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Introduction

My research interests lie in the area of topology, including *algebraic topology* and *low-dimensional topology*, and interactions between the two. The two main themes of my previous research work are concerned with studying:

- The homology of *moduli spaces of submanifolds* of an ambient manifold (such as configuration spaces or spaces of links in $\mathbb{R}^3$), as well as *diffeomorphism groups* of manifolds, via the phenomenon of homological stability. This also includes *motivic cohomology* of configuration spaces on smooth algebraic varieties, and *configuration-section spaces* – where the complement of a configuration is not just "empty space", but is equipped with a "field" with singularities at the point-particles.
- The representation theory of the fundamental groups of such moduli spaces (which includes examples such as *surface braid groups*, *loop braid groups* and *mapping class groups*) through topologically-defined *homological representations*.

As well as being central objects in topology, these moduli spaces have important applications in many other domains. A key example is the *Mumford conjecture* [Mum83] – a statement from algebraic geometry about Riemann's moduli spaces of algebraic curves – which was proven by Madsen and Weiss [MW07], using purely topological methods. More recently, homological stability for certain *configuration-mapping spaces* was proven by Ellenberg, Venkatesh and Westerland [EVW16], from which they deduced a result in analytic number theory: an asymptotic version of the *Cohen-Lenstra conjecture* for function fields. (See also §7 below.)

Overview

Concerning the *homology of moduli spaces of submanifolds*:

(a) I proved [Pal13] in my thesis that the *oriented configuration spaces* $C_n^+(M)$ on open, connected manifolds M are homologically stable: a point in this space is a configuration of points in M, equipped with the *non-local* data of an ordering modulo the action of the alternating group. Together with J. Miller [MP15b], we then identified a space modelling the limiting homology of these spaces, as the number of particles goes to infinity, lifting the classical result of D. McDuff [McD75] and G. Segal [Seg73; Seg79] (which concerns *unordered* configuration spaces). Along the way, we also generalised their *group-completion theorem* [MS76] to the setting of twisted-homology-equivalences [MP15a]. — See §1 for more details.

(b) The unordered configuration spaces $C_n(M)$ on a *closed*, connected manifold M are known *not* to stabilise in general, and represent a much more difficult and complicated case. Together with F. Cantero [CP15], we proved that there are nevertheless some more subtle *stability and periodicity phenomena* in the homology of configuration spaces on closed manifolds. For example, the homology of $C_n(M)$ with coefficients in $\mathbb{Z}[\frac{1}{2}]$ stabilises for $\dim(M)$ odd, whereas, for $\dim(M)$ even, their mod-p homology is unstable but *periodic* in a stable range (with an explicit period depending only on p and $\chi(M)$). — See §3 for more details of these results.

(c) Another direction in which I have taken these ideas, which opens the door to a wide variety of interesting new examples, is to generalise configuration spaces to *moduli spaces of disconnected submanifolds*, where point particles are replaced with embedded submanifolds of a specified diffeomorphism type and isotopy class (which may be parametrised, oriented, unoriented, etc.). Under a certain restriction on the codimension, I have proven [Pal18a] that these moduli spaces are also *homologically stable* as the number of components of the submanifold goes to infinity. As a corollary, I also proved [Pal18b] homological stability for:

- moduli spaces of *manifolds with conical singularities*, with respect to the number of singularities of a given type,
- the *symmetric diffeomorphism groups* of any sequence of manifolds obtained by iterating the operation of "parametric connected sum", an operation which generalises both ordinary connected sum and surgery.

See §4 for more details of these results.

(d) *Configuration-section spaces* model point-particles moving in a background field that is undefined (and may be singular) at the particles themselves. Particular examples of these include the classical *Hurwitz spaces*, which classify branched coverings of the 2-disc. In joint work with U. Tillmann [PT20a], we proved homological stability for configuration-section spaces, subject to a condition on the permitted "charges" of the particles. This has some intersection with, but is mostly complementary to, a homological stability result of Ellenberg, Venkatesh and Westerland [EVW16], which they used to deduce an asymptotic version of the *Cohen-Lenstra conjecture* for function fields.
In parallel, we also studied the natural monodromy action of $\pi_1(C_k(M))$ on the fibres of configuration-mapping spaces on M, and obtained explicit formulas when the dimension of M is at least 3 [PT20b]. See §7 for more details of both of these results.

(e) Together with G. Horel [HP20], we have proven an analogue of classical homological stability for the *motivic* and *étale motivic* cohomology of configuration schemes on a smooth scheme X. See §8.

Concerning *motion groups* (the fundamental groups of moduli spaces of submanifolds) and *mapping class groups*:

(f) I have proven [Pal18c] that the unordered configuration spaces $C_n(M)$ are homologically stable with coefficients in any *polynomial twisted coefficient system* (this in particular includes a choice of representation of $\pi_1(C_n(M))$ for each n). — See §2 for more details.

(g) In joint work with A. Soulié [PS19], we have set up a general topological construction of representations of motion groups (fundamental groups of moduli spaces of submanifolds) and mapping class groups. This generalises and unifies several known constructions, including the *Lawrence-Bigelow representations* and the *Long-Moody construction*. It also produces many new families of representations, in particular for the loop-braid groups. See §5 for more details.

(h) The *Lawrence-Bigelow representations*, and other representations of mapping class groups of surfaces, come in many different "flavours", depending on which part of the boundary of the surface is removed, and which type of homology (such as locally finite homology, ordinary homology relative to the boundary, etc.) is considered. In joint work with C. Anghel [AP20], we have investigated the fundamental relationships between these different flavours, establishing various non-degenerate pairings and embeddings between them. See §9 for details.

In addition, I have also studied *multi-crossing diagrams* for links, in joint work with C. Adams and J. Hoste: our main result is the construction of a complete set of Reidemeister moves for *triple-crossing diagrams*. — See §6 for more details.

1. Configuration spaces with non-local structure

— *Homological stability for oriented configuration spaces; a twisted group-completion theorem; stable homology of oriented configuration spaces.* —

For a space M, the nth ordered configuration space $\widetilde{C}_n(M)$ is defined to be the subspace of M^n consisting of all n-tuples of pairwise distinct points in M. The symmetric group $\mathfrak{S}_n$ acts on this space, and we define

$$C_n(M) = \widetilde{C}_n(M)/\mathfrak{S}_n \qquad \text{and} \qquad C_n^+(M) = \widetilde{C}_n(M)/A_n,$$

where $A_n < \mathfrak{S}_n$ is the alternating group. These are called, respectively, the *unordered* configuration space on M and the *oriented* configuration space on M. It is a classical result, going back to McDuff [McD75] and Segal [Seg73; Seg79], that the sequence $C_n(M)$, when M is a connected, open manifold, is *homologically* *stable*. This means that, for each degree i, there are isomorphisms $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ once n is sufficiently large (depending on i).

Aside. The condition that M is connected is clearly necessary, as one can see by considering H_0. On the other hand, the condition that M is open is not obviously necessary, and the situation in this case is much more subtle. This is the subject of another part of my previous work, see §3.

The ordered configuration spaces $\widetilde{C}_n(M)$ are not homologically stable: for example, the first homology of $\widetilde{C}_n(\mathbb{R}^2)$ is the abelianisation of the pure braid group, which is $\mathbb{Z}^{\binom{n}{2}}$. This raises the question of whether there is an intermediate covering space between $\widetilde{C}_n(M)$ and $C_n(M)$ for which homological stability still holds. I proved in [Pal13] that the answer is positive for the oriented configuration spaces $C_n^+(M)$, which doubly cover the unordered configuration spaces $C_n(M)$:

Theorem ([Pal13]) *The natural stabilisation map $C_n^+(M) \to C_{n+1}^+(M)$ induces isomorphisms on homology in degrees $* \leqslant \frac{n-5}{3}$ and surjections in degrees $* \leqslant \frac{n-2}{3}$.*

The theorem also holds more generally for *labelled configuration spaces*, where each point is equipped with a "label" in some fixed, path-connected parameter space X. The so-called *slope* of the stability range here is $\frac{1}{3}$ (and one may calculate explicitly for certain M to see that this is sharp), in contrast to the slope of $\frac{1}{2}$ that holds for the unordered configuration spaces.

Note. The integral homology of $C_n^+(M)$ may be interpreted as a certain twisted homology group $H_*(C_n(M);\mathbb{Z}[\mathbb{Z}/2])$, where $\pi_1(C_n(M))$ acts on the group ring of $\mathbb{Z}/2$ via the natural projection $\pi_1(C_n(M)) \to \mathfrak{S}_n$ followed by the sign homomorphism. This is an example of an *abelian twisted coefficient system* on $C_n(M)$, and is a precursor (in a special case) of the notion of *abelian homological stability*, which has been developed more recently by Randal-Williams and Wahl [RW17] and Krannich [Kra19]. See §2 for more twisted homological stability results.

This result leads to the question of whether one can identify the *stable homology* of the sequence $\{C_n^+(M) \mid n \in \mathbb{N}\}$, in other words the colimit $\lim_{n\to\infty} H_*(C_n^+(M))$, in terms of other well-understood spaces. In joint work with **Jeremy Miller**, we answered this question positively by lifting the classical *scanning map* [Seg73; McD75] to a homology equivalence between appropriate covering spaces.

- First, in our paper [MP15a], we generalise the McDuff-Segal *group-completion theorem* [MS76] as well as McDuff's *homology fibration criterion* [McD75, §5] to the setting of homology with twisted coefficients (more precisely to a setting where we consider homology with respect to all twisted coefficient systems in a fixed class $\mathfrak{C}$ that is closed under pullbacks).
- Using these tools, we proved in [MP15b] a new kind of scanning result, lifting the classical scanning map to covering spaces and showing that it remains a homology equivalence after doing so. This identifies the stable homology of oriented configuration spaces on M with the homology of an explicit double cover of the section space of a certain bundle over M:

Theorem ([MP15b]) *Writing $\dot{T}M \to M$ for the fibrewise one-point compactified tangent bundle of M and $\Gamma_c(\dot{T}M \to M)_\circ$ for its space of compactly-supported sections of degree zero, we have:*

$$\lim_{n\to\infty} H_*(C_n^+(M)) \;\cong\; H_*(\widetilde{\Gamma}_c(\dot{T}M \to M)_\circ), \tag{1}$$

for a certain explicit double cover $\widetilde{\Gamma}_c(\dot{T}M \to M)_\circ \longrightarrow \Gamma_c(\dot{T}M \to M)_\circ$.

This double cover may be defined as the connected covering space of $\Gamma_c = \Gamma_c(\dot{T}M \to M)_\circ$ corresponding to the projection

$$\pi_1(\Gamma_c) \to H_1(\Gamma_c) \cong H_1(C_2(M)) \to H_1(C_2(\mathbb{R}^\infty)) \cong \mathbb{Z}/(2).$$

Here, the isomorphism $H_1(\Gamma_c) \cong H_1(C_2(M))$ arises from homological stability and the identification of the stable homology for *unordered* configuration spaces, and the map $H_1(C_2(M)) \to H_1(C_2(\mathbb{R}^\infty))$ is induced by any embedding $M \hookrightarrow \mathbb{R}^\infty$. For example, when $M = \mathbb{R}^\infty$ the right-hand side of (1) is the universal cover of (one component of) the infinite loopspace $\Omega^\infty S^\infty = Q\mathbb{S}^0$. When $M = \mathbb{R}^2$ it is the unique connected double cover of $\Omega^2 S^3$.

2. Twisted homological stability

— *Twisted homological stability for configuration spaces; different notions of the degree of twisted coefficient systems.* —

As well as the notion of abelian homological stability mentioned in the previous section, another sense in which a sequence of spaces (or groups) can satisfy *twisted homological stability* is with respect to a so-called *polynomial twisted coefficient system*. This consists of a choice of local coefficient system on each space X_n, together with additional morphisms of local coefficient systems between them, organised into a functor $\mathscr{C} \to \mathsf{Ab}$, where the automorphism groups of $\mathscr{C}$ are the fundamental groups $\pi_1(X_n)$. In addition, the *degree* of this functor (defined using extra structure on $\mathscr{C}$) is required to be finite.

Many families of groups $G = \{G_n\}$ are known to be homologically stable in this sense (where we set $X_n = BG_n$ in the above paragraph), for appropriately-defined categories $\mathscr{C}_G$, for example the symmetric groups $\mathfrak{S}_n$, braid groups β_n, general linear groups, automorphism groups of free groups $\mathrm{Aut}(F_n)$ and mapping class groups of surfaces and of 3-manifolds.[1] In [Pal18c], I proved the first such result for a sequence of *spaces*, namely the unordered configuration spaces $C_n(M)$ on any connected, open manifold M. Note that, when $\dim(M) \geqslant 3$, these spaces are not aspherical (in contrast to the case of surfaces), so this does not reduce to a statement about the homology of their fundamental groups. When $\dim(M) = 2$, these configuration spaces are aspherical, so the result may be thought of as twisted homological stability for their fundamental groups, the surface braid groups $B_n(M) = \pi_1(C_n(M))$.

Theorem ([Pal18c]) *Let M be an open, connected manifold and let T be a twisted coefficient system for $\{C_n(M)\}$. This includes in particular the data of a local coefficient system T_n for each $C_n(M)$, as well as a homomorphism*

$$H_*(C_n(M);T_n) \longrightarrow H_*(C_{n+1}(M);T_{n+1}).$$

This homomorphism is split-injective in all degrees, and, if T has finite degree d, it is an isomorphism in the range $* \leqslant \frac{n-d}{2}$.

This theorem also generalises to configuration spaces with labels in any path-connected space X, as in the previous section.

Twisted coefficient systems. In connection with these results, I have also explored in more depth [Pal17] the notion of *twisted coefficient system* (a.k.a. finite-degree or polynomial functor), and in particular the *degree* of a twisted coefficient system. The main results of [Pal17] are:

- A comparison and unification of various different notions of "finite-degree" functor $\mathscr{C} \to \mathscr{A}$, where $\mathscr{A}$ is an abelian category and $\mathscr{C}$ is a category with various kinds of additional structure.
- The development of a functorial construction of (*injective* or *partial*) *braid categories*, which were used in [Pal18c] as the domain of definition of twisted coefficient systems, and of finite-degree functors on such braid categories.

3. Configurations on closed manifolds

— *Stability phenomena for configuration spaces on closed manifolds.* —

When M is closed, homological stability for the unordered configuration spaces $C_n(M)$ is not true in general, for example one may calculate that $H_1(C_n(S^2);\mathbb{Z}) \cong \mathbb{Z}/(2n-2)$, which does not stabilise as $n \to \infty$. Moreover, the stabilisation maps mentioned in the theorem in §1 do not exist, since these depend on adding a new configuration point in M "near infinity". In joint work with **Federico Cantero** [CP15], we prove three main results which show that the homology of configuration spaces on closed manifolds exhibits a large amount of stability despite these issues.

(1) When the Euler characteristic of M is zero, we construct *replication maps* $C_n(M) \to C_{\lambda n}(M)$ for any integer $\lambda \geqslant 2$, and prove that they induce homological stability after inverting λ:

[1] Symmetric groups: [Bet02]; braid groups: [CF13; RW17; Pal18c]; general linear groups: [Dwy80; Kal80]; automorphism groups of free groups: [RW17]; mapping class groups of surfaces: [Iva93; CM09; Bol12; RW17]; mapping class groups of 3-manifolds: [RW17]. Note that these are references for the proofs of *twisted* homological stability; in each case, homological stability with untwisted coefficients was known earlier.

Theorem ([CP15]) *These maps induce isomorphisms on $H_i(-,\mathbb{Z}[\frac{1}{\lambda}])$ in the range $2i \leqslant \lambda$.*

Note. A construction related to our replication maps has also been used in the paper [Ber+06], in which they use something similar to a replication map in §3 to build a crossed simplicial group out of the configuration spaces on any given manifold M that admits a non-vanishing vector field.

(2) When the manifold M is odd-dimensional, the configuration spaces $C_n(M)$ do in fact satisfy homological stability after inverting 2.

Theorem ([CP15]) *When* $\dim(M)$ *is odd, there are isomorphisms*

$$H_i(C_n(M);\mathbb{Z}[\tfrac{1}{2}]) \cong H_i(C_{n+1}(M);\mathbb{Z}[\tfrac{1}{2}]) \qquad \text{and} \qquad H_i(C_n(M);\mathbb{Z}) \cong H_i(C_{n+2}(M);\mathbb{Z})$$

in the range $2i \leqslant n$, induced by a zigzag of maps.

This strengthens a result of [BM14].

(3) When the manifold M is even-dimensional, and $\mathbb{F}$ is a field of characteristic 0 or 2, it is known by the work of many people [BCT89; ML88; Chu12; Ran13; BM14; Knu17] that homological stability holds for $C_n(M)$ with coefficients in $\mathbb{F}$, even when M is closed. When $\mathbb{F}$ has odd characteristic p, however, this is false, as one can see from the example of $M = S^2$ mentioned above. In fact:

$$H_1(C_n(S^2);\mathbb{F}) \cong \begin{Bmatrix} \mathbb{F} & p \mid n-1 \\ 0 & p \nmid n-1 \end{Bmatrix} \quad \text{for } n \geqslant 2.$$

From this example we see that the first homology of $C_n(S^2)$ is not stable, but it is p-periodic and takes on only 2 different values. Our third result is that this phenomenon holds in general, when the Euler characteristic χ of M is non-zero. Write $a = \nu_p(\chi)$ for the p-adic valuation of χ, in other words $\chi = p^a b$ with b coprime to p.

Theorem ([CP15]) *Suppose that* $\dim(M)$ *is even. For each fixed i, the sequence*

$$H_i(C_n(M);\mathbb{F}) \quad \text{for } n \geqslant 2i \tag{2}$$

is p^{a+1}-periodic and takes on at most $a+2$ values. Moreover, if $\chi \equiv 1$ mod p then the above sequence is 1-periodic, i.e. homological stability holds with coefficients in $\mathbb{F}$.

The p^{a+1}-periodicity result is similar to a theorem of [Nag15], although his estimate of the period is different, namely a power of p depending on i rather than on χ. This result was later improved by [KM16] to p-periodicity, independent of i or χ. Combining this with (a slightly more precise statement of) our result, a corollary is that in fact the sequence (2) above takes on only two different values.

4. Moduli spaces of disconnected submanifolds

— *Moduli spaces of disconnected submanifolds; symmetric diffeomorphism groups; manifolds with conical singularities; partitioned braid groups.* —

Instead of configurations of points in M (closed 0-dimensional submanifolds), one may consider configurations of closed submanifolds of M of higher dimension, which are diffeomorphic to the disjoint union of finitely many copies of a fixed ("model") manifold L. In the article [Pal18a] I proved that *moduli spaces of disconnected submanifolds* of this kind are also homologically stable as the number of components goes to infinity, just as in the classical setting of points [Seg73; McD75; Seg79] – under a certain hypothesis on the relative dimensions of the manifolds involved.

Let $\bar{M}$ be a connected manifold with non-empty boundary and of dimension at least 2, and denote its interior by M. Also fix a closed manifold L and an embedding $\iota_0 \colon L \hookrightarrow \partial\bar{M}$. Choose a self-embedding $e \colon \bar{M} \hookrightarrow \bar{M}$ which is isotopic to the identity and such that $e(\iota_0(L))$ is contained in the interior $M \subset \bar{M}$. We then obtain a sequence of pairwise-disjoint embeddings of L into M by defining $\iota_n := e^n \circ \iota_0$ for $n \geqslant 0$.

Let $nL = \{1,\ldots,n\} \times L$ and write $\iota_{1,\ldots,n} \colon nL \hookrightarrow M$ for the embedding $(i,x) \mapsto \iota_i(x)$.

Definition Define $C_{nL}(M)$ to be the path-component of

$$\mathrm{Emb}(nL,M)/\mathrm{Diff}(nL)$$

containing $[\iota_{1,\dots,n}]$. Here, the embedding space is given the Whitney topology and $C_{nL}(M)$ the quotient topology. There is a natural *stabilisation map*

$$C_{nL}(M) \longrightarrow C_{(n+1)L}(M) \tag{3}$$

defined by first adjoining the embedding ι_0 to a given embedding $nL \hookrightarrow M$, to obtain a new embedding $(n+1)L \hookrightarrow \bar{M}$, and then composing with the self-embedding e. In symbols, this may be written as $[\phi] \mapsto [\phi^+]$, where $\phi^+(i,x) = e \circ \phi(i,x)$ for $1 \leqslant i \leqslant n$ and $\phi^+(n+1,x) = \iota_1(x)$.

Theorem ([Pal18a]) *Assume that the dimensions $m = \dim(M)$ and $\ell = \dim(L)$ satisfy*

$$2\ell \leqslant m-3. \tag{4}$$

Then (3) induces isomorphisms on homology in degrees $* \leqslant \frac{n-2}{2}$ and surjections in degrees $* \leqslant \frac{n}{2}$.

This theorem may be extended further:

- There is a more general version of this setting, in which the submanifolds $L \subset M$ are parametrised modulo a subgroup of $\mathrm{Diff}(L)$ and come equipped with labels in some bundle over $\mathrm{Emb}(L,M)$. The theorem proved in [Pal18a] includes this more general setting.
- This setting is compatible with the techniques of §1 above, so we also have homological stability for "oriented" (in the sense of §1) versions of these moduli spaces, in which the submanifolds $L \subset M$ are ordered modulo even permutations.

Applications to diffeomorphism groups. In the sequel [Pal18b], I used homological stability for the moduli spaces $C_{nL}(M)$ (and their more refined versions mentioned in the first point above) to prove homological stability for:

- Symmetric diffeomorphism groups, with respect to parametric connected sum.
- Diffeomorphism groups of manifolds with conical singularities, with respect to the number of singularities.

Definition Given two embeddings $L \hookrightarrow M$ and $L \hookrightarrow Q$ with isomorphic normal bundles, one may cut out a tubular neighbourhood of each embedding and glue the resulting boundaries to obtain the *parametric connected sum* $M \sharp_L Q$. If L is a point this corresponds to the ordinary connected sum of M and Q. Other examples include the following.

- If $L \hookrightarrow Q$ is the canonical embedding $S^k \hookrightarrow S^m$, where $k \leqslant m = \dim(M)$, then $M \sharp_L Q$ is the result of a k-surgery on M.
- If $L \hookrightarrow Q$ is the embedding $S^1 \hookrightarrow \mathbf{T} \hookrightarrow \mathbf{T} \cup_{p/q} \mathbf{T} = L(p,q)$, where $\mathbf{T} = D^2 \times S^1$ denotes the solid torus, then $M \sharp_L Q$ is the result of a Dehn surgery of slope p/q on the 3-manifold M.

If we now iterate the operation $-\sharp_L Q$ many times, using a different copy of Q and a disjoint embedding $L \hookrightarrow M$ each time (but always using the same copy of M), we obtain a sequence

$$M \qquad M\sharp_L Q \qquad M\sharp_L Q \sharp_L Q \qquad M\sharp_L Q\sharp_L Q\sharp_L Q \qquad \cdots \tag{5}$$

of manifolds, which we abbreviate to $M \sharp_L^n Q$, the *nth iterated parametric connected sum*.

A diffeomorphism of $M \sharp_L^n Q$ is called *symmetric* if it fixes the boundary of M and preserves the decomposition of $M \sharp_L^n Q$ into pieces of the form $M \smallsetminus n\mathscr{T}(L)$ and $Q \smallsetminus \mathscr{T}(L)$, where $\mathscr{T}(L)$ is a tubular neighbourhood of L in M or Q. The corresponding subgroup

$$\Sigma\mathrm{Diff}(M \sharp_L^n Q) \;\leqslant\; \mathrm{Diff}(M \sharp_L^n Q)$$

is called the *symmetric diffeomorphism group* of $M \sharp_L^n Q$.[2]

[2] A mild technical condition has been elided from the definition of symmetric diffeomorphism group and in the statement of the theorem below, in order to simplify the discussion.

Theorem ([Pal18b]) *If M is connected and has non-empty boundary and $\dim(L) \leqslant \frac{1}{2}(\dim(M)-3)$, the sequence*

$$\cdots \longrightarrow B\Sigma\mathrm{Diff}(M\sharp_L^n Q) \longrightarrow B\Sigma\mathrm{Diff}(M\sharp_L^{n+1} Q) \longrightarrow \cdots$$

of (classifying spaces of) symmetric diffeomorphism groups is homologically stable.

This generalises a result of Tillmann [Til16], which corresponds to the case $L = \text{point}$ (i.e. the usual connected sum operation).

Informal definition Fix an $(m-1)$-dimensional manifold T. Let $\mathrm{cone}(T) = (T \times [0,\infty))/(T \times \{0\})$ be the open cone on T. An m-dimensional *manifold with conical T-singularities* is a space M that is locally homeomorphic to $\mathrm{cone}(T)$, together with a smooth atlas on the subset $M_{mfd} \subseteq M$ of locally Euclidean points of M. A *diffeomorphism* of M is a homeomorphism $M \to M$ that restricts to a diffeomorphism $M_{mfd} \to M_{mfd}$ and is of the form $\mathrm{cone}(\varphi)$ for some diffeomorphism $\varphi \colon T \to T$ near each point of the discrete subset $M \smallsetminus M_{mfd} \subseteq M$. These form a subgroup

$$\mathrm{Diff}^T(M) \;\leqslant\; \mathrm{Homeo}(M).$$

For example, we may construct a manifold with conical singularities by collapsing a tubular neighbourhood $\mathscr{T}(L)$ of any submanifold $L \subset M$. The quotient $M_L = M/\mathscr{T}(L)$ is then a manifold with a single conical $\partial\mathscr{T}(L)$-singularity. In particular, using the setting at the beginning of this subsection, we may collapse a tubular neighbourhood of each submanifold $\iota_i(L) \subset M$ for $1 \leqslant i \leqslant n$, to obtain a manifold $M_{n\cdot L}$ with (precisely n) conical $\partial\mathscr{T}(L)$-singularities.

Theorem ([Pal18b]) *If M is connected and has non-empty boundary and $\dim(L) \leqslant \frac{1}{2}(\dim(M)-3)$, the sequence of classifying spaces $B\mathrm{Diff}^{\partial\mathscr{T}(L)}(M_{n\cdot L})$ is homologically stable.*

Special values of ℓ and m. The condition (4) on the relative dimensions of L and M excludes some interesting special cases of the moduli space $C_{nL}(M)$.

One such special case is $\ell = 1$, $m = 3$, in other words, moduli spaces of links in a 3-manifold. If one considers moduli spaces of *unlinks* in a 3-manifold M, then this is known to be homologically stable as the number of components of the unlink goes to infinity, by a result of Kupers [Kup20]. However, the techniques of Kupers do not generalise to moduli spaces of non-trivial links (even if the components are pairwise unlinked), so this question is open for general links.

Another special case is $\ell = 0$, $m = 2$. If L is a point, this corresponds to configuration spaces of points on a surface M, for which homological stability is known classically. However, the only assumption that we have made about L is that it is closed, not necessarily connected, so we could also take $L = \{1,\ldots,\xi\}$ for any positive integer ξ. The moduli space $C_{nL}(M)$ is then the covering space of $C_{n\xi}(M)$ with

$$p_\xi(n) = \frac{(n\xi)!}{n!\,(\xi!)^n}$$

sheets, whose fibres correspond to all ways of partitioning the $n\xi$ points of a configuration into n subsets of size ξ. Since configuration spaces on M are aspherical (M is a connected surface with non-empty boundary), this is equivalent to studying the corresponding index-$p_\xi(n)$ subgroup of the surface braid group $B_{n\xi}(M)$, called the *partitioned surface braid group* $B_{\xi|n}(M)$, consisting of all braids that preserve a given partition $n\xi = \xi + \xi + \cdots + \xi$ of their endpoints. In joint work with **TriThang Tran**, we have shown that homological stability holds also in this special case, corresponding to $(\ell,m) = (0,2)$.

Theorem ([PT14]) *Let M be a connected surface with non-empty boundary. Then the sequence of partitioned surface braid groups $B_{\xi|n}(M)$ is homologically stable as $n \to \infty$, for any fixed $\xi \geqslant 1$.*

This recovers the classical case of homological stability for configuration spaces of points when $\xi = 1$, and is new for $\xi \geqslant 2$.

5. Homological representations of motion groups and mapping class groups

— *A unified functorial construction of representations of motion groups and mapping class groups; new representations of loop braid groups.* —

Mapping class groups, braid groups and more generally *motion groups* (fundamental groups of moduli spaces of submanifolds) typically have "wild" representation theory. It is therefore very useful to be able to construct representations of these groups by topological or geometric means, in order to be able to understand them with topological or geometric tools. As one very important example, Lawrence [Law90] and Bigelow [Big04] constructed families of linear representations of the classical braid groups starting from actions on the twisted homology of configuration spaces — these were used by Bigelow [Big01] and Krammer [Kra02] to prove the linearity of the braid groups.

In joint work with **Arthur Soulié**, we give a unified construction of such topological representations, which:

- applies simultaneously to all mapping class groups and motion groups in a given dimension d,
- produces a much wider family of representations,
- produces interesting representations also over non-commutative rings.

In more detail, in each dimension d, we construct a large family of representations of a category $\mathfrak{U}\mathscr{D}_d$ whose automorphism groups contain all mapping class groups and motion groups in dimension d. There are three parameters that one may vary in the construction:

- a submanifold $Z \subset \mathbb{R}^d$,
- two integers $\ell \geqslant 2$ and $i \geqslant 0$.

Each of these parameters may be varied to obtain interesting representations. For example, the family of Lawrence-Bigelow representations depends on an integer parameter $k \geqslant 1$; our construction recovers this in the case when Z is a 0-dimensional manifold of size k (and $d = 2$, $\ell = 2$, $i = k$). In dimensions $d \geqslant 3$, it becomes interesting to take higher-dimensional submanifolds of $\mathbb{R}^d$ for Z, in particular in the case of the *loop-braid groups* (the group of motions of an n-component unlink in $\mathbb{R}^3$).

The parameter $\ell \geqslant 2$ controls the ground ring over which the representation is defined: it is the group-ring of a group of nilpotency class at most $\ell - 1$. There are cases where the output of our construction is independent of ℓ (hence the ground ring is commutative), but there are also many interesting cases where we obtain an *infinite tower of representations* as $\ell \to \infty$. In particular, the family of Lawrence-Krammer-Bigelow representations is the $\ell = 2$ term of such a tower.

The parameter $i \geqslant 0$ controls the degree in which we take homology. In the case of the Lawrence-Bigelow representations, there is only one interesting degree in which we can take homology (the homology in other degrees being trivial). However, more generally there can be many interesting degrees in which to take homology. For example, if we consider the family of loop-braid groups and the "naive" analogue of the Lawrence-Bigelow representations (taking Z to be a 0-dimensional manifold of size k), then there are non-trivial homology groups in all degrees $k \leqslant i \leqslant 2k$.

There is also an iterative version of our construction, which recovers, in the case of the classical braid groups, the *Long-Moody construction* [Lon94].

As a sample of our construction, applied to the (extended and non-extended) loop-braid groups, we have the following. The family of loop-braid groups LB_n naturally forms a category $\mathfrak{U}\mathscr{L}\beta$, and similarly the family of extended loop-braid groups LB'_n forms a category $\mathfrak{U}\mathscr{L}\beta'$. For an integer $k \geqslant 1$, if we take either $Z = k$ or $Z = U_k$ (an unlink with k components), as well as $\ell = 2$ and $i = k$, we obtain:

Theorem ([PS19]) *Our construction specialises to define representations:*

$$\begin{aligned}
\mathfrak{L}_1(1,\mathscr{L}\beta)&\colon \mathfrak{U}\mathscr{L}\beta \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}]} & \mathfrak{L}_1(1,\mathscr{L}\beta')&\colon \mathfrak{U}\mathscr{L}\beta' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}/2]}\\
\mathfrak{L}_k(k,\mathscr{L}\beta)&\colon \mathfrak{U}\mathscr{L}\beta \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)]} & \mathfrak{L}_k(k,\mathscr{L}\beta')&\colon \mathfrak{U}\mathscr{L}\beta' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^2]}\\
\mathfrak{L}_1(U_1,\mathscr{L}\beta)&\colon \mathfrak{U}\mathscr{L}\beta \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)^2]} & \mathfrak{L}_1(U_1,\mathscr{L}\beta')&\colon \mathfrak{U}\mathscr{L}\beta' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^3]}\\
\mathfrak{L}_k(U_k,\mathscr{L}\beta)&\colon \mathfrak{U}\mathscr{L}\beta \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)^4]} & \mathfrak{L}_k(U_k,\mathscr{L}\beta')&\colon \mathfrak{U}\mathscr{L}\beta' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[(\mathbb{Z}/2)^5]}\\
\mathfrak{L}_1^+(U_1,\mathscr{L}\beta)&\colon \mathfrak{U}\mathscr{L}\beta \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^2]} & \mathfrak{L}_1^+(U_1,\mathscr{L}\beta')&\colon \mathfrak{U}\mathscr{L}\beta' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)]}\\
\mathfrak{L}_k^+(U_k,\mathscr{L}\beta)&\colon \mathfrak{U}\mathscr{L}\beta \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^3\oplus(\mathbb{Z}/2)]} & \mathfrak{L}_k^+(U_k,\mathscr{L}\beta')&\colon \mathfrak{U}\mathscr{L}\beta' \longrightarrow \mathrm{Mod}_{\mathbb{Z}[\mathbb{Z}^2\oplus(\mathbb{Z}/2)^2]},
\end{aligned}$$

where $k \geqslant 2$.

For example, this means that, if we take Z to be an unoriented unlink with two components, we obtain representations of all loop-braid groups LB_n, defined over the group-ring $\mathbb{Z}[\mathbb{Z}\oplus(\mathbb{Z}/2)^4]$, which may be thought of as a Laurent polynomial ring in 5 variables x_1,x_2,x_3,x_4,x_5 modulo the ideal generated by x_i^2-1 for $i=2,3,4,5$. If we quotient also by x_1^2-1, then these representations extend to the extended loop-braid groups $LB'_n \supset LB_n$.

Moreover, using non-stopping results for lower central series of *partitioned tripartite welded braid groups* from [DPS21], we obtain *infinite towers of representations* when we consider all $\ell \geqslant 2$ in the cases $Z \in \{2, U_2, U_3\}$, that is, a configuration of two points or an unlink with either two or three components (with one exception: when $Z=2$ and we construct representations of $\mathfrak{U}\mathscr{L}\beta'$, where the tower stops at $\ell = 3$).

We also compute explicitly the matrices of the representations encoded by the two functors $\mathfrak{L}_1(1,\mathscr{L}\beta)$ and $\mathfrak{L}_1(1,\mathscr{L}\beta')$. These extend the *reduced Burau representations* of the classical braid groups to LB_n and LB'_n respectively. The extension to LB_n is straightforward, whereas the extension to LB'_n is more indirect: we must first reduce the ground ring $\mathbb{Z}[\mathbb{Z}]$ to $R=\mathbb{Z}[\mathbb{Z}/2]$, then extend the underlying R-module R^{n-1} of the reduced Burau representation to a larger, non-free R-module and then extend the action of LB_n on R^{n-1} to an action of LB'_n on this larger module. This extension (in particular the matrices defining it) does not seem algebraically obvious, but it arises naturally as part of our general topological construction. The variant of our construction using *reduced* homology also defines extensions of the *unreduced* Burau representations to LB_n and LB'_n; these are more straightforward again, and in the case of LB_n they were already introduced by Vershinin in [Ver01, §4] by assigning explicit matrices to generators.

6. Higher crossing diagrams in knot theory

— *Reidemeister moves for triple-crossing link diagrams; relations between different n-crossing numbers for links.* —

Another theme of my previous work is in knot theory (which one may think of as the study of π_0 of the moduli space of 1-dimensional submanifolds of the 3-sphere), and more precisely with the representation of links by diagrams in the plane (or 2-sphere). A classical link diagram is an immersion of a 1-manifold into the plane, which is an embedding except at a finite number of double points, where the 1-manifold must intersect itself transversely, together with additional data at each intersection point specifying which strand passes "over" the other at that point. For a given link, such a diagram is unique up to ambient isotopy and the well-known *Reidemeister moves*.

In joint work with **Colin Adams** and **Jim Hoste** [AHP19], we study instead *triple-crossing diagrams*, which consist of an immersed 1-manifold in the plane, which is an embedding except at a finite number of points, at which *exactly three* strands must intersect transversely (plus additional data at each intersection point specifying which strands pass "over" others). We introduce an analogue of the Reidemeister moves for such diagrams, consisting of:

- Analogues of the (classical) I- and II-moves, which may be thought of as surgeries supported on a small subdisc of a diagram.
- The *trivial pass move*, which consists of cutting a strand and re-attaching it through another part of the diagram without introducing any new crossings. This may be thought of as a surgery on an annular neighbourhood of a diagram.

- Two families of moves, called the *band moves* and *basepoints moves*, which each consist of a surgery supported on a pair of disjoint subdiscs of the diagram – or equivalently a disjoint pair of simultaneous surgeries (supported on subdiscs of the diagram) that are appropriately compatible.

Definition For a link L in S^3, a *maximal nonsplit sublink* of L is a sublink L_1 of L such that (a) there exists an embedded 2-sphere in $S^3 \smallsetminus L$ separating L_1 and $L \smallsetminus L_1$, and (b) there does *not* exist any embedded 2-sphere in $S^3 \smallsetminus L_1$ separating L_1 into two smaller sublinks.

A *relative orientation* of L is an orientation of L modulo orientation-reversal of each maximal nonsplit sublink. Equivalently, this is a choice, for each maximal nonsplit sublink L_1 of L, of an orientation of L_1 modulo complete reversal (reversing every component of L_1 simultaneously).

For example, a relative orientation of a *knot* (or more generally a *totally split* link) is no information. On the other hand, the Hopf link has exactly two relative orientations (in contrast, it has four orientations).

Theorem ([AHP19]) *A triple-crossing diagram determines a relatively oriented link. Any two triple-crossing diagrams representing the same relatively oriented link differ by a finite sequence of I-moves, II-moves, trivial pass moves, band moves, basepoints moves and ambient isotopy. In other words:*

$$\Big\{\textit{Triple-crossing diagrams}\Big\}\big/\langle \textit{I, II, trivial pass, band, basepoint, isotopy}\rangle \;\cong\; \Big\{\textit{relatively oriented links}\Big\}$$

The notion of triple-crossing diagram may be generalised to any integer $n \geqslant 2$, with $n = 2$ corresponding to the classical notion of link diagram. The *n-crossing number* $c_n(L)$ of a link L is then the smallest number of crossings among all n-crossing diagrams of that link. One can ask how the sequence $\{c_n(L) \mid n \in \mathbb{N}\}$ behaves for each L, and which relations between the crossing numbers hold for all links L (or for all but finitely many links L). For example, it is not hard to show that $c_n(L) \geqslant c_{n+2}(L)$ for all n and L, and this inequality is known be strict for $n = 2$. We prove that, with a few small exceptions, it is also strict for $n = 3$.

Theorem ([AHP19]) *Let L be a non-split link that is neither the unlink nor the Hopf link. Then*

$$c_3(L) > c_5(L). \tag{6}$$

Note that this inequality is clearly false for the unlink and the Hopf link: for these links, $c_3(L)$ and $c_5(L)$ are both equal to 0 and both equal to 1, respectively.

Also note that, if L can be written as the union of maximal nonsplit sublinks $L_1, \ldots, L_k$, then $c_n(L) = c_n(L_1) + \cdots + c_n(L_k)$. Thus the strict inequality (6) holds whenever L has at least one maximal nonsplit sublink that is nontrivial and not the Hopf link.

7. Configuration-section spaces

— *Homological stability for configuration-section spaces; formulas for monodromy actions of configuration-mapping bundles.* —

Configuration-section spaces on a manifold M equipped with a bundle $E \to M$ are a kind of "non-local" configuration space (in a difference sense from the non-locality of §1), whose elements consists of a finite configuration in M together with a section of $E \to M$ on the *complement* of the configuration. Such spaces may be thought of physically as spaces of "fields" with singularities.

One often considers subspaces where the behaviour of the field (section) is constrained in a neighbourhood of the singularities (particles) – this may be thought of as restricting the allowed "charges" of the particles. More precisely, if the bundle $E \to M$ is trivial with fibre X, we fix a set $c \subseteq [S^{d-1}, X]$ of homotopy classes of maps $S^{d-1} \to X$, where d is the dimension of M. The sections are then required to restrict to an element of c, up to homotopy, in a neighbourhood of each particle (they are undefined at the particles themselves). [For non-trivial bundles $E \to M$, the definition is similar but a little more delicate.]

When $M = \mathbb{R}^2$ and the fibre X of the bundle is BG for a discrete group G, these are the *Hurwitz spaces*, classifying branched coverings of the 2-disc with deck transformation group G (and prescribed monodromy, if we impose a condition $c \subseteq [S^1, BG] = \mathrm{Conj}(G)$). These spaces have important connections with number theory through recent work of Ellenberg, Venkatesh and Westerland [EVW16], who proved an asymptotic version of the *Cohen-Lenstra conjecture* for function fields via a certain rational homological stability result for Hurwitz spaces.

In joint work with **Ulrike Tillmann**, we have proven another homological stability result for configuration-section spaces [PT20a], which is in a sense both more and less general than that of Ellenberg, Venkatesh and Westerland. It is *more* general in the sense that it holds for any bundle over any connected, open manifold M, but it is also *less* general in the sense that we assume a stronger condition on the allowed "charges" of the particles.

Theorem ([PT20a]) *Let M be a connected manifold of dimension $d \geqslant 2$ with basepoint $* \in \partial M$ and let $\xi \colon E \to M$ be a fibre bundle whose fibre over $*$ we denote by X. Assume that*

$$\text{the preimage of } c \text{ under } [S^{d-1}, X] \twoheadleftarrow \pi_{d-1}(X) \text{ is a single element,} \tag{7}$$

so c corresponds to a fixed point of the $\pi_1(X)$-action on $\pi_{d-1}(X)$. Then the stabilisation maps of configuration-section spaces

$$\mathrm{C}\Gamma_k^{c,*}(M;\xi) \longrightarrow \mathrm{C}\Gamma_{k+1}^{c,*}(M;\xi) \tag{8}$$

induce isomorphisms on $H_i(-;\mathbb{Z})$ in the range $k \geqslant 2i+4$ and surjections in the range $k \geqslant 2i+2$. With field coefficients, these ranges may be improved to $k \geqslant 2i+2$ and $k \geqslant 2i$ respectively.

Let us now assume that the bundle $E \to M$ is trivial with fibre X; in this case we write $\mathrm{CMap}_k^{c,*}(M;X)$ and call these *configuration-mapping spaces*. They fit into a natural fibre sequence

$$\mathrm{Map}^{c,*}(M \smallsetminus k \text{ points}, X) \longrightarrow \mathrm{CMap}_k^{c,*}(M;X) \longrightarrow C_k(\mathring{M}) \tag{9}$$

(the right-hand map is a "*configuration-mapping bundle*"), which is obtained functorially from the universal fibre sequence

$$M \smallsetminus k \text{ points} \longrightarrow U_k(M) \longrightarrow C_k(\mathring{M}). \tag{10}$$

In [PT20b], we obtain explicit formulas, when the dimension of M is at least 3, for the monodromy action $\pi_1(C_k(M)) \to \pi_0(\mathrm{hAut}(M \smallsetminus k \text{ points}))$ of the universal fibre sequence (10), from which we also deduce explicit formulas for the monodromy action of the fibre sequence (9). A special case of this is as follows:

Theorem ([PT20b]) *If $d = \dim(M) \geqslant 3$ and M satisfies at least one of the following conditions:*

- *M is simply-connected, or*
- *the handle-dimension of M is at most $d-2$;*

then the "point-pushing" action of $\gamma = (\alpha_1, \ldots, \alpha_k; \sigma) \in \pi_1(C_k(M)) \cong \pi_1(M)^k \rtimes \Sigma_k$ on the mapping space $\mathrm{Map}^{c,*}(M \smallsetminus z, X) \simeq \mathrm{Map}_*(M,X) \times (\Omega_c^{d-1}X)^k$ is given as follows:

$$(\alpha_1, \ldots, \alpha_k; \sigma) \cdot (f, g_1, \ldots, g_k) = (f, \bar{g}_1, \ldots, \bar{g}_k), \tag{11}$$

where $\bar{g}_i = f_*(\alpha_i).g_{\sigma(i)}.\mathrm{sgn}(\alpha_i)$, and

- *for an element $\alpha \in \pi_1(M)$ we write $\mathrm{sgn}(\alpha) = +1$ if α lifts to a loop in the orientation double cover of M and $\mathrm{sgn}(\alpha) = -1$ otherwise,*
- *$\pi_1(X)$ acts up to homotopy on $\Omega_c^{d-1}X$ in the natural way,*
- *$\{\pm 1\}$ acts on $\Omega_c^{d-1}X$ through the involution given by precomposition with a reflection of S^{d-1} in a hyperplane containing the basepoint.*

As a corollary, we obtain a precise description of the set of path-components $\pi_0(\mathrm{CMap}_k^{c,*}(M,X))$ of the configuration-mapping space, under the above conditions.

8. Motivic homological stability

— *Stability of motivic and étale cohomology of configuration schemes.* —

In joint work with **Geoffroy Horel** [HP20], we have proven that the *motivic and étale cohomology* of configuration spaces on smooth algebraic varieties (or schemes) is stable.

In more detail, if X is a smooth scheme over a number field K, one may ask whether the corresponding unordered configuration schemes $C_n(X)$ exhibit stability in their étale or motivic cohomology – lifting the classical stability for the singular (co)homology of the complex manifolds $C_n(X_{\mathbb{C}})$, where $X_{\mathbb{C}}$ denotes the complex points of X.

Theorem ([HP20]) *Suppose that X may be written as $Y - D$, where Y is a smooth scheme and D is a non-empty, closed, smooth subscheme admitting a K-point. If the étale motive of X is mixed Tate and the complex manifold $X_{\mathbb{C}}$ is connected, then there are maps of étale motivic cohomology groups*

$$H^{p,q}_{et}(C_{n+1}(X);\Lambda) \longrightarrow H^{p,q}_{et}(C_n(X);\Lambda)$$

that are isomorphisms for $p \leqslant n/2$ and any coefficient ring Λ. In the case when $X = \mathbb{A}^d$ is affine space, there are analogous maps of motivic cohomology groups

$$H^{p,q}(\widetilde{C}_{n+1}(\mathbb{A}^d);\Lambda) \longrightarrow H^{p,q}(\widetilde{C}_n(\mathbb{A}^d);\Lambda)$$

that are isomorphisms for $p \leqslant n/2$ and any coefficient ring Λ. (Here, $\widetilde{C}_n$ is a stacky version of the unordered configuration scheme C_n.)

9. Lawrence-Bigelow representations

In joint work with **Cristina Anghel** [AP20], we have investigated the fundamental relationships (in terms of non-degenerate pairings, embeddings and isomorphisms) between the many different flavours of homological representations of mapping class groups – with the *Lawrence-Bigelow representations* of the braid groups being our motivating example. Understanding the relationships between these representations is important for applications in quantum topology (see for example [Ang20]).

For any surface Σ equipped with a decomposition $\partial_{\mathrm{in}}\Sigma \cup \partial_{\mathrm{out}}\Sigma$ of its boundary (a *surface triad*), we think of $\partial_{\mathrm{in}}\Sigma$ as its *inner* "free" boundary and $\partial_{\mathrm{out}}\Sigma$ as its *outer* "fixed" boundary, and consider the mapping class group

$$\Gamma(\Sigma) = \pi_0(\mathrm{Diff}(\Sigma, \partial_{\mathrm{out}}\Sigma)).$$

For example, if $\Sigma = \Sigma_{0,n+1}$ with n of its boundary-components considered as *inner* and one considered as *outer*, this is isomorphic to the braid group on n strands. Choosing a local system $\mathscr{L}$ (defined over a ring R) on the configuration space $C_k(\Sigma)$ that is preserved by the action of $\Gamma(\Sigma)$, we obtain homological $\Gamma(\Sigma)$-representations

$$H^\bullet(C_\circ),$$

where $C_\circ$ is either $C_{\mathrm{out}} = C_k(\Sigma \smallsetminus \partial_{in}\Sigma)$ or $C_{\mathrm{in}} = C_k(\Sigma \smallsetminus \partial_{out}\Sigma)$ and $H^\bullet$ denotes one of

- ordinary homology H, twisted by $\mathscr{L}$,
- homology relative to the boundary H^∂, twisted by $\mathscr{L}$,
- locally-finite (Borel-Moore) homology H^{lf}, twisted by $\mathscr{L}$,
- locally-finite homology of the associated covering space $\widetilde{C}_k(\Sigma) \to C_k(\Sigma)$ (in the case where $\mathscr{L}$ arises from such a covering), denoted $H^{lf,\sim}$.

For example, when $\Sigma = \Sigma_{0,n+1}$ we may take $\mathscr{L}$ to be the local system arising from the covering corresponding to the kernel of the homomorphism

$$\varphi\colon \pi_1(C_k(\Sigma)) \longrightarrow \begin{cases} \mathbb{Z}\{c\} & k=1 \\ \mathbb{Z}\{c,x\} & k \geqslant 2 \end{cases}$$

given by $\varphi(\gamma) = ic + jx$, where i is the total winding number of γ around the n inner boundary components and j is the *writhe* of the braid obtained from γ by filling in the n inner boundary components with discs. In this example the B_n-representations above are 8 different flavours of the *Lawrence-Bigelow representations*.

Theorem ([AP20]) (i) *For $\circ \in \{\mathrm{in}, \mathrm{out}\}$, there is a non-degenerate pairing*

$$H^{lf}(C_\circ) \otimes H^\partial(C_\circ) \longrightarrow R.$$

Moreover, these representations are free as R-modules. We describe explicit bases such that the pairing above is given by the identity matrix.

(ii) *Under a mild condition on the local system $\mathscr{L}$, there are embeddings of representations*

$$H^\partial(C_{\mathrm{in}}) \longrightarrow H^{lf}(C_{\mathrm{out}})$$
$$H^\partial(C_{\mathrm{out}}) \longrightarrow H^{lf}(C_{\mathrm{in}}).$$

When $k \geqslant 2$ this implies that $H^{lf}(C_{\mathrm{out}})$ and $H^{lf}(C_{\mathrm{in}})$ are reducible. In fact, with respect to the explicit bases that we describe, the matrices of these embeddings are diagonal, and their diagonal entries are products of quantum factorials in R.

(iii) *For $\circ \in \{\mathrm{in}, \mathrm{out}\}$, there is a natural injective map*

$$H^{lf}(C_\circ) \longrightarrow H^{lf,\sim}(C_\circ).$$

If $\mathscr{B}$ denotes a free basis for $H^{lf}(C_\circ)$ as a module over $R = k[G]$, then $H^{lf,\sim}(C_\circ)$ is a direct sum over $\mathscr{B}$ of copies of the completion $k[[G]] = \prod_G k$ of $k[G] = \bigoplus_G k$.

References

[AHP19] C. Adams, J. Hoste and M. Palmer. *Triple-crossing number and moves on triple-crossing link diagrams*. *Journal of Knot Theory and Its Ramifications* 28.11 (2019), p. 1940001 (↑ 9, 10).

[Ang20] C. A.-M. Anghel. *Coloured Jones and Alexander polynomials as topological intersections of cycles in configuration spaces*. ArXiv:2002.09390. 2020 (↑ 12).

[AP20] C. A.-M. Anghel and M. Palmer. *Lawrence-Bigelow representations, bases and duality*. ArXiv:2011.02388. 2020 (↑ 2, 12).

[BCT89] C.-F. Bödigheimer, F. Cohen and L. Taylor. *On the homology of configuration spaces*. *Topology* 28.1 (1989), pp. 111–123 (↑ 5).

[Ber+06] A. J. Berrick, F. R. Cohen, Y. L. Wong and J. Wu. *Configurations, braids, and homotopy groups*. *J. Amer. Math. Soc.* 19.2 (2006), pp. 265–326 (↑ 5).

[Bet02] S. Betley. *Twisted homology of symmetric groups*. *Proc. Amer. Math. Soc.* 130.12 (2002), 3439–3445 (electronic) (↑ 4).

[Big01] S. J. Bigelow. *Braid groups are linear*. *J. Amer. Math. Soc.* 14.2 (2001), 471–486 (electronic) (↑ 8).

[Big04] S. Bigelow. *Homological representations of the Iwahori-Hecke algebra*. *Geom. Topol. Monogr.* 7 (2004), pp. 493–507 (↑ 8).

[BM14] M. Bendersky and J. Miller. *Localization and homological stability of configuration spaces*. *Q. J. Math.* 65.3 (2014), pp. 807–815 (↑ 5).

[Bol12] S. K. Boldsen. *Improved homological stability for the mapping class group with integral or twisted coefficients*. *Math. Z.* 270.1-2 (2012), pp. 297–329 (↑ 4).

[CF13] T. Church and B. Farb. *Representation theory and homological stability*. *Adv. Math.* 245 (2013), pp. 250–314 (↑ 4).

[Chu12] T. Church. *Homological stability for configuration spaces of manifolds*. *Invent. Math.* 188.2 (2012), pp. 465–504 (↑ 5).

[CM09] R. L. Cohen and I. Madsen. *Surfaces in a background space and the homology of mapping class groups*. *Algebraic geometry—Seattle 2005. Part 1*. Vol. 80. Proc. Sympos. Pure Math. Providence, RI: Amer. Math. Soc., 2009, pp. 43–76 (↑ 4).

[CP15] F. Cantero and M. Palmer. *On homological stability for configuration spaces on closed background manifolds*. *Doc. Math.* 20 (2015), pp. 753–805 (↑ 1, 4, 5).

[DPS21] J. Darné, M. Palmer and A. Soulié. *When the lower central series stops*. Preprint in progress, to be posted on the arXiv soon. 2021 (↑ 9).

[Dwy80] W. G. Dwyer. *Twisted homological stability for general linear groups*. *Ann. of Math. (2)* 111.2 (1980), pp. 239–251 (↑ 4).

[EVW16] J. S. Ellenberg, A. Venkatesh and C. Westerland. *Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields*. *Ann. of Math. (2)* 183.3 (2016), pp. 729–786 (↑ 1, 2, 10).

[HP20] G. Horel and M. Palmer. *Motivic homological stability of configuration spaces*. ArXiv:2007.13718. 2020 (↑ 2, 11, 12).

[Iva93] N. V. Ivanov. *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*. *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*. Vol. 150. Contemp. Math. Providence, RI: Amer. Math. Soc., 1993, pp. 149–194 (↑ 4).

[Kal80] W. van der Kallen. *Homology stability for linear groups*. *Invent. Math.* 60.3 (1980), pp. 269–295 (↑ 4).

[KM16] A. Kupers and J. Miller. *Sharper periodicity and stabilization maps for configuration spaces of closed manifolds*. *Proc. Amer. Math. Soc.* 144.12 (2016), pp. 5457–5468 (↑ 5).

[Knu17] B. Knudsen. *Betti numbers and stability for configuration spaces via factorization homology*. *Algebr. Geom. Topol.* 17.5 (2017), pp. 3137–3187 (↑ 5).

[Kra02] D. Krammer. *Braid groups are linear*. *Ann. of Math. (2)* 155.1 (2002), pp. 131–156 (↑ 8).

[Kra19] M. Krannich. *Homological stability of topological moduli spaces*. *Geom. Topol.* 23.5 (2019), pp. 2397–2474 (↑ 3).

[Kup20] A. Kupers. *Homological stability for unlinked circles in a 3-manifold*. *Q. J. Math.* 71.1 (2020), pp. 335–358 (↑ 7).

[Law90] R. J. Lawrence. *Homological representations of the Hecke algebra*. *Comm. Math. Phys.* 135.1 (1990), pp. 141–191 (↑ 8).

[Lon94] D. D. Long. *Constructing representations of braid groups*. *Comm. Anal. Geom.* 2.2 (1994), pp. 217–238 (↑ 8).

[McD75] D. McDuff. *Configuration spaces of positive and negative particles*. *Topology* 14 (1975), pp. 91–107 (↑ 1–3, 5).

[ML88] R. J. Milgram and P. Löffler. *The structure of deleted symmetric products*. *Braids (Santa Cruz, CA, 1986)*. Vol. 78. Contemp. Math. Providence, RI: Amer. Math. Soc., 1988, pp. 415–424 (↑ 5).

[MP15a] J. Miller and M. Palmer. *A twisted homology fibration criterion and the twisted group-completion theorem*. *Q. J. Math.* 66.1 (2015), pp. 265–284 (↑ 1, 3).

[MP15b] J. Miller and M. Palmer. *Scanning for oriented configuration spaces*. *Homology, Homotopy and Applications* 17.1 (2015), pp. 35–66 (↑ 1, 3).

[MS76] D. McDuff and G. Segal. *Homology fibrations and the "group-completion" theorem*. *Invent. Math.* 31.3 (1975/76), pp. 279–284 (↑ 1, 3).

[Mum83] D. Mumford. *Towards an enumerative geometry of the moduli space of curves*. *Arithmetic and geometry, Vol. II*. Vol. 36. Progr. Math. Boston, MA: Birkhäuser Boston, 1983, pp. 271–328 (↑ 1).

[MW07] I. Madsen and M. Weiss. *The stable moduli space of Riemann surfaces: Mumford's conjecture*. *Ann. of Math. (2)* 165.3 (2007), pp. 843–941 (↑ 1).

[Nag15] R. Nagpal. *FI-modules and the cohomology of modular representations of symmetric groups*. ArXiv:1505.04294. 2015 (↑ 5).

[Pal13] M. Palmer. *Homological stability for oriented configuration spaces*. *Trans. Amer. Math. Soc.* 365.7 (2013), pp. 3675–3711 (↑ 1, 3).

[Pal17] M. Palmer. *A comparison of twisted coefficient systems*. ArXiv:1712.06310. 2017 (↑ 4).

[Pal18a] M. Palmer. *Homological stability for moduli spaces of disconnected submanifolds, I*. ArXiv:1805.03917. 2018. To appear in Algebraic and Geometric Topology. (↑ 2, 5, 6).

[Pal18b] M. Palmer. *Stability for moduli spaces of manifolds with conical singularities*. ArXiv:1807.07558. 2018 (↑ 2, 6, 7).

[Pal18c] M. Palmer. *Twisted homological stability for configuration spaces*. *Homology, Homotopy and Applications* 20.2 (2018), pp. 145–178 (↑ 2, 4).

[PS19] M. Palmer and A. Soulié. *Topological representations of motion groups and mapping class groups – a unified functorial construction*. ArXiv:1910.13423. 2019 (↑ 2, 9).

[PT14] M. Palmer and T. Tran. *Homological stability for subgroups of surface braid groups*. ArXiv:1410.0923. 2014. To appear in Homology, Homotopy and Applications. NB: the first arXiv version of this article had a single author; I became a co-author at a later stage. (↑ 7).

[PT20a] M. Palmer and U. Tillmann. *Homology of configuration-mapping and -section spaces*. ArXiv:2007.11607. 2020 (↑ 2, 11).

[PT20b] M. Palmer and U. Tillmann. *Point-pushing actions and configuration-mapping spaces*. ArXiv:2007.11613. 2020 (↑ 2, 11).

[Ran13] O. Randal-Williams. *Homological stability for unordered configuration spaces*. *Q. J. Math.* 64.1 (2013), pp. 303–326 (↑ 5).

[RW17] O. Randal-Williams and N. Wahl. *Homological stability for automorphism groups*. *Adv. Math.* 318 (2017), pp. 534–626 (↑ 3, 4).

[Seg73] G. Segal. *Configuration-spaces and iterated loop-spaces*. *Invent. Math.* 21 (1973), pp. 213–221 (↑ 1–3, 5).

[Seg79] G. Segal. *The topology of spaces of rational functions*. *Acta Math.* 143.1-2 (1979), pp. 39–72 (↑ 1, 2, 5).

[Til16] U. Tillmann. *Homology stability for symmetric diffeomorphism and mapping class groups*. *Math. Proc. Cambridge Philos. Soc.* 160.1 (2016), pp. 121–139 (↑ 7).

[Ver01] V. V. Vershinin. *On homology of virtual braids and Burau representation*. Vol. 10. 5. Knots in Hellas '98, Vol. 3 (Delphi). 2001, pp. 795–812 (↑ 9).