}^*) \arrow[r, "\text{isom. on } \pi_* \text{ for } * \leq (n-2)i-n+2"] \arrow[dd, "\text{``Linearisation'' (page 7)}"'] & \boxed{\text{homotopy pullback}} \arrow[r] \arrow[d] & h\mathrm{Aut}_\partial(\mathrm{con}(W_{g,1}^*; i)) \arrow[d] \\
& f\mathrm{Imm}_\partial^\times(W_{g,1}^*, W_{g,1}^*) \arrow[r] \arrow[d, equal] & h\mathrm{Aut}_\partial(\mathrm{con}^{loc}(W_{g,1}^*; i)) \arrow[d, equal] \\
X \ltimes Y \arrow[r] & X^* \ltimes Y^* & X^* \ltimes Z^*
\end{tikzcd}$$

$X = h\mathrm{Aut}_\partial(W_{g,1})$

$X^* = h\mathrm{Aut}_\partial(W_{g,1}^*)$

$Y = \mathrm{Map}_*(W_{g,1}, SO(2n))$

$Y^* = \mathrm{Map}_*(W_{g,1}^*, SO(2n))$

$Z^* = \mathrm{Map}_*(W_{g,1}^*, h\mathrm{Aut}(\mathrm{con}(\mathbb{R}^{2n}; i)))$

<u>Thm</u> [S.-Tillmann-Weiss; Weiss]

The map

$$\begin{tikzcd}
B\mathrm{Diff}_\partial(W_{g,1}^*) \arrow[rr, "\text{(page 6)}"] \arrow[dr, "\text{(top horizontal map above)}"'] & & Bh\mathrm{Aut}_\partial(W_{g,1}) \\
& Bh\mathrm{Aut}_\partial(\mathrm{con}(W_{g,1}^*; i)) \arrow[ur, dotted] &
\end{tikzcd}$$

significance: $\kappa(\mathcal{L}_{n+1})$ pulls back along this map

factors like:

(after killing the homotopy groups of $Bh\mathrm{Aut}_\partial(W_{g,1})$ in degrees $\geq (n-2)i - n$).

$((n-2)i-n+2)$-connected.

$$\begin{tikzcd}[column sep=small]
 & \kappa(h_{n\times n}) & & & & \bar{\kappa}(h_{n\times n}) \arrow[llll] \\
\mathbb{Q}[p_1,p_2,\dots] \arrow[d, phantom, "\cup" description] & & & & & \\
(\text{decomp.}) \arrow[r, dashed, "\kappa"] \arrow[d] & BDiff_\partial(W_{g,1}^*) \arrow[r] \arrow[d] & \boxed{\text{hty pullb}} \arrow[r] \arrow[d] & BhAut_\partial(con(W_{g,1}^*;i)) \arrow[r] \arrow[d] \arrow[loop above, "id"] & BhAut_\partial(W_{g,1}) & \\
(\text{decomp.}) \arrow[r, dashed, "\tilde{\kappa}"] \arrow[d, phantom, "\cap" description] & B(X\ltimes Y) \arrow[r] \arrow[loop below, "id\ltimes f_\#"] & B(X^*\ltimes Y^*) \arrow[r] \arrow[loop below, "id\ltimes f_\#"] & B(X^*\ltimes Z^*) \arrow[loop below, "id\ltimes id"] & & \\
H^*(BSO(2n)) \arrow[d, equal] & & & & & \\
\mathbb{Q}[e,p_1,\dots,p_{n-1}] \arrow[loop below, "f^*"] & & & & &
\end{tikzcd}$$

$X^* = hAut_\partial(W_{g,1}^*)$

$Y^* = Map_*(W_{g,1}^*, SO(2n))$

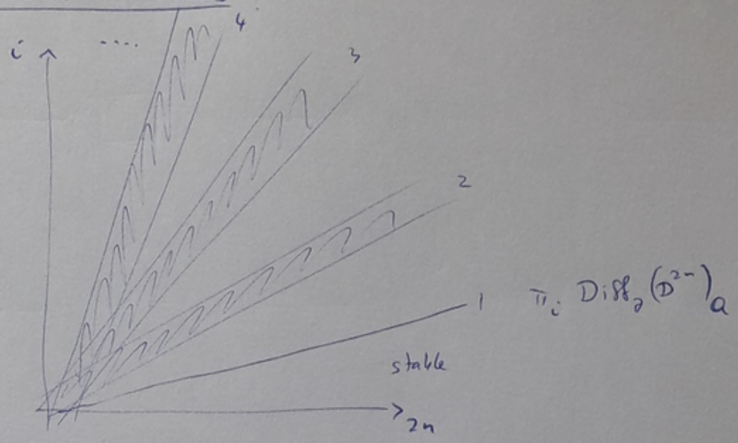
$Z^* = Map_*(W_{g,1}^*, hAut(con(\mathbb{R}^{2n};i)))$

- Localise the whole diagram at $\mathbb{Q}$.
- Fact: $BSO(2n)_{\mathbb{Q}} \simeq \underset{e}{K(\mathbb{Q},2n)} \times \prod_{i=1}^{n-1} \underset{p_i}{K(\mathbb{Q},4i)}$ $\qquad \frac{n+1}{4} \leq a,b \leq n.$
- Choose $f: BSO(2n)_{\mathbb{Q}} \longrightarrow BSO(2n)_{\mathbb{Q}}$
  - nullhomotopic on the $K(\mathbb{Q},4b)$ factor (p_b)
  - identity on all other factors
- Then f^*: $e \mapsto e$
  $p_j \mapsto p_j \quad (j \neq b)$
  $p_b \mapsto 0$
- Technical theorem [Weiss]

$$\begin{tikzcd}
BSO(2n)_{\mathbb{Q}} \arrow[d, "f"'] \arrow[dr] & \\
BSO(2n)_{\mathbb{Q}} \arrow[r] & BhAut(con(\mathbb{R}^{2n};i))_{\mathbb{Q}}
\end{tikzcd}$$

commutes in degrees $\lesssim 8n$ (i.e. if we kill hty groups above degree $8n$). $\square$

The work of Kupers - Randal - Williams

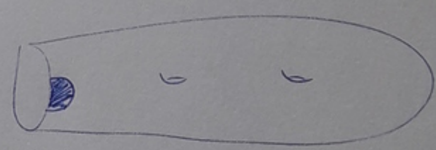


Thm [KRW] '20 Above the stable range & outside of these "bands", $\pi_* \operatorname{Diff}_\partial(D^{2n})_{\mathbb{Q}}$ is given precisely by the (duals of) exotic Pontryagin classes.

Idea:

Weiss fiber sequence

$$\operatorname{Diff}_\partial(D^{2n}) \longrightarrow \operatorname{Diff}_\partial(W_{g,1}) \longrightarrow \operatorname{Emb}_\partial(W_{g,1}^*, W_{g,1}^*)$$



① Calculate $\pi_* L_i \operatorname{Emb}_\partial(W_{g,1}^*, W_{g,1}^*)$ explicitly

$\boxed{\text{Emb. Calc}} \Rightarrow \pi_* \operatorname{Emb}_\partial(W_{g,1}^*, W_{g,1}^*)$ is concentrated in the "bands"

② [Galatius - Randal-Williams] : $H^*(B\operatorname{Diff}_\partial(W_{g,1}))$ well-understood.

X hty associative H-space $\longrightarrow$ [Cartan-Serre]

$$\pi_*(X)_{\mathbb{Q}} \cong PH_*(X)$$

[Milnor-Moore]

$$H_*(X) \cong \mathcal{U}(PH_*(X))$$
$$\cong \mathcal{U}(\pi_*(X)_{\mathbb{Q}})$$

X nilpotent $\longrightarrow$ unstable Adams SS

$$\boxed{(\text{gen. } H_*) \Rightarrow \pi_*}$$

But $BDiff_\partial(W_{g,1})$ is not nilpotent!

$$\pi_1(-) \cong \pi_0 Diff_\partial(W_{g,1}) \xrightarrow[\text{action on } H_n]{} \text{f.i. subgroup of } \begin{cases} Sp_{2g}(\mathbb{Z}) & n \text{ odd} \\ O_{g,g}(\mathbb{Z}) & n \text{ even} \end{cases}$$

Let $Tor_\partial(W_{g,1}) := \ker\left(Diff_\partial(W_{g,1}) \longrightarrow Aut(H_n(--))\right)$

Then $BTor_\partial(W_{g,1}) \longrightarrow BDiff_\partial(W_{g,1})$ is a covering space, so $\pi_{\geq 2}(-)$ agree.

Thm [KRW] • $BTor_\partial(W_{g,1})$ is nilpotent

• analogue of [GRW] for $H^*(BTor_\partial(W_{g,1}))$
(hom stab' + stable H_*)

- concentrated in "bands"
- $\longleftrightarrow$ exotic P. classes.

unstable Adams SS $\Rightarrow$ $\square$.