Previous research

Introduction and overview

My research interests lie in the area of topology, including *algebraic topology* and *low-dimensional topology*, and interactions between the two. The two main themes of my previous research work are concerned with studying:

(A) The homology of *moduli spaces of submanifolds* of an ambient manifold (such as configuration spaces or spaces of links in $\mathbb{R}^3$), as well as *diffeomorphism groups* of manifolds, via the phenomenon of homological stability.

(B) The representation theory of the fundamental groups of such moduli spaces (which includes examples such as *surface braid groups*, *loop braid groups* and *mapping class groups*) through geometrically-defined "homological" representations, and in particular different kinds of *functoriality* for such representations, including topological quantum field theories.

As well as being interesting geometrical objects in their own right, these objects are also relevant for many diverse areas of topology. For example, classifying spaces of diffeomorphism groups control the characteristic classes associated to the corresponding manifold bundles, via their cohomology. Configuration spaces are ubiquitous in topology, appearing in connection with mapping spaces in homotopy theory, functor calculus and operads. Braid groups and motion groups are related to mapping class groups of surfaces and to knot theory, including variants such as virtual knot theory, where their representations play a fundamental role in constructing and understanding knot and link invariants.

An important tool for studying moduli spaces of submanifolds is *homological stability*. This is the phenomenon where a sequence $\{X_n \mid n \in \mathbb{N}\}$ of spaces has the property that its homology $H_i(X_n)$ in any fixed degree i is eventually independent of n. This can be calculationally very useful when combined with a computation of the stable limit of the homology groups. The philosophy is that the homology of the family $\{X_n\}$ of spaces typically carries a topological structure that is not visible at the level of the individual spaces, but becomes visible in the limit when $n \to \infty$, making it amenable to explicit calculations, and homological stability allows one to leverage this fact.

An important example of this is homological stability for the mapping class groups of orientable surfaces $\mathrm{Mod}(\Sigma_{g,1})$, due to Harer [Har85], together with the theorem of Madsen and Weiss [MW07], which completely determines the rational cohomology of the mapping class groups in a stable range, thereby proving the Mumford Conjecture [Mum83]. It can also have powerful theoretical consequences: for example, homological stability for certain examples of *configuration-mapping spaces* was used in [EVW16] to prove results in number theory related to the Cohen-Lenstra conjecture.

In my previous work, I have carried out one or both steps of this programme for a number of moduli spaces of interest, fitting into theme **(A)** of my research programme. In particular:

(a) I proved [Pal13] in my thesis that the *oriented configuration spaces* $C_n^+(M)$ on open, connected manifolds M are homologically stable: a point in this space is a configuration of points in M, equipped with the *non-local* data of an ordering modulo the action of the alternating group. Together with J. Miller [MP15b], we then identified a space modelling the limiting homology of these spaces, as the number of particles goes to infinity, completing step (ii) of the above programme, and lifting the classical result of D. McDuff [McD75] and G. Segal [Seg73; Seg79] (which concerns *unordered* configuration spaces). Along the way, we also generalised their *group-completion theorem* [MS76] to the setting of twisted-homology-equivalences [MP15a]. — See §1 for more details.

(b) The unordered configuration spaces $C_n(M)$ on a *closed*, connected manifold M are known *not* to stabilise in general, and represent a much more difficult and complicated case. Together with F. Cantero [CP15], we proved that there are nevertheless some more subtle *stability and periodicity phenomena* in the homology of configuration spaces on closed manifolds. For example, the homology of $C_n(M)$ with coefficients in $\mathbb{Z}[\frac{1}{2}]$ stabilises for $\dim(M)$ odd, whereas, for $\dim(M)$ even, their mod-p homology is unstable but *periodic* in a stable range (with an explicit period depending only on p and $\chi(M)$). — See §3 for more details of these results.

(c) Another direction in which I have taken these ideas, which opens the door to a wide variety of interesting new examples, is to generalise configuration spaces to *moduli spaces of disconnected submanifolds*, where point particles are replaced with embedded submanifolds of a specified diffeomorphism type and isotopy class (which may be parametrised, oriented, unoriented, etc.). Under a certain restriction on the codimension, I have proven [Pal18a] that these moduli spaces are also *homologically stable* as the number of components of the submanifold goes to infinity. As a corollary, I also proved [Pal18b] homological stability for:
   - the diffeomorphism groups of *manifolds with conical singularities*, with respect to the number of singularities of a given type,
   - the *symmetric diffeomorphism groups* of any sequence of manifolds obtained by iterating the operation of "parametrised connected sum", an operation which generalises both ordinary connected sum and surgery.

   See §4 for more details of these results.

Related to theme **(B)** of my research programme:

(d) I have proven [Pal18c] that the unordered configuration spaces $C_n(M)$ are homologically stable with coefficients in any *polynomial twisted coefficient system* (this in particular includes a choice of representation of $\pi_1(C_n(M))$ for each n). — See §2 for more details.

(e) In joint work with A. Soulié, we have set up a unified framework for topologically constructing representations of motion groups (fundamental groups of moduli spaces of disconnected submanifolds), extending several known constructions, due to R. Lawrence, S. Bigelow, D. Long and J. Moody, as well as obtaining new families of representations, including analogues, for the *loop braid groups*, of the *Lawrence-Bigelow representations* for the classical braid groups.
   Moreover, our construction automatically gives not only a family of representations of, for example, the braid groups or loop braid groups, but a functor on a category having these groups as its automorphism groups, as well as a richer structure that allows one to define the *degree* of such a functor. In particular, we extend the *Lawrence-Bigelow representations* of the braid groups to functors of this kind and prove that they have finite degree, and deduce that the homology of the braid groups with coefficients twisted by the Lawrence-Bigelow representations is stable. — See §5 for more details.

In addition to these two main themes, I have also studied *multi-crossing diagrams* for links, in joint work with C. Adams and J. Hoste, where our main result is the construction of a complete set of Reidemeister moves for *triple-crossing diagrams*. — See §6 for more details.

1. Configuration spaces with non-local structure

— *Homological stability for oriented configuration spaces; a twisted group-completion theorem; stable homology of oriented configuration spaces.* —

For a space M, the nth ordered configuration space $\widetilde{C}_n(M)$ is defined to be the subspace of M^n consisting of all n-tuples of pairwise distinct points in M. The symmetric group $\mathfrak{S}_n$ acts on this space, and we define

$$C_n(M) = \widetilde{C}_n(M)/\mathfrak{S}_n \qquad \text{and} \qquad C_n^+(M) = \widetilde{C}_n(M)/A_n,$$

where $A_n < \mathfrak{S}_n$ is the alternating group. These are called, respectively, the *unordered* configuration space on M and the *oriented* configuration space on M. It is a classical result, going back to McDuff [McD75] and Segal [Seg73; Seg79], that the sequence $C_n(M)$, when M is a connected, open manifold, is *homologically stable*. This means that, for each degree i, there are isomorphisms $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ once n is sufficiently large (depending on i).

Aside. The condition that M is connected is clearly necessary, as one can see by considering H_0. On the other hand, the condition that M is open is not obviously necessary, and the situation in this case is much more subtle. This is the subject of another part of my previous work, see §3.

The ordered configuration spaces $\widetilde{C}_n(M)$ are not homologically stable: for example, the first homology of $\widetilde{C}_n(\mathbb{R}^2)$ is the abelianisation of the pure braid group, which is $\mathbb{Z}^{\binom{n}{2}}$. This raises the question of whether there is an intermediate covering space between $\widetilde{C}_n(M)$ and $C_n(M)$ for which homological stability still holds. I proved in [Pal13] that the answer is positive for the oriented configuration spaces $C_n^+(M)$, which doubly cover the unordered configuration spaces $C_n(M)$:

Theorem ([Pal13]) *The natural stabilisation map $C_n^+(M) \to C_{n+1}^+(M)$ induces isomorphisms on homology in degrees $* \leqslant \frac{n-5}{3}$ and surjections in degrees $* \leqslant \frac{n-2}{3}$.*

The theorem also holds more generally for *labelled configuration spaces*, where each point is equipped with a "label" in some fixed, path-connected parameter space X. The so-called *slope* of the stability range here is $\frac{1}{3}$ (and one may calculate explicitly for certain M to see that this is sharp), in contrast to the slope of $\frac{1}{2}$ that holds for the unordered configuration spaces.

Note. The integral homology of $C_n^+(M)$ may be interpreted as a certain twisted homology group $H_*(C_n(M);\mathbb{Z}[\mathbb{Z}/2])$, where $\pi_1(C_n(M))$ acts on the group ring of $\mathbb{Z}/2$ via the natural projection $\pi_1(C_n(M)) \to \mathfrak{S}_n$ followed by the sign homomorphism. This is an example of an *abelian twisted coefficient system* on $C_n(M)$, and is a precursor (in a special case) of the notion of *abelian homological stability*, which has been developed more recently by Randal-Williams and Wahl [RW17] and Krannich [Kra17]. See §2 for more twisted homological stability results.

This result leads to the question of whether one can identify the *stable homology* of the sequence $\{C_n^+(M) \mid n \in \mathbb{N}\}$, in other words the colimit $\lim_{n\to\infty} H_*(C_n^+(M))$, in terms of other well-understood spaces. In joint work with **Jeremy Miller**, we answered this question positively by lifting the classical *scanning map* [Seg73; McD75] to a homology equivalence between appropriate covering spaces.

- First, in our paper [MP15a], we generalise the McDuff-Segal *group-completion theorem* [MS76] as well as McDuff's *homology fibration criterion* [McD75, §5] to the setting of homology with twisted coefficients (more precisely to a setting where we consider homology with respect to all twisted coefficient systems in a fixed class $\mathfrak{C}$ that is closed under pullbacks).
- Using these tools, we proved in [MP15b] a new kind of scanning result, lifting the classical scanning map to covering spaces and showing that it remains a homology equivalence after doing so. This identifies the stable homology of oriented configuration spaces on M with the homology of an explicit double cover of the section space of a certain bundle over M:

Theorem ([MP15b]) *Writing $\dot{T}M \to M$ for the fibrewise one-point compactified tangent bundle of M and $\Gamma_c(\dot{T}M \to M)_\circ$ for its space of compactly-supported sections of degree zero, we have:*

$$\lim_{n\to\infty} H_*(C_n^+(M)) \;\cong\; H_*(\widetilde{\Gamma}_c(\dot{T}M \to M)_\circ), \tag{1}$$

for a certain explicit double cover $\widetilde{\Gamma}_c(\dot{T}M \to M)_\circ \longrightarrow \Gamma_c(\dot{T}M \to M)_\circ$.

This double cover may be defined as the connected covering space of $\Gamma_c = \Gamma_c(\dot{T}M \to M)_\circ$ corresponding to the projection

$$\pi_1(\Gamma_c) \to H_1(\Gamma_c) \cong H_1(C_2(M)) \to H_1(C_2(\mathbb{R}^\infty)) \cong \mathbb{Z}/(2).$$

Here, the isomorphism $H_1(\Gamma_c) \cong H_1(C_2(M))$ arises from homological stability and the identification of the stable homology for *unordered* configuration spaces, and the map $H_1(C_2(M)) \to H_1(C_2(\mathbb{R}^\infty))$ is induced by any embedding $M \hookrightarrow \mathbb{R}^\infty$. For example, when $M = \mathbb{R}^\infty$ the right-hand side of (1) is the universal cover of (one component of) the infinite loopspace $\Omega^\infty S^\infty = Q\mathbb{S}^0$. When $M = \mathbb{R}^2$ it is the unique connected double cover of $\Omega^2 S^3$.

2. Twisted homological stability

— *Twisted homological stability for configuration spaces; different notions of the degree of twisted coefficient systems.* —

As well as the notion of abelian homological stability mentioned in the previous section, another sense in which a sequence of spaces (or groups) can satisfy *twisted homological stability* is with respect to a so-called *polynomial twisted coefficient system*. This consists of a choice of local coefficient system on each space X_n, together with additional morphisms of local coefficient systems between them, organised into a functor $\mathscr{C} \to \mathsf{Ab}$, where the automorphism groups of $\mathscr{C}$ are the fundamental groups $\pi_1(X_n)$. In addition, the *degree* of this functor (defined using extra structure on $\mathscr{C}$) is required to be finite.

Many families of groups $G = \{G_n\}$ are known to be homologically stable in this sense (where we set $X_n = BG_n$ in the above paragraph), for appropriately-defined categories $\mathscr{C}_G$, for example the symmetric groups $\mathfrak{S}_n$, braid groups β_n, general linear groups, automorphism groups of free groups $\mathrm{Aut}(F_n)$ and mapping class groups of surfaces and of 3-manifolds.[1] In [Pal18c], I proved the first such result for a sequence of *spaces*, namely the unordered configuration spaces $C_n(M)$ on any connected, open manifold M. Note that, when $\dim(M) \geqslant 3$, these spaces are not aspherical (in contrast to the case of surfaces), so this does not reduce to a statement about the homology of their fundamental groups. When $\dim(M) = 2$, these configuration spaces are aspherical, so the result may be thought of as twisted homological stability for their fundamental groups, the surface braid groups $B_n(M) = \pi_1(C_n(M))$.

Theorem ([Pal18c]) *Let M be an open, connected manifold and let T be a twisted coefficient system for $\{C_n(M)\}$. This includes in particular the data of a local coefficient system T_n for each $C_n(M)$, as well as a homomorphism*

$$H_*(C_n(M);T_n) \longrightarrow H_*(C_{n+1}(M);T_{n+1}).$$

This homomorphism is split-injective in all degrees, and, if T has finite degree d, it is an isomorphism in the range $* \leqslant \frac{n-d}{2}$.

This theorem also generalises to configuration spaces with labels in any path-connected space X, as in the previous section.

Twisted coefficient systems. In connection with these results, I have also explored in more depth [Pal17] the notion of *twisted coefficient system* (a.k.a. finite-degree or polynomial functor), and in particular the *degree* of a twisted coefficient system. The main results of [Pal17] are:

- A comparison and unification of various different notions of "finite-degree" functor $\mathscr{C} \to \mathscr{A}$, where $\mathscr{A}$ is an abelian category and $\mathscr{C}$ is a category with various kinds of additional structure.
- The development of a functorial construction of (*injective* or *partial*) *braid categories*, which were used in [Pal18c] as the domain of definition of twisted coefficient systems, and of finite-degree functors on such braid categories.

3. Configurations on closed manifolds

— *Stability phenomena for configuration spaces on closed manifolds.* —

When M is closed, homological stability for the unordered configuration spaces $C_n(M)$ is not true in general, for example one may calculate that $H_1(C_n(S^2);\mathbb{Z}) \cong \mathbb{Z}/(2n-2)$, which does not stabilise as $n \to \infty$. Moreover, the stabilisation maps mentioned in the theorem in §1 do not exist, since these depend on adding a new configuration point in M "near infinity". In joint work with **Federico Cantero** [CP15], we prove three main results which show that the homology of configuration spaces on closed manifolds exhibits a large amount of stability despite these issues.

(1) When the Euler characteristic of M is zero, we construct *replication maps* $C_n(M) \to C_{\lambda n}(M)$ for any integer $\lambda \geqslant 2$, and prove that they induce homological stability after inverting λ:

[1] Symmetric groups: [Bet02]; braid groups: [CF13; RW17; Pal18c]; general linear groups: [Dwy80; Kal80]; automorphism groups of free groups: [RW17]; mapping class groups of surfaces: [Iva93; CM09; Bol12; RW17]; mapping class groups of 3-manifolds: [RW17]. Note that these are references for the proofs of *twisted* homological stability; in each case, homological stability with untwisted coefficients was known earlier.

Theorem ([CP15]) *These maps induce isomorphisms on* $H_i(-,\mathbb{Z}[\frac{1}{\lambda}])$ *in the range* $2i \leqslant \lambda$.

Note. A construction related to our replication maps has also been used in the paper [Ber⁺06], in which they use something similar to a replication map in §3 to build a crossed simplicial group out of the configuration spaces on any given manifold M that admits a non-vanishing vector field.

(2) When the manifold M is odd-dimensional, the configuration spaces $C_n(M)$ do in fact satisfy homological stability after inverting 2.

Theorem ([CP15]) *When* $\dim(M)$ *is odd, there are isomorphisms*

$$H_i(C_n(M);\mathbb{Z}[\tfrac{1}{2}]) \cong H_i(C_{n+1}(M);\mathbb{Z}[\tfrac{1}{2}]) \qquad \text{and} \qquad H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+2}(M);\mathbb{Z})$$

in the range $2i \leqslant n$*, induced by a zigzag of maps.*

This strengthens a result of [BM14].

(3) When the manifold M is even-dimensional, and $\mathbb{F}$ is a field of characteristic 0 or 2, it is known by the work of many people [BCT89; ML88; Chu12; Ran13; BM14; Knu17] that homological stability holds for $C_n(M)$ with coefficients in $\mathbb{F}$, even when M is closed. When $\mathbb{F}$ has odd characteristic p, however, this is false, as one can see from the example of $M = S^2$ mentioned above. In fact:

$$H_1(C_n(S^2);\mathbb{F}) \cong \begin{Bmatrix} \mathbb{F} & p \mid n-1 \\ 0 & p \nmid n-1 \end{Bmatrix} \quad \text{for } n \geqslant 2.$$

From this example we see that the first homology of $C_n(S^2)$ is not stable, but it is p-periodic and takes on only 2 different values. Our third result is that this phenomenon holds in general, when the Euler characteristic χ of M is non-zero. Write $a = \nu_p(\chi)$ for the p-adic valuation of χ, in other words $\chi = p^a b$ with b coprime to p.

Theorem ([CP15]) *Suppose that* $\dim(M)$ *is even. For each fixed i, the sequence*

$$H_i(C_n(M);\mathbb{F}) \quad \text{for } n \geqslant 2i \tag{2}$$

is p^{a+1}*-periodic and takes on at most* $a+2$ *values. Moreover, if* $\chi \equiv 1 \bmod p$ *then the above sequence is* 1*-periodic, i.e. homological stability holds with coefficients in* $\mathbb{F}$.

The p^{a+1}-periodicity result is similar to a theorem of [Nag15], although his estimate of the period is different, namely a power of p depending on i rather than on χ. This result was later improved by [KM16] to p-periodicity, independent of i or χ. Combining this with (a slightly more precise statement of) our result, a corollary is that in fact the sequence (2) above takes on only two different values.

4. Moduli spaces of disconnected submanifolds

— *Moduli spaces of disconnected submanifolds; symmetric diffeomorphism groups; manifolds with conical singularities; partitioned braid groups.* —

Instead of configurations of points in M (closed 0-dimensional submanifolds), one may consider configurations of closed submanifolds of M of higher dimension, which are diffeomorphic to the disjoint union of finitely many copies of a fixed ("model") manifold L. In the recent preprint [Pal18a], I proved that *moduli spaces of disconnected submanifolds* of this kind are also homologically stable as the number of components goes to infinity, just as in the classical setting of points [Seg73; McD75; Seg79] – under a certain hypothesis on the relative dimensions of the manifolds involved.

Let $\bar{M}$ be a connected manifold with non-empty boundary and of dimension at least 2, and denote its interior by M. Also fix a closed manifold L and an embedding $\iota_0 \colon L \hookrightarrow \partial\bar{M}$. Choose a self-embedding $e\colon \bar{M} \hookrightarrow \bar{M}$ which is isotopic to the identity and such that $e(\iota_0(L))$ is contained in the interior $M \subset \bar{M}$. We then obtain a sequence of pairwise-disjoint embeddings of L into M by defining $\iota_n := e^n \circ \iota_0$ for $n \geqslant 0$.

Let $nL = \{1,\dots,n\} \times L$ and write $\iota_{1,\dots,n} \colon nL \hookrightarrow M$ for the embedding $(i,x) \mapsto \iota_i(x)$.

Definition Define $C_{nL}(M)$ to be the path-component of

$$\mathrm{Emb}(nL,M)/\mathrm{Diff}(nL)$$

containing $[\iota_{1,\dots,n}]$. Here, the embedding space is given the Whitney topology and $C_{nL}(M)$ the quotient topology. There is a natural *stabilisation map*

$$C_{nL}(M) \longrightarrow C_{(n+1)L}(M) \tag{3}$$

defined by adjoining the embedding ι_0 to a given embedding $nL \hookrightarrow M$, to obtain a new embedding of the form $(n+1)L \hookrightarrow \bar{M}$, and then composing with the self-embedding e. In symbols, this may be written as $[\phi] \mapsto [\phi^+]$, where $\phi^+(i,x) = e \circ \phi(i,x)$ for $1 \leqslant i \leqslant n$ and $\phi^+(n+1,x) = \iota_1(x)$.

Theorem ([Pal18a]) *Assume that the dimensions $m = \dim(M)$ and $\ell = \dim(L)$ satisfy*

$$2\ell \leqslant m-3. \tag{4}$$

Then (3) induces isomorphisms on homology in degrees $* \leqslant \frac{n-2}{2}$ and surjections in degrees $* \leqslant \frac{n}{2}$.

This theorem may be extended further:

- There is a more general version of this setting, in which the submanifolds $L \subset M$ are parametrised modulo a subgroup of $\mathrm{Diff}(L)$ and come equipped with labels in some bundle over $\mathrm{Emb}(L,M)$. The theorem proved in [Pal18a] includes this more general setting.
- This setting is compatible with the techniques of §1 above, so we also have homological stability for "oriented" (in the sense of §1) versions of these moduli spaces, in which the submanifolds $L \subset M$ are ordered modulo even permutations.

Applications to diffeomorphism groups. In the sequel [Pal18b], I used homological stability for the moduli spaces $C_{nL}(M)$ (and their more refined versions mentioned in the first point above) to prove homological stability for:

- Symmetric diffeomorphism groups, with respect to parametrised connected sum.
- Diffeomorphism groups of manifolds with conical singularities, with respect to the number of singularities.

Definition Given two embeddings $L \hookrightarrow M$ and $L \hookrightarrow Q$ with isomorphic normal bundles, one may cut out a tubular neighbourhood of each embedding and glue the resulting boundaries to obtain the *parametrised connected sum* $M \sharp_L Q$. If L is a point this corresponds to the ordinary connected sum of M and Q. Other examples include the following.

- If $L \hookrightarrow Q$ is the canonical embedding $S^k \hookrightarrow S^m$, where $k \leqslant m = \dim(M)$, then $M \sharp_L Q$ is the result of a k-surgery on M.
- If $L \hookrightarrow Q$ is the embedding $S^1 \hookrightarrow \mathbf{T} \hookrightarrow \mathbf{T} \cup_{p/q} \mathbf{T} = L(p,q)$, where $\mathbf{T} = D^2 \times S^1$ denotes the solid torus, then $M \sharp_L Q$ is the result of a Dehn surgery of slope p/q on the 3-manifold M.

If we now iterate the operation $-\sharp_L Q$ many times, using a different copy of Q and a disjoint embedding $L \hookrightarrow M$ each time (but always using the same copy of M), we obtain a sequence

$$M \qquad M \sharp_L Q \qquad M \sharp_L Q \sharp_L Q \qquad M \sharp_L Q \sharp_L Q \sharp_L Q \qquad \cdots \tag{5}$$

of manifolds, which we abbreviate to $M \sharp_L^n Q$, the *n-th iterated parametrised connected sum*.

A diffeomorphism of $M \sharp_L^n Q$ is called *symmetric* if it fixes the boundary of M and preserves the decomposition of $M \sharp_L^n Q$ into pieces of the form $M \smallsetminus n\mathscr{T}(L)$ and $Q \smallsetminus \mathscr{T}(L)$, where $\mathscr{T}(L)$ is a tubular neighbourhood of L in M or Q. The corresponding subgroup

$$\Sigma\mathrm{Diff}(M \sharp_L^n Q) \;\leqslant\; \mathrm{Diff}(M \sharp_L^n Q)$$

is called the *symmetric diffeomorphism group* of $M \sharp_L^n Q$.[2]

[2] A mild technical condition has been elided from the definition of symmetric diffeomorphism group and in the statement of the theorem below, in order to simplify the discussion.

Theorem ([Pal18b]) *If M is connected and has non-empty boundary and $\dim(L) \leqslant \frac{1}{2}(\dim(M)-3)$, the sequence*

$$\cdots \longrightarrow B\Sigma\mathrm{Diff}(M\sharp_L^n Q) \longrightarrow B\Sigma\mathrm{Diff}(M\sharp_L^{n+1} Q) \longrightarrow \cdots$$

of (classifying spaces of) symmetric diffeomorphism groups is homologically stable.

This generalises a result of Tillmann [Til16], which corresponds to the case $L =$ point (i.e. the usual connected sum operation).

Informal definition Fix an $(m-1)$-dimensional manifold T. Let $\mathrm{cone}(T) = (T \times [0,\infty))/(T \times \{0\})$ be the open cone on T. An m-dimensional *manifold with conical T-singularities* is a space M that is locally homeomorphic to $\mathrm{cone}(T)$, together with a smooth atlas on the subset $M_{mfd} \subseteq M$ of locally Euclidean points of M. A *diffeomorphism* of M is a homeomorphism $M \to M$ that restricts to a diffeomorphism $M_{mfd} \to M_{mfd}$ and is of the form $\mathrm{cone}(\varphi)$ for some diffeomorphism $\varphi\colon T \to T$ near each point of the discrete subset $M \smallsetminus M_{mfd} \subseteq M$. These form a subgroup

$$\mathrm{Diff}^T(M) \leqslant \mathrm{Homeo}(M).$$

For example, we may construct a manifold with conical singularities by collapsing a tubular neighbourhood $\mathscr{T}(L)$ of any submanifold $L \subset M$. The quotient $M_L = M/\mathscr{T}(L)$ is then a manifold with a single conical $\partial\mathscr{T}(L)$-singularity. In particular, using the setting at the beginning of this subsection, we may collapse a tubular neighbourhood of each submanifold $\iota_i(L) \subset M$ for $1 \leqslant i \leqslant n$, to obtain a manifold $M_{n \cdot L}$ with (precisely n) conical $\partial\mathscr{T}(L)$-singularities.

Theorem ([Pal18b]) *If M is connected and has non-empty boundary and $\dim(L) \leqslant \frac{1}{2}(\dim(M)-3)$, the sequence of classifying spaces $B\mathrm{Diff}^{\partial\mathscr{T}(L)}(M_{n\cdot L})$ is homologically stable.*

Special values of ℓ and m. The condition (4) on the relative dimensions of L and M excludes some interesting special cases of the moduli space $C_{nL}(M)$.

One such special case is $\ell = 1$, $m = 3$, in other words, moduli spaces of links in a 3-manifold. If one considers moduli spaces of *unlinks* in a 3-manifold M, then this is known to be homologically stable as the number of components of the unlink goes to infinity, by a result of Kupers [Kup13]. However, the techniques of Kupers do not generalise to moduli spaces of non-trivial links (even if the components are pairwise unlinked), so this question is open for general links.

Another special case is $\ell = 0$, $m = 2$. If L is a point, this corresponds to configuration spaces of points on a surface M, for which homological stability is known classically. However, the only assumption that we have made about L is that it is closed, not necessarily connected, so we could also take $L = \{1,\ldots,\xi\}$ for any positive integer ξ. The moduli space $C_{nL}(M)$ is then the covering space of $C_{n\xi}(M)$ with

$$p_\xi(n) = \frac{(n\xi)!}{n!(\xi!)^n}$$

sheets, whose fibres correspond to all ways of partitioning the $n\xi$ points of a configuration into n subsets of size ξ. Since configuration spaces on M are aspherical (M is a connected surface with non-empty boundary), this is equivalent to studying the corresponding index-$p_\xi(n)$ subgroup of the surface braid group $B_{n\xi}(M)$, called the *partitioned surface braid group* $B_{\xi|n}(M)$, consisting of all braids that preserve a given partition $n\xi = \xi + \xi + \cdots + \xi$ of their endpoints. In joint work with **TriThang Tran**, we have shown that homological stability holds also in this special case, corresponding to $(\ell, m) = (0,2)$.

Theorem ([PT14]) *Let M be a connected surface with non-empty boundary. Then the sequence of partitioned surface braid groups $B_{\xi|n}(M)$ is homologically stable as $n \to \infty$, for any fixed $\xi \geqslant 1$.*

This recovers the classical case of homological stability for configuration spaces of points when $\xi = 1$, and is new for $\xi \geqslant 2$.

5. Homological representations of motion groups

— *A unified functorial construction of representations; new representations of loop braid groups; polynomiality and twisted homological stability.* —

The braid groups B_n (on $n \geqslant 3$ strands) are known to have "wild" representation theory, so there is no easy classification system for their representations.[3] It is therefore useful to be able to construct representations of the braid groups *topologically*, so that they may be studied using geometry and topology.

Important examples of topologically-defined representations of braid groups are given by the *Lawrence-Bigelow representations*, introduced by R. Lawrence in 1990 and later generalised by S. Bigelow to a construction that inputs a B_m-representation V over a ring k and outputs a B_n-representation over $k[t^{\pm 1}]$ for all n. The original construction $\mathfrak{LB}_{m,n}$ of Lawrence corresponds to the case $V = k = \mathbb{Z}[q^{\pm 1}] = \mathbb{Z}[\mathbb{Z}]$, where B_m acts through its abelianisation, for $m \geqslant 2$, and $V = k = \mathbb{Z}$, for $m = 1$. The significance of this family of representations is that the sequence $\mathfrak{LB}_{2,n}$ of representations was used by S. Bigelow and D. Krammer to prove (independently) that the braid groups are *linear*.

Other examples of topologically-defined representations of braid groups come from the *Long-Moody construction*, which inputs a B_m-representation V over k and outputs a B_{m-1}-representation over k. Both of these constructions are topological, induced by the action (up to homotopy) of B_n on a space equipped with a certain local system.

Unified construction of homological representations. Braid groups are simultaneously examples of *motion groups* and of *mapping class groups*; other examples of these are surface braid groups, loop braid groups, mapping class groups of surfaces and automorphism groups of free groups.

In joint work with **Arthur Soulié**, we set up a general framework for constructing "*homological representations*" of motion groups and mapping class groups that recovers the constructions of Lawrence-Bigelow and Long-Moody for the classical braid groups (so in a sense it unifies these constructions, as well as extending them to a much wider setting). Moreover, the construction produces "coherent" families of representations, in the sense that they extend to a functor on a category with object set $\mathbb{N}$ and whose automorphism groups are the family of groups under consideration (e.g. braid groups on any number of strands). The richer structure of this category may then be used

(i) to organise the representation theory of the family of groups, and
(ii) to prove twisted homological stability results, via a certain notion of polynomiality.

Informally, our construction may be summarised as follows:

Theorem ([PS19]) *Let $\mathscr{G} = \{\pi_1(C_{nL}(M))\}_{n\geqslant 1}$ be a family of motion groups, considered as a groupoid, and let $\mathfrak{U}\mathscr{G}$ be the associated "homogeneous category" of* [RW17] *(whose underlying groupoid $(\mathfrak{U}\mathscr{G})^{\sim}$ is $\mathscr{G}$). There is then a construction, taking as input a functor from a certain topological category of embeddings between manifolds to the category of covering spaces, and outputting a representation of the category $\mathfrak{U}\mathscr{G}$. Moreover, there is an iterative variant of this in which the construction itself is twisted by a representation of the category $\mathfrak{U}\mathscr{G}$, giving an endofunctor of the category of representations of $\mathfrak{U}\mathscr{G}$.*

Based on this construction, we obtain the following results for the classical braid groups.

Theorem ([PS19]) *If $\mathscr{G} = \beta$ is the family of braid groups and $m \geqslant 2$,*[4] *we obtain functors*

$$\mathfrak{LB}_m \colon \mathfrak{U}\beta \longrightarrow \mathbb{Z}[q^{\pm}, t^{\pm}]\text{-Mod}$$

that, on objects, recover the Lawrence-Bigelow representations $\mathfrak{LB}_{m,n}$ of B_n. Moreover, the functor $\mathfrak{LB}_m$ is polynomial of degree m. Hence, from [RW17]*, we deduce that the twisted homology groups $H_i(B_n; \mathfrak{LB}_{m,n})$ are stable (independent of n for $n \gg i$) for any fixed m.*

[3] More precisely, their representation theory is "wild" in the sense that the representation theory of the free group F_2 may be embedded into the representation theory of B_n for any $n \geqslant 3$. This also implies that the representation theory of B_n (for any fixed $n \geqslant 3$) contains the representation theory of all finite groups, and that there are k-parameter families of irreducible representations of B_n for arbitrarily large k.

[4] These statements also hold for $m = 1$, but in this case the ground ring is $\mathbb{Z}[t^{\pm}]$ instead of $\mathbb{Z}[q^{\pm}, t^{\pm}]$; in other words, we set $q = 1$.

Using the iterative version of our construction above, we also recover the Long-Moody construction as an endofunctor

$$\mathfrak{LM}\colon \mathrm{Functors}(\mathfrak{U}\beta, k\text{-Mod}) \longrightarrow \mathrm{Functors}(\mathfrak{U}\beta, k\text{-Mod})$$

for any ring k. Moreover, we construct a new family $\{\mathfrak{LM}_m\}_{m\geqslant 1}$ *of "higher Long-Moody constructions" with* $\mathfrak{LM}_1 = \mathfrak{LM}$.

For the loop braid groups LB_n and extended loop braid groups LB_n^{ext}, we obtain:

Theorem ([PS19]) *If* $\mathcal{G} = L\beta = \{LB_n\}_{n\geqslant 1}$ *is the family of loop braid groups and* $\mathcal{G} = L\beta^{\mathrm{ext}} = \{LB_n^{\mathrm{ext}}\}_{n\geqslant 1}$ *the family of extended loop braid groups, we obtain, for any integers* $m \geqslant 2$,[5] *functors*

$$\mathfrak{LB}_m^{\alpha}\colon \mathfrak{U}L\beta \longrightarrow \mathbb{Z}[q^{\pm}, s^{\pm}, t^{\pm}]/(s^2)\text{-Mod},$$
$$\mathfrak{LB}_m^{\beta}\colon \mathfrak{U}(L\beta^{\mathrm{ext}}) \longrightarrow \mathbb{Z}[q^{\pm}, r^{\pm}, s^{\pm}, t^{\pm}]/(q^2, r^2, s^2, t^2)\text{-Mod},$$
$$\mathfrak{LB}_m^{\gamma}\colon \mathfrak{U}(L\beta^{\mathrm{ext}}) \longrightarrow \mathbb{Z}[q^{\pm}, s^{\pm}, t^{\pm}]/(s^2, t^2)\text{-Mod}.$$

In particular, these give coherent families of representations, over rings of Laurent polynomials, of the loop braid groups $\{LB_n\}_{n\geqslant 1}$ *(for the first) and of the extended loop braid groups* $\{LB_n^{\mathrm{ext}}\}_{n\geqslant 1}$ *(for the second and third), which are analogues of the Lawrence-Bigelow representations of the classical braid groups* $\{B_n\}_{n\geqslant 1}$.

6. Higher crossing diagrams in knot theory

— *Reidemeister moves for triple-crossing link diagrams; relations between different n-crossing numbers for links.* —

The last theme of my previous work is in knot theory (which one may think of as the study of π_0 of the moduli space of 1-dimensional submanifolds of the 3-sphere), and more precisely with the representation of links by diagrams in the plane (or 2-sphere). A classical link diagram is an immersion of a 1-manifold into the plane, which is an embedding except at a finite number of double points, where the 1-manifold must intersect itself transversely, together with additional data at each intersection point specifying which strand passes "over" the other at that point. For a given link, such a diagram is unique up to ambient isotopy and the well-known *Reidemeister moves*.

In joint work with **Colin Adams** and **Jim Hoste** [AHP17], we study instead *triple-crossing diagrams*, which consist of an immersed 1-manifold in the plane, which is an embedding except at a finite number of points, at which *exactly three* strands must intersect transversely (plus additional data at each intersection point specifying which strands pass "over" others). We introduce an analogue of the Reidemeister moves for such diagrams, consisting of:

- Analogues of the (classical) I- and II-moves, which may be thought of as surgeries supported on a small subdisc of a diagram.
- The *trivial pass move*, which consists of cutting a strand and re-attaching it through another part of the diagram without introducing any new crossings. This may be thought of as a surgery on an annular neighbourhood of a diagram.
- Two families of moves, called the *band moves* and *basepoints moves*, which each consist of a surgery supported on a pair of disjoint subdiscs of the diagram.

Definition For a link L in S^3, a *maximal nonsplit sublink* of L is a sublink L_1 of L such that (a) there exists an embedded 2-sphere in $S^3 \smallsetminus L$ separating L_1 and $L \smallsetminus L_1$, and (b) there does *not* exist any embedded 2-sphere in $S^3 \smallsetminus L_1$ separating L_1 into two smaller sublinks.

A *relative orientation* of L is an orientation of L modulo orientation-reversal of each maximal nonsplit sublink. Equivalently, this is a choice, for each maximal nonsplit sublink L_1 of L, of an orientation of L_1 modulo complete reversal (reversing every component of L_1 simultaneously).

Theorem ([AHP17]) *A triple-crossing diagram determines a relatively oriented link. Any two triple-crossing diagrams representing the same relatively oriented link differ by a finite sequence of I-moves, II-moves, trivial pass moves, band moves, basepoints moves and ambient isotopy.*

[5] These statements also hold for $m = 1$, but in this case we set $q = s = 1$.

The notion of triple-crossing diagram may be generalised to any integer $n \geqslant 2$, with $n = 2$ corresponding to the classical notion of link diagram. The *n-crossing number* $c_n(L)$ of a link L is then the smallest number of crossings among all n-crossing diagrams of that link. One can ask how the sequence $\{c_n(L) \mid n \in \mathbb{N}\}$ behaves for each L, and which relations between the crossing numbers hold for all links L (or for all but finitely many links L). For example, it is not hard to show that $c_n(L) \geqslant c_{n+2}(L)$ for all n and L, and this inequality is known be strict for $n = 2$. We prove that, with a few small exceptions, it is also strict for $n = 3$.

Theorem ([AHP17]) *Let L be a non-split link that is neither the unlink nor the Hopf link. Then*

$$c_3(L) > c_5(L). \tag{6}$$

Note that this inequality is clearly false for the unlink and the Hopf link: for these links, $c_3(L)$ and $c_5(L)$ are both equal to 0 and both equal to 1, respectively.

Also note that, if L can be written as the union of maximal nonsplit sublinks $L_1, \ldots, L_k$, then $c_n(L) = c_n(L_1) + \cdots + c_n(L_k)$. Thus the strict inequality (6) holds whenever L has at least one maximal nonsplit sublink that is nontrivial and not the Hopf link.

— References —

[AHP17] C. Adams, J. Hoste and M. Palmer. *Triple-crossing number and moves on triple-crossing link diagrams*. ArXiv:1706.09333v1. 2017. To appear in Journal of Knot Theory and Its Ramifications. (↑ 9, 10).

[BCT89] C.-F. Bödigheimer, F. Cohen and L. Taylor. *On the homology of configuration spaces*. *Topology* 28.1 (1989), pp. 111–123 (↑ 5).

[Ber+06] A. J. Berrick, F. R. Cohen, Y. L. Wong and J. Wu. *Configurations, braids, and homotopy groups*. *J. Amer. Math. Soc.* 19.2 (2006), pp. 265–326 (↑ 5).

[Bet02] S. Betley. *Twisted homology of symmetric groups*. *Proc. Amer. Math. Soc.* 130.12 (2002), 3439–3445 (electronic) (↑ 4).

[BM14] M. Bendersky and J. Miller. *Localization and homological stability of configuration spaces*. *Q. J. Math.* 65.3 (2014), pp. 807–815. {arxiv:1212.3596} (↑ 5).

[Bol12] S. K. Boldsen. *Improved homological stability for the mapping class group with integral or twisted coefficients*. *Math. Z.* 270.1-2 (2012), pp. 297–329. {arxiv:0904.3269} (↑ 4).

[CF13] T. Church and B. Farb. *Representation theory and homological stability*. *Adv. Math.* 245 (2013), pp. 250–314. {arxiv:1008.1368} (↑ 4).

[Chu12] T. Church. *Homological stability for configuration spaces of manifolds*. *Invent. Math.* 188.2 (2012), pp. 465–504. {arxiv:1103.2441} (↑ 5).

[CM09] R. L. Cohen and I. Madsen. *Surfaces in a background space and the homology of mapping class groups*. *Algebraic geometry—Seattle 2005. Part 1*. Vol. 80. Proc. Sympos. Pure Math. Providence, RI: Amer. Math. Soc., 2009, pp. 43–76. {arxiv:math/0601750} (↑ 4).

[CP15] F. Cantero and M. Palmer. *On homological stability for configuration spaces on closed background manifolds*. *Doc. Math.* 20 (2015), pp. 753–805. {arxiv:1406.4916} (↑ 2, 4, 5).

[Dwy80] W. G. Dwyer. *Twisted homological stability for general linear groups*. *Ann. of Math. (2)* 111.2 (1980), pp. 239–251 (↑ 4).

[EVW16] J. S. Ellenberg, A. Venkatesh and C. Westerland. *Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields*. *Ann. of Math. (2)* 183.3 (2016), pp. 729–786. {arxiv:0912.0325} (↑ 1).

[Har85] J. L. Harer. *Stability of the homology of the mapping class groups of orientable surfaces*. *Ann. of Math. (2)* 121.2 (1985), pp. 215–249 (↑ 1).

[Iva93] N. V. Ivanov. *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*. *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*. Vol. 150. Contemp. Math. Providence, RI: Amer. Math. Soc., 1993, pp. 149–194 (↑ 4).

[Kal80] W. van der Kallen. *Homology stability for linear groups*. *Invent. Math.* 60.3 (1980), pp. 269–295 (↑ 4).

[KM16] A. Kupers and J. Miller. *Sharper periodicity and stabilization maps for configuration spaces of closed manifolds*. *Proc. Amer. Math. Soc.* 144.12 (2016), pp. 5457–5468. {arxiv:1509.06410} (↑ 5).

[Knu17] B. Knudsen. *Betti numbers and stability for configuration spaces via factorization homology*. *Algebr. Geom. Topol.* 17.5 (2017), pp. 3137–3187. {arxiv:1405.6696} (↑ 5).

[Kra17] M. Krannich. *Homological stability of topological moduli spaces*. ArXiv:1710.08484v2. v2: 2019, v1: 2017. To appear in Geometry & Topology (↑ 3).

[Kup13] A. Kupers. *Homological stability for unlinked circles in a 3-manifold*. ArXiv:1310.8580v3. v3: 2017, v1: 2013 (↑ 7).

[McD75] D. McDuff. *Configuration spaces of positive and negative particles*. *Topology* 14 (1975), pp. 91–107 (↑ 1–3, 5).

[ML88] R. J. Milgram and P. Löffler. *The structure of deleted symmetric products*. *Braids (Santa Cruz, CA, 1986)*. Vol. 78. Contemp. Math. Providence, RI: Amer. Math. Soc., 1988, pp. 415–424 (↑ 5).

[MP15a] J. Miller and M. Palmer. *A twisted homology fibration criterion and the twisted group-completion theorem*. *Q. J. Math.* 66.1 (2015), pp. 265–284. {arxiv:1409.4389} (↑ 1, 3).

[MP15b] J. Miller and M. Palmer. *Scanning for oriented configuration spaces*. *Homology, Homotopy and Applications* 17.1 (2015), pp. 35–66. {arxiv:1306.6896} (↑ 1, 3).

[MS76] D. McDuff and G. Segal. *Homology fibrations and the "group-completion" theorem*. *Invent. Math.* 31.3 (1975/76), pp. 279–284 (↑ 1, 3).

[Mum83] D. Mumford. *Towards an enumerative geometry of the moduli space of curves*. *Arithmetic and geometry, Vol. II*. Vol. 36. Progr. Math. Boston, MA: Birkhäuser Boston, 1983, pp. 271–328 (↑ 1).

[MW07] I. Madsen and M. Weiss. *The stable moduli space of Riemann surfaces: Mumford's conjecture*. *Ann. of Math. (2)* 165.3 (2007), pp. 843–941. {arxiv:math/0212321} (↑ 1).

[Nag15] R. Nagpal. *FI-modules and the cohomology of modular representations of symmetric groups*. ArXiv:1505.04294v1. 2015 (↑ 5).

[Pal13] M. Palmer. *Homological stability for oriented configuration spaces*. *Trans. Amer. Math. Soc.* 365.7 (2013), pp. 3675–3711. {arxiv:1106.4540} (↑ 1, 3).

[Pal17] M. Palmer. *A comparison of twisted coefficient systems*. ArXiv:1712.06310v1. 2017 (↑ 4).

[Pal18a] M. Palmer. *Homological stability for moduli spaces of disconnected submanifolds, I*. ArXiv:1805.03917v2. v2: 2019, v1: 2018. To appear in Algebraic and Geometric Topology. (↑ 2, 5, 6).

[Pal18b] M. Palmer. *Homological stability for moduli spaces of disconnected submanifolds, II - Symmetric diffeomorphism groups and parametrised connected sum*. ArXiv:1807.07558v1. 2018 (↑ 2, 6, 7).

[Pal18c] M. Palmer. *Twisted homological stability for configuration spaces*. *Homology, Homotopy and Applications* 20.2 (2018), pp. 145–178. {arxiv:1308.4397} (↑ 2, 4).

[PS19] M. Palmer and A. Soulié. *A unified functorial construction of homological representations of families of groups*. ArXiv:1910.13423. 2019 (↑ 8, 9).

[PT14] M. Palmer and T. Tran. *Homological stability for subgroups of surface braid groups*. arXiv:1410.0923v1. 2014. To appear in Homology, Homotopy and Applications. NB: the first arXiv version of this article had a single author; I became a co-author at a later stage. (↑ 7).

[Ran13] O. Randal-Williams. *Homological stability for unordered configuration spaces*. *Q. J. Math.* 64.1 (2013), pp. 303–326. {arxiv:1105.5257} (↑ 5).

[RW17] O. Randal-Williams and N. Wahl. *Homological stability for automorphism groups*. *Adv. Math.* 318 (2017), pp. 534–626. {arxiv:1409.3541} (↑ 3, 4, 8).

[Seg73] G. Segal. *Configuration-spaces and iterated loop-spaces*. *Invent. Math.* 21 (1973), pp. 213–221 (↑ 1–3, 5).

[Seg79] G. Segal. *The topology of spaces of rational functions*. *Acta Math.* 143.1-2 (1979), pp. 39–72 (↑ 1, 2, 5).

[Til16] U. Tillmann. *Homology stability for symmetric diffeomorphism and mapping class groups*. *Math. Proc. Cambridge Philos. Soc.* 160.1 (2016), pp. 121–139. {arxiv:1510.07564} (↑ 7).