# Lawrence representations

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**Abstract**

A quick description of Ruth Lawrence's construction of representations of surface braid groups.

## 1. A general construction

Let $X$ be a path-connected, locally path-connected, semi-locally simply-connected space, and let $x_0 \in X$ be a basepoint. Now choose a surjective group homomorphism

$$\phi \colon \pi_1(X, x_0) \longrightarrow Q,$$

and denote its kernel by $H = \ker(\phi) \leqslant \pi_1(X, x_0) = G$. This normal subgroup corresponds to a regular covering space $\xi \colon \widetilde{X} \to X$ with deck transformation group $\mathcal{D}(\xi) \cong N_H(G)/H = G/H \cong Q$. The action of $Q$ on $\widetilde{X}$ induces an action of the group-ring $\mathbb{Z}[Q]$ on the homology groups $H_i(\widetilde{X};\mathbb{Z})$ of $\widetilde{X}$, i.e., $H_i(\widetilde{X};\mathbb{Z})$ is a $\mathbb{Z}[Q]$-module.

Now let $B$ be another group, acting on $X$ by basepoint-preserving homeomorphisms, i.e., a group homomorphism

$$\theta \colon B \longrightarrow \mathrm{Homeo}_{x_0}(X).$$

This induces an action of $B$ on $\pi_1(X, x_0) = G$, i.e., a homomorphism $a_\theta \colon B \to \mathrm{Aut}(G)$, by the formula $a_\theta(b)([\gamma]) = [b \circ \gamma]$. Fix a basepoint $\widetilde{x}_0 \in \widetilde{X}$ such that $\xi(\widetilde{x}_0) = x_0$ and assume that:

(a) $a_\theta$ preserves the subgroup $H \leqslant G$. In other words: $a_\theta(b)(H) \leqslant H$ for all $b \in B$.

It then follows that:

(i) There are well-defined actions $a_\theta^{\mathrm{r}} \colon B \to \mathrm{Aut}(H)$ and $\bar{a}_\theta \colon B \to \mathrm{Aut}(Q)$.

(ii) The quotient homomorphism $\phi$ is *equivariant*, meaning that $\phi \circ a_\theta(b) = \bar{a}_\theta(b) \circ \phi$ for all $b \in B$.

(iii) There is a unique action $\widetilde{\theta} \colon B \to \mathrm{Homeo}_{\widetilde{x}_0}(\widetilde{X})$ such that $\xi \circ \widetilde{\theta}(b) = \theta(b) \circ \xi$ for all $b \in B$.

(iv) For any deck transformation $\psi \in \mathcal{D}(\xi) = Q$ we have $\widetilde{\theta}(b) \circ \psi = \bar{a}_\theta(\psi) \circ \widetilde{\theta}(b)$ for all $b \in B$.

From fact (iii) we get a well-defined induced action of $B$ on the homology groups of $\widetilde{X}$:

$$B \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H_i(\widetilde{X};\mathbb{Z})). \tag{1}$$

The notation $\mathrm{Aut}_{\mathbb{Z}}(-)$ means the group of automorphisms of $(-)$ as a $\mathbb{Z}$-module, i.e., as an abelian group. However, as we noticed above, the group $H_i(\widetilde{X};\mathbb{Z})$ has more structure than this: it is a $\mathbb{Z}[Q]$-module. Let's now make the additional assumption that:

(b) The action $\bar{a}_\theta$ of $B$ on $Q$ is trivial.

This implies, from fact (iv) above, that $\widetilde{\theta}(b)$ commutes with all deck transformations, for all $b \in B$. The induced action on the homology group $H_i(\widetilde{X};\mathbb{Z})$ of $\widetilde{X}$ therefore commutes with its structure as a $\mathbb{Z}[Q]$-module. In other words, the action (1) above is actually an action through automorphisms of $H_i(\widetilde{X};\mathbb{Z})$ that are $\mathbb{Z}[Q]$*-module automorphisms*, not just $\mathbb{Z}$-module automorphisms. So we may write this induced action as:

$$B \longrightarrow \mathrm{Aut}_{\mathbb{Z}[Q]}(H_i(\widetilde{X};\mathbb{Z})).$$

**Summary.** The upshot of this construction is that, if we have an action $\theta \colon B \to \mathrm{Homeo}_{x_0}(X)$ such that the induced action $a_\theta$ of $B$ on $\pi_1(X, x_0)$ commutes with our chosen surjective homomorphism $\phi \colon \pi_1(X, x_0) \twoheadrightarrow Q$ (i.e., $\phi \circ a_\theta(b) = \phi$ for all $b \in B$), then we get a well-defined induced action

$$h_{\theta,i} \colon B \longrightarrow \mathrm{Aut}_{\mathbb{Z}[Q]}(H_i(\widetilde{X};\mathbb{Z})). \tag{2}$$

## 2. Interlude: locally path-connected groups

Let $B$ be a topological group, i.e.: $B$ is a topological space and an abstract group, and the multiplication operation $\cdot\colon B\times B \to B$ and the inverse operation $(-)^{-1}\colon B \to B$ are both continuous. Then $\pi_0(B)$, the set of path-components of $B$, inherits a group structure from $B$. Also, if $\phi\colon B \to B'$ is a continuous group homomorphism, then it induces a group homomorphism $\pi_0(\phi)\colon \pi_0(B) \to \pi_0(B')$ between these abstract groups. Let us consider $\pi_0(B)$ to be a topological group by giving it the discrete topology. (Note: *any* abstract group may be viewed as a topological group by giving it the discrete topology.) The function

$$\pi_B\colon B \longrightarrow \pi_0(B),$$

taking a point in $B$ to the path-component that it lies in, is a group homomorphism. It is not always continuous, but if we assume that $B$ is locally path-connected, then it is continuous. If we have a continuous group homomorphism $\phi\colon B \to B'$ between locally path-connected groups, then

$$\pi_0(\phi) \circ \pi_B \;=\; \pi_{B'} \circ \phi.$$

In the special case where $B'$ is discrete, then $\pi_{B'}$ is the identity $B' \to \pi_0(B') = B'$, so the above equality reduces to $\pi_0(\phi) \circ \pi_B = \phi$.

**Summary.** Any continuous group homomorphism $\phi$ from a locally path-connected group $B$ to a discrete group $B'$ factors as

$$\phi \;=\; \pi_0(\phi) \circ \pi_B\colon B \longrightarrow \pi_0(B) \longrightarrow \pi_0(B') = B'.$$

## 3. A continuous version of the construction

Now let $B$ be a locally path-connected topological group, and assume that it acts continuously on $X$ through basepoint-preserving homeomorphisms. This just means that we have a *continuous* group homomorphism

$$\theta\colon B \longrightarrow \mathrm{Homeo}_{x_0}(X),$$

where $\mathrm{Homeo}_{x_0}(X)$ is given the subspace topology induced from the compact-open topology on $\mathrm{Map}(X, X)$.

There is again an induced action $a_\theta$ of $B$ on $\pi_1(X, x_0)$, and we choose a surjective homomorphism $\phi\colon \pi_1(X, x_0) \twoheadrightarrow Q$ that is invariant under this action, i.e., $\phi \circ a_\theta(b) = \phi$ for all $b \in B$. As before, the construction then gives us an induced action

$$h_{\theta,i}\colon B \longrightarrow \mathrm{Aut}_{\mathbb{Z}[Q]}(H_i(\widetilde{X}; \mathbb{Z})).$$

Since $B$ is locally path-connected, and $\pi_1(X, x_0)$ and $\mathrm{Aut}_{\mathbb{Z}[Q]}(H_i(\widetilde{X}; \mathbb{Z}))$ are both discrete, the actions $a_\theta$ and $H_\theta$ both factor through $\pi_0(B)$, as explained in the previous section:

$$\pi_0(a_\theta)\colon \pi_0(B) \longrightarrow \pi_1(X, x_0)$$
$$\pi_0(h_{\theta,i})\colon \pi_0(B) \longrightarrow \mathrm{Aut}_{\mathbb{Z}[Q]}(H_i(\widetilde{X}; \mathbb{Z})).$$

So it's enough to check that $\phi$ is invariant under the action of $\pi_0(B)$, i.e., $\phi \circ \pi_0(a_\theta)([b]) = \phi$ for all $[b] \in \pi_0(B)$. Equivalently, it's enough to check that $\phi \circ a_\theta(b) = \phi$ for at least one point $b$ in each path-component of $B$.

**Summary of the construction.** Let $X$ be a path-connected, locally path-connected and semi-locally simply-connected space, with basepoint $x_0$. Let $\phi\colon \pi_1(X, x_0) \twoheadrightarrow Q$ be a surjective group homomorphism and let $\xi\colon \widetilde{X} \to X$ be the path-connected covering associated to $\ker(\phi) \leqslant \pi_1(X, x_0)$.

Let $B$ be a locally path-connected group acting continuously on $X$ through basepoint-preserving homeomorphisms, i.e., there is a continuous group homomorphism $\theta\colon B \to \mathrm{Homeo}_{x_0}(X)$. Assume that $\phi$ is invariant under this action, i.e., $\phi([\gamma]) = \phi([\theta(b) \circ \gamma])$ for all $[\gamma] \in \pi_1(X, x_0)$ and all $b \in B$ (it's enough to check this for at least one $b$ in each path-component of $B$). Then there is a well-defined action

$$\pi_0(h_{\theta,i})\colon \pi_0(B) \longrightarrow \mathrm{Aut}_{\mathbb{Z}[Q]}(H_i(\widetilde{X}; \mathbb{Z})).$$

## 4. Application to configuration spaces on punctured discs

Let $\mathbb{D}_n$ denote the surface $\mathbb{D}^2 - \{p_1, \ldots, p_n\}$, where $p_1, \ldots, p_n$ are $n$ distinct points in the interior of the closed disc $\mathbb{D}^2$. Let $C_m(\mathbb{D}_n)$ denote the configuration space of $m$ unordered points in $\mathbb{D}^n$, namely

$$\{(x_1, \ldots, x_m) \in (\mathbb{D}_n)^m \mid x_i \neq x_j \text{ for } i \neq j\}/{\sim},$$

where the equivalence relation is $(x_1, \ldots, x_m) \sim (y_1, \ldots, y_m) \Leftrightarrow \{x_1, \ldots, x_m\} = \{y_1, \ldots, y_m\}$. Fix $m$ distinct points $\bar{x}_1, \ldots, \bar{x}_m$ in the boundary of the disc. We will use the configuration $\{\bar{x}_1, \ldots, \bar{x}_m\}$ as the basepoint $x_0$ of $X = C_m(\mathbb{D}_n)$.

**The quotient of the fundamental group.** First assume that $m = 1$, so $X = \mathbb{D}_n$. In this case, $\pi_1(X, x_0)$ is the free group on $n$ letters, and we can define a surjective group homomorphism as follows:

$$\phi\colon \pi_1(X, x_0) \cong F_n \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z} = Q,$$

where the first map is the abelianisation and the second is the sum map $(a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n$.

When $m \geqslant 2$ we define $\phi$ differently, and $Q$ is $\mathbb{Z}^2$ instead of $\mathbb{Z}$. First note that there are continuous maps

$$C_m(\mathbb{D}_n) \longrightarrow C_m(\mathbb{D}^2) \qquad \text{and} \qquad C_m(\mathbb{D}_n) \longrightarrow C_{m+n}(\mathbb{D}^2).$$

The first one takes a configuration in $\mathbb{D}_n$ to itself, which is in particular a configuration in $\mathbb{D}^2$. The second one takes a configuration $\{x_1, \ldots, x_m\}$ in $\mathbb{D}_n$ to the configuration $\{x_1, \ldots, x_m, p_1, \ldots, p_n\}$ in $\mathbb{D}^2$. We therefore get two homomorphisms

$$T\colon \pi_1(X, x_0) \longrightarrow B_m \longrightarrow \mathbb{Z} \qquad \text{and} \qquad R\colon \pi_1(X, x_0) \longrightarrow B_{m+n} \longrightarrow \mathbb{Z}$$

constructed from these maps by applying $\pi_1$ and composing with the abelianisation. One can think of $T(\gamma)$ as counting the total number of half-twists that occur in a path $\gamma$ of configurations of $m$ points in $\mathbb{D}^2$, where a *half-twist* means something like one of the standard generators of the braid group (consisting of a pair of adjacent strands crossing each other, and no other crossings). Half-twists with the opposite orientation count negatively. On the other hand, $R(\gamma)$ counts the total number of half-twists that occur between pairs of configuration points, and also between configuration points and puncture points (the $p_i$ are called "puncture points"). In principle, it also counts half-twists between pairs of puncture points, but of course these remain fixed, so there are zero of these. So $R(\gamma) - T(\gamma)$ is the total number (counted with signs) of half-twists that occur between configuration points and puncture points in the path $\gamma$ of configurations in $\mathbb{D}_n$. This is twice the total number of times that a configuration point winds around a puncture point, which must be an integer, so $R(\gamma) - T(\gamma)$ is always even. We can therefore define a homomorphism

$$W\colon \pi_1(X, x_0) \longrightarrow \mathbb{Z}$$

by $W(\gamma) = \frac{1}{2}(R(\gamma) - T(\gamma))$. Finally, we define $\phi$ to be the following homomorphism:

$$\phi\colon \pi_1(X, x_0) \longrightarrow \mathbb{Z}^2 = Q \qquad ; \qquad \gamma \mapsto (T(\gamma), W(\gamma)).$$

**The braid group action.** We now take $B$ to be the topological group $\mathrm{Homeo}_\partial(\mathbb{D}^2; \{p_1, \ldots, p_n\})$ of self-homeomorphisms of the disc $\mathbb{D}^2$ that restrict to the identity on its boundary and take the subset $\{p_1, \ldots, p_n\}$ to itself (not necessarily by the identity). This acts on $X = C_m(\mathbb{D}_n)$ by acting on each point in a configuration separately, i.e.,

$$b \cdot \{x_1, \ldots, x_m\} \;=\; \{b(x_1), \ldots, b(x_m)\} \qquad \text{for} \qquad b \in \mathrm{Homeo}_\partial(\mathbb{D}^2; \{p_1, \ldots, p_n\}),$$

and it preserves the basepoint $\{\bar{x}_1, \ldots, \bar{x}_m\} \in C_m(\mathbb{D}_n)$ since it acts by the identity on the boundary. This is moreover a *continuous* group action (and $B$ is locally path-connected – in fact, it is locally contractible).

Now, $\pi_0(B)$ is the braid group $B_n$ on $n$ strands (one of the equivalent definitions of $B_n$ is precisely as the *mapping class group of the* 2*-disc with n marked points*, i.e., $\pi_0(\mathrm{Homeo}_\partial(\mathbb{D}^2; \{p_1, \ldots, p_n\}))$).

There is an induced action of $\pi_0(B) = B_n$ on $\pi_1(X, x_0)$, and it turns out (—exercise!—) that the homomorphism $\phi$ defined above (for both cases $m = 1$ and $m \geqslant 2$) is invariant under this action.

Hence the general construction (in the case $i = m$) produces an action

$$\pi_0(B) = B_n \longrightarrow \mathrm{Aut}_{\mathbb{Z}[\mathbb{Z}]}(H_1(\widetilde{\mathbb{D}}_n; \mathbb{Z})) \qquad \text{for } m = 1$$
$$\pi_0(B) = B_n \longrightarrow \mathrm{Aut}_{\mathbb{Z}[\mathbb{Z}^2]}(H_m(\widetilde{C}_m(\mathbb{D}_n); \mathbb{Z})) \qquad \text{for } m \geqslant 2.$$

The group-ring $\mathbb{Z}[\mathbb{Z}]$ can be thought of equivalently as the ring $\mathbb{Z}[x^{\pm 1}]$ of Laurent polynomials in one variable, and similarly $\mathbb{Z}[\mathbb{Z}^2]$ is the ring $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ of Laurent polynomials in two variables. So we get a representation of $B_n$ over one of these rings, depending on whether $m = 1$ or $m \geqslant 2$. *This is the Lawrence construction of braid group representations.*

**The Burau representation.** For $m = 1$, it turns out that the $\mathbb{Z}[\mathbb{Z}]$-module $H_1(\widetilde{\mathbb{D}}_n; \mathbb{Z})$ is isomorphic to the free module $\mathbb{Z}[\mathbb{Z}]^{n-1}$. The Lawrence representation at $m = 1$ can therefore be described in terms of matrices over $\mathbb{Z}[\mathbb{Z}]$:

$$B_n \longrightarrow GL_{n-1}(\mathbb{Z}[\mathbb{Z}]).$$

This turns out to be exactly the reduced Burau representation. (The unreduced Burau representation can also be recovered by a slight modification of Lawrence's construction; see below.)

**The Lawrence-Krammer-Bigelow representation.** For $m = 2$ it also turns out (although the proof is much longer) that the $\mathbb{Z}[\mathbb{Z}^2]$-module $H_2(\widetilde{C}_2(\mathbb{D}_n); \mathbb{Z})$ is a free module, this time of rank $\binom{n}{2}$. So the Lawrence representation at $m = 2$ can be described in terms of matrices over $\mathbb{Z}[\mathbb{Z}^2]$:

$$B_n \longrightarrow GL_{\binom{n}{2}}(\mathbb{Z}[\mathbb{Z}^2]). \tag{3}$$

This is (by definition) the *Lawrence-Krammer-Bigelow representation.*

## 5. Variations of the construction

Lawrence's construction may also be varied by applying it to configuration spaces of points in a punctured surface (with boundary) $F$, instead of $\mathbb{D}^2$. This leads to representations of *surface braid groups* $B_n(F)$. There are other variations of the construction. Instead of ordinary homology, we could use *Borel-Moore homology*, or take homology relative to the boundary (the configuration spaces $C_m(\mathbb{D}_n)$ are all manifolds with boundary – or, more precisely, with "corners"), or part of the boundary, or a combination. One of these variations, using Borel-Moore homology relative to a point on the boundary of $\mathbb{D}^2$, recovers the unreduced Burau representation, as mentioned above.

## 6. Sources

The original paper in which Ruth Lawrence introduced these constructions (described in a different way to here) is [Law90]. In [Big01] and [Kra02], Bigelow and Krammer independently proved that the Lawrence-Krammer-Bigelow representation is faithful, i.e., the homomorphism (3) above is injective. This immediately implies that the braid groups are linear, since $GL_k(\mathbb{Z}[\mathbb{Z}^2])$ can be embedded into $GL_k(\mathbb{R})$ by viewing $\mathbb{Z}[\mathbb{Z}^2]$ as a ring of Laurent polynomials $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ and sending $x$ and $y$ to algebraically independent transcendental real numbers. The description in §4 of the homomorphisms $T$, $R$ and $W$ comes from §2 of [Bud05]. The fact that $H_2(\widetilde{C}_2(\mathbb{D}_n); \mathbb{Z})$ is a free module of rank $\binom{n}{2}$ over $\mathbb{Z}[\mathbb{Z}^2]$ is proved as Proposition 3.6 in [PP02] and as Theorem 4.1 in [Big03].

## References


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