Let $G = \{M \in GL_n(\mathbb{Z}) \mid M \text{ is unitriangular}\}$. For $k \geqslant 0$ let U_k be the set of $(n \times n)$-matrices $A = \{a_{ij}\}$ over $\mathbb{Z}$ with entries $a_{ij} = 0$ if $j < i + k$, and for $k \geqslant 1$ let H_k be the subgroup

$$H_k = \{I + A \mid A \in U_k\} \leqslant GL_n(\mathbb{Z}).$$

By definition, $G = H_1$. Note that $U_k \supseteq U_{k+1}$ and $H_k \geqslant H_{k+1}$. Also note that, for $k \geqslant n$, we have $U_k = \{0\}$ and hence $H_k = \{I\}$. We will show that the derived series of G is $G^{(i)} = H_{2^i}$ and therefore that G is solvable, with derived length $\lceil \log_2(n) \rceil$.

Some easy facts to check:

(a) For all $j, k \geqslant 0$, if $X \in U_j$ and $Y \in U_k$ then $X + Y \in U_{\mathsf{min}(j,k)}$ and $XY \in U_{j+k}$.
(b) If X is any $(n \times n)$-matrix such that $X^\ell = 0$ for some ℓ (i.e. X is nilpotent), then $I + X$ is invertible and $(I + X)^{-1} = I - X(I + X)^{-1}$.
(c) As a consequence, if X, Y are both nilpotent, then

$$[I + X, I + Y] = I + XY\overline{Y} - YX\overline{X}\,\overline{Y} - XYX\overline{X}\,\overline{Y}, \tag{1}$$

where $\overline{X} = (I + X)^{-1}$ and $\overline{Y} = (I + Y)^{-1}$.
(d) By fact (a), every $X \in U_1$ is nilpotent, so the above formulas apply to any $X, Y \in U_1$.
(e) The subgroup H_k is generated by the set $\{I + E_{i,i+k} \mid 1 \leqslant i \leqslant n - k\}$, where all entries of $E_{i,j}$ are 0 except the one at position (i, j), which is 1.

Given any $A, B \in G = H_1$, write $A = I + X$ and $B = I + Y$ for $X, Y \in U_1$. Then:

$$[A, B] = [I + X, I + Y] = \text{formula (1)}.$$

Since $X, Y \in U_1$ and $\overline{X}, \overline{Y} \in U_0$, fact (a) implies that $[A, B] \in H_2$. Hence:

$$[G, G] \leqslant H_2. \tag{2}$$

More generally, if $A, B \in H_k$ and we write $A = I + X$ and $B = I + Y$, facts (c) and (a) tell us that

$$[A, B] = I + (\text{something in } U_{2k}) \in H_{2k}.$$

So we have:

$$[H_k, H_k] \leqslant H_{2k}. \tag{3}$$

By an inductive argument, fact (3) implies that $G^{(i)} \leqslant H_{2^i}$, which is one half of what we wanted. To make this into an equality, we need to turn (3) into an equality. Using fact (e), it's enough to show that each generator $I + E_{i,i+2k}$ of H_{2k} can be written as a commutator $[A, B]$ for some $A, B \in H_k$. One can check that:

$$[I + E_{i,i+k}, I + E_{i+k,i+2k}] \;=\; I + E_{i,i+2k}, \tag{4}$$

so we can take $A = I + E_{i,i+k}$ and $B = I + E_{i+k,i+2k}$. This finishes the proof.

Side note: we could alternatively use the generating set given in fact (e) also for the first half of the proof, i.e. to prove the inclusion $\leqslant$ in (3). In general, the following is true:

(f) Suppose that Γ is a group, $S \subseteq \Gamma$ is a generating set for Γ and $N \triangleleft \Gamma$ is a normal subgroup, such that $[s, t] \in N$ for all $s, t \in S$. Then $[\Gamma, \Gamma] \leqslant N$. *This can be proved using the following commutator identities, which hold for any x, y, z in any group*:
$[x, zy] = [x, y](y^{-1}[x, z]y) \qquad [x, y^{-1}] = y[y, x]y^{-1}$
$[xz, y] = (z^{-1}[x, y]z)[z, y] \qquad [x^{-1}, y] = x[y, x]x^{-1}$

Using facts (a) and (b), we can show that $H_{2k} \triangleleft H_k$. Now using equation (4), together with:

$$[I + E_{i,i+k}, I + E_{j,j+k}] = I \qquad \text{for } j \neq i \pm k, \tag{5}$$

$$[I + E_{j+k,j+2k}, I + E_{j,j+k}] = I - E_{j,j+2k}, \tag{6}$$

we see that the assumptions of fact (f) are true, with $\Gamma = H_k$, $S =$ the generating set of fact (e) and $N = H_{2k}$. So fact (f) tells us that $[H_k, H_k] \leqslant H_{2k}$. This completes the alternative proof of (3).

Note: *For the larger group $\widehat{G}$ of all invertible upper triangular matrices over $\mathbb{Z}$, which contains G as an index-2^n normal subgroup, see here.*