Homology of configuration-mapping and -section spaces

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22^{nd} July 2020

Abstract

For a given bundle $\xi\colon E \to M$ over a manifold, *configuration-section spaces* on ξ parametrise finite subsets $z \subseteq M$ equipped with a section of ξ defined on $M \smallsetminus z$, with prescribed "charge" in a neighbourhood of the points z. These spaces may be interpreted physically as spaces of fields that are permitted to be singular at finitely many points, with constrained behaviour near the singularities. As a special case, they include the *Hurwitz spaces*, which parametrise branched covering spaces of the 2-disc with specified deck transformation group.

We prove that configuration-section spaces are homologically stable (with $\mathbb{Z}$ coefficients) whenever the underlying manifold M is connected and has non-empty boundary and the charge is "small" in a certain sense, and describe a model for the stable homology. This has a partial intersection with the work on Hurwitz spaces of Ellenberg, Venkatesh and Westerland.

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1. Introduction

Configuration spaces of points in manifolds have been intensively studied in topology and geometry, and may be interpreted physically as a model for particles moving in a background space. In *labelled* configuration spaces, each particle is equipped with an additional parameter, taking values in a fixed space X or, more generally, in a bundle over the underlying manifold. A more physically relevant setting corresponds to equipping not the particles, but instead their *complement*, with a map to X or a section of a bundle over the underlying manifold (this viewpoint is suggested in [Seg14], for example). For maps to a fixed space X, these are the *configuration-mapping spaces*, introduced in [EVW]. Since these spaces are intended to model particles moving in physical fields, which typically take values in a (possibly non-trivial) bundle over the underlying manifold, one is naturally led to consider, more generally, *configuration-section spaces*, which we introduce in §3.

Roughly, configuration-mapping spaces are defined as follows. Given a d-dimensional manifold M, a space X and a set $c \subseteq [S^{d-1}, X]$ of unbased homotopy classes of maps from S^{d-1} to X, a

2020 *Mathematics Subject Classification*: 55R80, 57N65, 55N25
Key words and phrases: Configuration-mapping spaces, configuration-section spaces, homological stability, stable homology, group-completion, polynomial functors, braid categories, Swiss cheese operads.

point of the k-th *configuration-mapping space*

$$\mathrm{CMap}_k^c(M;X)$$

consists of a subset $z \subset \mathring{M}$ of the interior of M of cardinality k and a continuous map $f\colon M \smallsetminus z \to X$. Moreover, we require that the restriction of f to a small punctured neighbourhood of each point of z lies in one of the homotopy classes in c. The homotopy class of the germ of f near $p \in z$ may be thought of as the "*charge*" (or "*monodromy*" or "*singularity type*") of the particle p, and c is therefore the set of allowed charges of the particles in the system being modelled. For a subset $D \subseteq \partial M$ (usually a point or a disc) and a basepoint $* \in X$, one may also impose the boundary condition that $f(D) = \{*\}$. See §2 for precise definitions, including how to topologise this set, and §3 for the generalisation to configuration-section spaces.

Some examples. In the special case $d = 2$ and $X = BG$ (so c is a subset of $[S^1, BG] = \mathrm{Conj}(G)$) the space $\mathrm{CMap}_k^c(M;BG)$, up to homotopy equivalence, parametrises branched coverings of the surface M with deck transformation group G and with monodromy around the branch points lying in c. In particular, when $M = D^2$, these are the *Hurwitz spaces* associated to the pair (G, c); see Remark 2.14 and Example 3.27 for more details. The main result of [EVW16] is a rational homological stability theorem for Hurwitz spaces when G is a finite group and c is a single conjugacy class that generates G and is "non-splitting" (*cf.* Remark 1.1 below).

As mentioned above, the physical motivation for studying configuration-section spaces is their interpretation as spaces of particles moving in fields defined on their complement. For example, we may take X to be the Eilenberg-MacLane space $K(\mathbb{Z}, d)$ (in this case we have $[S^{d-1}, X] = \{*\}$ so $c = \{*\}$) and consider the configuration-mapping space $\mathrm{CMap}_k^{c,*}(D^d; K(\mathbb{Z}, d))$. Its points consist of a configuration $z \subset \mathbb{R}^d \cong \mathring{D}^d$ together with a based map $f\colon D^d \smallsetminus z \to K(\mathbb{Z}, d)$, which may be thought of (*non-canonically*) as associating a "phase" in $S^1 \simeq \Omega^{d-1}K(\mathbb{Z}, d)$ to each particle $p \in z$. However, this description of f as a separate phase for each particle cannot be made consistent as the configuration z varies, so it is really modelling some *non-local* data associated to the whole configuration of particles. In particular, when $d = 3$, this could be a model for an asymptotic part of the moduli space of magnetic monopoles of total charge k in $\mathbb{R}^3$ (*cf.* [AH88, Proposition 3.12] and [Seg97]). See Example 3.30 for further details.

Further examples, which are configuration-section spaces but *not* configuration-mapping spaces (in general), include configuration spaces where the complement of the configuration is equipped with a *tangential structure* (e.g. orientation, spin, etc) that need not extend to the whole of M (see Example 3.28), or with a tuple of linearly independent vector fields (see Example 3.29).

Homological stability. Our main result is that configuration-section spaces are *homologically stable*, subject to a condition on the "charge" c.

Theorem A *Let M be a connected manifold of dimension $d \geqslant 2$ with basepoint $* \in \partial M$. Let X be a based space and choose an element $g \in \pi_{d-1}(X)$ that is fixed under the natural action of $\pi_1(X)$. Set $c = \{\{g\}\} \subseteq \pi_{d-1}(X)/\pi_1(X) = [S^{d-1}, X]$. Then there are stabilisation maps*

$$\mathrm{CMap}_k^{c,*}(M;X) \longrightarrow \mathrm{CMap}_{k+1}^{c,*}(M;X) \tag{1.1}$$

inducing isomorphisms on $H_i(-;\mathbb{Z})$ in the range $k \geqslant 2i+4$ and surjections in the range $k \geqslant 2i+2$. With field coefficients, these ranges may be improved to $k \geqslant 2i+2$ and $k \geqslant 2i$ respectively.

See Theorem 8.5 for the precise statement, including the generalisation to configuration-section spaces, where the analogue of the "charge" c is slightly more subtle to define.

This is an analogue of classical homological stability ([Seg73; McD75; Seg79]; see also [Ran13b]) for ordinary (unordered) configuration spaces on a connected, open manifold.

Remark 1.1 (*Relation with the result of* [EVW16].) In the case of Hurwitz spaces mentioned above, our assumption is that $c = \{\{g\}\} \subseteq \mathrm{Conj}(G)$ is a single conjugacy class of size 1 (corresponding to an element g of the centre of G). The result of [EVW16], by contrast, allows larger conjugacy classes, although it is specific to the setting of Hurwitz spaces $\mathrm{Hur}^c_{G,k} \simeq \mathrm{CMap}^{c,*}_k(D^2; BG)$. These two results are therefore somewhat orthogonal in terms of generality – and indeed our methods are entirely different to those of Ellenberg, Venkatesh and Westerland.

In more detail, the main result of [EVW16] says that, in a stable range, the rational homology groups $H_i(\mathrm{Hur}^c_{G,k}; \mathbb{Q})$ are periodic with respect to k (with a period depending on the pair (G, c)), as long as G is finite, the conjugacy class c generates G and c is "non-splitting" in the sense that $H \cap c$ is either empty or a single conjugacy class for every subgroup $H \leqslant G$. On the other hand, specialising Theorem A to the case $(M = D^2, X = BG)$, we show that the integral homology groups $H_i(\mathrm{Hur}^c_{G,k}; \mathbb{Z})$ are constant (1-periodic) in a stable range, as long as $|c| = 1$. The intersection of our results is the case where G is finite cyclic and $c = \{g\}$ for a generator $g \in G$.

Split-injectivity. For ordinary configuration spaces, and configuration spaces where the points (rather than the complement) are labelled, the analogous stabilisation maps induce *split-injections* on homology in all degrees. This follows from an easy argument (see [McD75, page 103]; also [MT14, §4] for a diffeomorphism equivariant stable splitting) that depends on the existence of multi-valued maps $C_k(M) \dashrightarrow C_{k-r}(M)$ that "forget r points from the configuration in all $\binom{k}{r}$ possible ways".

However, such maps do not exist for configuration-section spaces, since in this setting one cannot simply forget a point; one must also extend the section on the complement of the configuration to the forgotten point, which is in general impossible. Indeed, we expect that split-injectivity on homology does *not* hold for configuration-section spaces in general. Nevertheless, we do have a partial positive result in this direction, for configuration-mapping spaces under additional hypotheses on the underlying manifold M: a certain map of spectral sequences, converging to the stabilisation map (1.1) on homology, is split-injective on E^2 pages. See Theorem 9.1 for the precise statement. This relies on a detailed study of the monodromy action associated to the fibration (1.2), which is carried out in the companion paper [PT].

Stable homology. It is well-known ([Seg73; McD75; Böd87]) that ordinary (unordered) configuration spaces on a connected, open manifold model function and more generally section spaces, the homology of which coincides with the stable homology of the configuration spaces. This remains true also in our case.

Theorem B *Let M be a connected manifold of dimension $d \geqslant 2$ with basepoint $* \in \partial M$ and $* \in L \subsetneq \partial M$ be a proper submanifold of the boundary. For any based space X and subset $c \subseteq [S^{d-1}, X]$, there exists a bundle $E_d(X, c)$ over M containing the trivial bundle $M \times X$ such that*

$$\mathrm{C}\Gamma^{c,*}(M, L; X) \longrightarrow \Gamma((M, L); E_d(X, c))$$

is a weak homotopy equivalence; here $\Gamma((M, L); E_d(X, c))$ denotes the space of sections that take $* \in \partial M$ to $* \in X$ and on $\overline{\partial M \smallsetminus L}$ take values in X.

This is the main result of the first part of [EVW]. In §10 below we extend their results to the setting of configuration-section spaces. Combining Theorem B with Theorem A and an application of the group completion theorem gives the following computation of the homology of finite configuration-section spaces.

Corollary C *Suppose that $d \geqslant 2$ and the subset $\widetilde{c}_D \subseteq \pi_{d-1}(X)$ has size 1. Then the scanning maps*

$$\mathrm{C}\Gamma^{c,*}_k(M; X) \longrightarrow \Gamma(M; E_d(X, c))_{[k]}$$

induce isomorphisms on $H_i(-; \mathbb{Z})$ in the range $k \geqslant 2i + 4$ and surjections in the range $k \geqslant 2i + 2$. With field coefficients, these ranges may be improved to $k \geqslant 2i + 2$ and $k \geqslant 2i$ respectively. Here the subscript $[k]$ on the right indicates those components that intersect non-trivially with the image.

See Theorem 10.12 and Corollary 10.16 for precise and more general versions of the above results.

Outline. The paper is organised as follows. Sections 2 and 3 contain the precise definitions of configuration-mapping spaces (recalled from [EVW]) and configuration-section spaces (a natural generalisation that we introduce); several different classes of examples are also discussed in section 3 (see Roadmap 3.1 for a more detailed plan of section 3). The structure on configuration-section spaces that we need, including the stabilisation maps, is encapsulated in the statement that they form an E_0*-module over an* E_{d-1}*-algebra* (*cf.* Remark 4.1 for the reasoning behind this terminology). In section 4, we first explain precisely what this means (recalling along the way several different flavours of Swiss cheese operads) and then define this structure on configuration-section spaces in appropriate models (Proposition 4.10). In section 5 we then define the (up-to-homotopy) monodromy action of the fundamental group $\pi_1(C_k(\mathring{M}))$ coming from the forgetful map

$$\mathrm{C\Gamma}_k^{c,*}(M;\xi) \longrightarrow C_k(\mathring{M}), \tag{1.2}$$

where $\mathrm{C\Gamma}_k^{c,*}(M;\xi)$ denotes the k-th configuration-section space associated to a bundle $\xi\colon E \to M$ and charge c. In section 7 we show that these actions, for $k \in \mathbb{N}$, extend to a *monodromy functor*

$$\mathcal{C}(M) \longrightarrow \mathrm{Ho}(\mathrm{Top}), \tag{1.3}$$

where $\mathcal{C}(M)$ is a certain *braid category* on M recalled in section 6. Theorem A is proved in section 8 (see Theorem 8.5 for the precise statement). First, by a spectral sequence argument, it suffices to prove *twisted* homological stability for ordinary configuration spaces with coefficients in the composition of the monodromy functor (1.3) with homology in any fixed degree q. To apply the known twisted homological stability results for configuration spaces, we prove (Proposition 8.6) that this composite functor is *polynomial* of degree q. In section 9 we discuss the extension of the monodromy functor to a larger braid category $\mathcal{B}_\sharp(M) \supseteq \mathcal{C}(M)$ and prove the partial split-injectivity result mentioned above (Theorem 9.1). Finally, in section 10, we extend (and correct) the arguments of [EVW] to the setting of configuration-section spaces to prove Theorem B and Corollary C on the stable homology of configuration-section spaces.

2. Configuration-mapping spaces

We begin by recalling the definition of *configuration-mapping spaces* from [EVW], which generalises the classical notion of *Hurwitz spaces* [Cle72; Hur91]. In fact, we slightly extend the definition of [EVW] by considering also *non-orientable* manifolds (we will generalise this further to *configuration-section spaces* in the next section).

Let M be a smooth, compact, connected manifold with non-empty boundary of dimension $d \geqslant 2$ and let X be a space. Also, let

$$c \subseteq [S^{d-1}, X]$$

be a non-empty subset of the set of (unbased) homotopy classes of maps $S^{d-1} \to X$. There is a $\mathbb{Z}/2$-action on the set $[S^{d-1}, X]$ given by precomposition by a reflection of the sphere, and, in the case when M is *non-orientable*, we assume that c is a collection of fixed points of this action (see Remark 2.3 for why). If M is *orientable*, there is no condition on c.

Remark 2.1 If X is path-connected, the set $[S^{d-1}, X]$ may be identified with the set of orbits $\pi_{d-1}(X)/\pi_1(X)$ for any choice of basepoint of X. When $d = 2$ this is the set of conjugacy classes of $\pi_1(X)$.

Definition 2.2 (*Configuration-mapping spaces, I.*) For a positive integer k, the underlying set of the *configuration-mapping space*

$$\mathrm{CMap}_k^c(M;X)$$

is the set of pairs (z, f), where z is a subset of the interior $\mathring{M}$ of M of cardinality k and f is a continuous map $M \smallsetminus z \to X$ with the following property: for any embedding $e\colon D^d \hookrightarrow M$ where $e(D^d) \cap z$ consists of a single point in the interior of $e(D^d)$, we have

$$[f \circ e \circ i] \in c, \tag{2.1}$$

where i denotes the inclusion $S^{d-1} \hookrightarrow D^d$.

Remark 2.3 The reason for assuming that c is a collection of fixed points of the involution on $[S^{d-1}, X]$, in the case when M is non-orientable, is the following. In the definition of configuration-mapping spaces above, the set c of homotopy classes of maps $S^{d-1} \to X$ is to be thought of as the set of permitted "monodromies" of the continuous map to X that is defined on the complement of a configuration in M. If M is non-orientable, the monodromy of any such map around a configuration point must *automatically* lie in a fixed point of $[S^{d-1}, X]$ under the involution. Hence, in this case, we may as well ignore the non-fixed points of $[S^{d-1}, X]$, since it is impossible for them to occur, and instead just consider subsets c of the set of fixed points of $[S^{d-1}, X]$ under the involution. This is illuminated in more detail in Example 3.22, in the context of *configuration-section spaces*.

To topologise the set of Definition 2.2, we will give a second definition of $\mathrm{CMap}_k^c(M; X)$ that has a natural topology, and then prove that there is a natural bijection between the two definitions. To do this, we first recall some auxiliary definitions and results.

Definition 2.4 ([Pal60; Cer61]) If G is a topological group with a continuous left-action on a space X, the action *admits local sections* if each $x \in X$ has an open neighbourhood U and a continuous map $\gamma\colon U \to G$ such that $\gamma(x').x = x'$ for each $x' \in U$.

The utility of this definition is given by the following result.

Proposition 2.5 ([Pal60, Theorem A]) *If X and Y are left G-spaces such that the action of G on Y admits local sections, and $f\colon X \to Y$ is G-equivariant, then f is a fibre bundle.*

Definition 2.6 For a smooth manifold-with-boundary M, we write $\mathrm{Diff}_\partial(M)$ for the topological group of self-diffeomorphisms of M that restrict to the identity on ∂M, equipped with the subspace topology induced by the smooth Whitney topology on $C^\infty(M, M)$, the space of all smooth self-maps of M.

For a smooth manifold (without boundary) M, we write $\mathrm{Diff}_c(M)$ for the topological group of self-diffeomorphisms φ of M such that are *compactly-supported*, meaning that $\{p \in M \mid \varphi(p) \neq p\}$ is relatively compact in M. This is topologised as follows. For a compact subset K of M, write $\mathrm{Diff}_K(M)$ for the topological group of self-diffeomorphisms φ of M such that $\{p \in M \mid \varphi(p) \neq p\} \subseteq K$, equipped with the subspace topology induced by the Whitney topology on $C^\infty(M, M)$. Note that, as a set, $\mathrm{Diff}_c(M)$ is the union of $\mathrm{Diff}_K(M)$ over all choices of K. We define the topology of $\mathrm{Diff}_c(M)$ to be the colimit of the topologies of $\mathrm{Diff}_K(M)$ over all choices of K.

Lemma 2.7 *If M is a smooth manifold without boundary, the continuous left-action of* $\mathrm{Diff}_c(M)$ *on $C_k(\mathring{M})$ admits local sections.*

Proof. Theorem B of [Pal60] implies, as a special case, that the action of $\mathrm{Diff}_\partial(M)$ on the *ordered* configuration space $F_k(\mathring{M})$ admits local sections. Then one may use this, and the fact that the covering map $F_k(\mathring{M}) \to C_k(\mathring{M})$ is $\mathrm{Diff}_\partial(M)$-equivariant, to construct local sections for the action of $\mathrm{Diff}_\partial(M)$ on $C_k(\mathring{M})$. Alternatively, the statement follows directly from Proposition 4.15 of [Pal20]. ∎

Definition 2.8 (*Configuration-mapping spaces, II.*) Fix a subset $\hat{z} \subseteq \mathring{M}$ of cardinality k and let $\mathrm{Map}_*^c(M \smallsetminus \hat{z}, X)$ denote the space of continuous maps $M \smallsetminus \hat{z} \to X$ satisfying the condition (2.1), equipped with the compact-open topology. Write $\mathrm{Diff}_\partial(M)$ for the group of diffeomorphisms of M that fix a neighbourhood of ∂M, equipped with the smooth Whitney topology, and let $\mathrm{Diff}_\partial(M, \hat{z})$ denote the subgroup of diffeomorphisms that fix $\hat{z}$ as a subset. This acts (on the right) on $\mathrm{Map}^c(M \smallsetminus \hat{z}, X)$ by precomposition, and on $\mathrm{Diff}_\partial(M)$ by right-multiplication, and we define:

$$\mathrm{CMap}_k^c(M; X) \;\coloneqq\; \frac{\mathrm{Map}^c(M \smallsetminus \hat{z}, X) \times \mathrm{Diff}_\partial(M)}{\mathrm{Diff}_\partial(M, \hat{z})}. \tag{2.2}$$

Lemma 2.9 *There is a bijection between the set defined in Definition 2.2 and the space* (2.2).

Proof. Consider the map

$$p\colon \mathrm{CMap}_k^c(M; X) \longrightarrow C_k(\mathring{M}) \tag{2.3}$$

given by $[f,\varphi] \mapsto \varphi(\hat{z})$. There is a continuous left-action of $\mathrm{Diff}_\partial(M)$ on $\mathrm{CMap}_k^c(M;X)$ induced by its action on itself by left-multiplication and (2.2), and on $C_k(\mathring{M})$ given by sending a configuration to its image under a diffeomorphism. The map (2.3) is equivariant with respect to these actions. The action of $\mathrm{Diff}_\partial(M)$ on $C_k(\mathring{M})$ admits local sections by Lemma 2.7 and hence (2.3) is a fibre bundle by Proposition 2.5.

The fibre of (2.3) over a configuration $z \in C_k(\mathring{M})$ is

$$p^{-1}(z) \;=\; \frac{\mathrm{Map}^c(M \smallsetminus \hat{z}, X) \times \mathrm{Diff}_\partial(M,\hat{z}) \cdot \varphi}{\mathrm{Diff}_\partial(M,\hat{z})}, \tag{2.4}$$

where $\mathrm{Diff}_\partial(M,\hat{z})\cdot\varphi$ denotes the coset of $\varphi \in \mathrm{Diff}_\partial(M)$ under the action of $\mathrm{Diff}_\partial(M,\hat{z})$, where φ is any diffeomorphism taking $\hat{z}$ to z. There is a canonical identification of (2.4) with $\mathrm{Map}^c(M \smallsetminus z, X)$ via $[f,\varphi] \mapsto f \circ \varphi^{-1}$. Hence a point of $\mathrm{CMap}_k^c(M;X)$ may be specified by its image under (2.3), namely an unordered configuration $z \in C_k(\mathring{M})$, together with an element of the fibre $p^{-1}(z)$, which is a continuous map $M \smallsetminus z \to X$ satisfying condition (2.1). This gives a natural bijection between the set $\mathrm{CMap}_k^c(M;X)$ defined in Definition 2.2, and the formal definition (2.2). □

Definition 2.10 (*Configuration-mapping spaces with a boundary condition.*) If we fix a subset $D \subseteq \partial M$ and a basepoint $* \in X$, we may define

$$\mathrm{CMap}_k^{c,D}(M;X)$$

to be the subspace of $\mathrm{CMap}_k^c(M;X)$ consisting of pairs (z,f) such that $f(p) = *$ for all $p \in D$. Equivalently (via the proof of Lemma 2.9), we replace $\mathrm{Map}^c(M \smallsetminus \hat{z}, X)$ in (2.2) with its subspace $\mathrm{Map}^c((M \smallsetminus \hat{z}, D),(X,*))$ of maps taking D to $\{*\}$. Typically, we will take $D = D^{d-1} \subseteq \partial M$ to be an embedded disc.

Definition 2.11 (*The associated fibre bundle.*) From Definition 2.8 and the proof of Lemma 2.9, we have a fibre bundle

$$p\colon \mathrm{CMap}_k^c(M;X) \longrightarrow C_k(\mathring{M}) \tag{2.5}$$

whose fibre over a configuration $z \in C_k(\mathring{M})$ is the space of maps $M \smallsetminus z \to X$ satisfying condition (2.1), and whose total space $\mathrm{CMap}_k^c(M;X)$ is the *configuration-mapping space*. For $D \subseteq \partial M$ and *based* spaces X, there are also restricted versions of the configuration-mapping space, from Definition 2.10,

$$\mathrm{CMap}_k^{c,D}(M;X) \subset \mathrm{CMap}_k^c(M;X),$$

corresponding to restricting each fibre of (2.5) to the space of maps of pairs $(M \smallsetminus z, D) \to (X,*)$ satisfying condition (2.1):

$$p\colon \mathrm{CMap}_k^{c,D}(M;X) \longrightarrow C_k(\mathring{M}), \tag{2.6}$$

with $p^{-1}(z) = \mathrm{Map}^c((M \smallsetminus z, D),(X,*))$.

Remark 2.12 Condition (2.1) depends only on the homotopy class of the map f, so the subspace $\mathrm{Map}^c(M \smallsetminus z, X)$ of $\mathrm{Map}(M \smallsetminus z, X)$ is a union of path-components (a similar statement also holds for the version with boundary condition on D).

Remark 2.13 When X is a point (so necessarily $c = [S^{d-1},X] = \{*\}$), we have, by definition (2.2), $\mathrm{CMap}_k^c(M;*) = \mathrm{Diff}_\partial(M)/\mathrm{Diff}_\partial(M,\hat{z})$. In this case, by (2.4), each fibre of the fibre bundle (2.3) is a single point (here we are essentially using the fact that $\mathrm{Diff}_\partial(M)$ acts transitively on $C_k(\mathring{M})$), so (2.3) is a homeomorphism (since fibre bundles are open maps):

$$\mathrm{CMap}_k^c(M;*) = \mathrm{Diff}_\partial(M)/\mathrm{Diff}_\partial(M,\hat{z}) \cong C_k(\mathring{M}),$$

identifying the configuration-mapping space with the usual (unordered) configuration space in this case.

Remark 2.14 When $M = D^2$ is the 2-disc with $D = I \subseteq \partial D^2$ an embedded interval in its boundary and $X = BG$ is the classifying space of a discrete group G (so c is a set of conjugacy classes of G), we have a homotopy equivalence:

$$\mathrm{CMap}_k^{c,I}(D^2;BG) \simeq \mathrm{Hur}_{G,k}^c,$$

(see [EVW, §5.8]), where $\mathrm{Hur}^c_{G,k}$ is the corresponding *Hurwitz space*, the moduli space (topologised appropriately) of the data (up to an appropriate notion of equivalence) consisting of (S, s_0, ν, i), where:

- S is a Riemann surface with basepoint $s_0 \in \partial S$
- $\nu\colon S \to D^2$ is a based covering map, branched at k point in the interior of D^2,
- $i\colon G \hookrightarrow \mathrm{Deck}(\nu)$ is an embedding of groups,

such that G acts transitively on the generic fibres of ν and the monodromy of ν around each of its branch points lies in one of the conjugacy classes in c. See also Example 3.27 and in particular the weak equivalences (3.20).

3. Configuration-section spaces

We now explain how to generalise this notion to *configuration-section spaces*, where we consider sections of a bundle over the complement of a configuration, instead of just a map to a fixed space. This includes, for example, moduli spaces of configurations whose complement is equipped with a *tangential structure* or with a *tuple of linearly independent vector fields* (which need not extend over the whole manifold).

Roadmap 3.1 (*Plan of the section.*) We define the underlying set of configuration-section spaces in Definition 3.2 and topologise it in Definition 3.6 (Lemma 3.7 gives the correspondence between the two definitions). A version with a boundary condition is given in Definition 3.9. Up to this point, however, these are the definitions of configuration-section spaces without any restriction on the allowed "charge" of the section near a particle of the configuration (this is the analogue of the subset $c \subseteq [S^{d-1}, X]$ for configuration-mapping spaces in the previous section, and can also be thought of as the allowed "monodromy" of the section near a particle, or as a "singularity condition" on the section). In order to define the appropriate analogue for configuration-section spaces, we first construct in Definition 3.11 the *covering space of local sections* $\Sigma(\xi) \to \mathring{M}$ associated to a bundle over a manifold $\xi\colon E \to M$. Any configuration-section of the bundle ξ determines a section of the associated covering space $\Sigma(\xi)$ (Construction 3.17 and Lemma 3.18), and this is then used in Definition 3.19 to define configuration-section spaces with prescribed "monodromy" or "charge". The forgetful fibration associated to this configuration-section space is described in Definition 3.21.

After this setting up of the theory, we discuss several families of examples. Example 3.22 first explains precisely how the notion of *configuration-section space* recovers the notion of *configuration-mapping space* when the bundle $\xi\colon E \to M$ is trivial (in fact, certain subtleties in the case where the base manifold M is non-orientable are explained more naturally from the configuration-section viewpoint). There is a slight variation of configuration-section spaces where the section is given only up to homotopy (Definition 3.23), which includes the example of *configurations equipped with a cohomology class on the complement* (Example 3.24). We then discuss the examples of *Hurwitz spaces* (Example 3.27), configurations with a tangential structure or tuple of linearly independent vector fields on their complement (Examples 3.28 and 3.29) and "*widely separated magnetic monopoles*" (Example 3.30).

Definition 3.2 (*Configuration-section spaces, without prescribed monodromy, I.*) Fix a smooth, connected d-manifold M (possibly with boundary) and a fibre bundle $\xi\colon E \to M$. For a non-negative integer k, the (unrestricted) *configuration-section space*, as a set, is given by

$$\mathrm{C\Gamma}_k(M;\xi) = \{(z,s) \mid z \subseteq \mathring{M} \text{ subset of cardinality } k,\ s\colon M \smallsetminus z \to E \text{ section of } \xi|_{M\smallsetminus z}\}. \quad (3.1)$$

We will topologise this, and construct the associated fibre bundle over $C_k(\mathring{M})$, in a similar way as for configuration-mapping spaces in §2.

Definition 3.3 For a smooth manifold-with-boundary M and fibre bundle $\xi\colon E \to M$, the automorphism group $\mathrm{Aut}_\partial(\xi)$ is the subgroup

$$\mathrm{Aut}_\partial(\xi) \ \leqslant\ \mathrm{Diff}_\partial(M) \times \mathrm{Homeo}(E)$$

of those pairs (φ, g) such that $\xi \circ g = \varphi \circ \xi$. For a subset $z \subseteq \mathring{M}$, we write $\mathrm{Aut}_\partial(\xi, z)$ for the subgroup of (φ, g) such that $\varphi(z) = z$. Similarly, for subsets $z, z' \subseteq \mathring{M}$ of the same cardinality, we write $\mathrm{Aut}_\partial(\xi, z \mapsto z')$ for the subgroup of (φ, g) such that $\varphi(z) = z'$. If $\partial M = \varnothing$, we write $\mathrm{Aut}_c(\xi)$ (resp. $\mathrm{Aut}_c(\xi, z)$ and $\mathrm{Aut}_c(\xi, z \mapsto z')$) if we replace $\mathrm{Diff}_\partial(M)$ with $\mathrm{Diff}_c(M)$ in the definition.

Remark 3.4 An element $(\varphi, g) \in \mathrm{Aut}_\partial(\xi)$ is determined by its second component $g \in \mathrm{Homeo}(E)$, since a self-homeomorphism of the total space E can descend to a self-diffeomorphism of the base manifold M in at most one way. Hence there is a continuous injection $\mathrm{Aut}_\partial(\xi) \hookrightarrow \mathrm{Homeo}(E)$, although the topology on $\mathrm{Aut}_\partial(\xi)$ is generally *finer* than the subspace topology induced by this injection.

Lemma 3.5 *If M is a smooth manifold-with-boundary and $\xi \colon E \to M$ is a fibre bundle, the continuous left-action of $\mathrm{Aut}_\partial(\xi)$ on $C_k(\mathring{M})$ given by $(\varphi, g) \colon z \mapsto \varphi(z)$ admits local sections.*

Proof. Let $z \in C_k(\mathring{M})$ and choose an embedded codimension-zero ball $B \subseteq \mathring{M}$ such that $z \subseteq \mathring{B}$. By Lemma 2.7, there is an open neighbourhood U of z in $C_k(\mathring{B})$ and a continuous map $\gamma \colon U \to \mathrm{Diff}_c(\mathring{B})$ such that $\gamma(z').z = z'$ for all $z' \in U$. Choose a trivialisation of $\xi|_{\mathring{B}}$. This induces a continuous group homomorphism $\mathrm{Diff}_c(\mathring{B}) \to \mathrm{Aut}_c(\xi|_{\mathring{B}})$. Now extending both the diffeomorphism of the underlying manifold and the homeomorphism of the total space by the identity on $M \smallsetminus \mathring{B}$ and on $\xi^{-1}(M \smallsetminus \mathring{B})$ defines a continuous group homomorphism $\mathrm{Aut}_c(\xi|_{\mathring{B}}) \to \mathrm{Aut}_\partial(\xi)$. Composing both of these with γ completes the construction of local sections for the action of $\mathrm{Aut}_\partial(\xi)$ on $C_k(\mathring{M})$. ◻

Definition 3.6 (*Configuration-section spaces, without prescribed monodromy, II.*) Fix a subset $\hat{z} \subseteq \mathring{M}$ of cardinality k, and define

$$\mathrm{C\Gamma}_k(M; \xi) \coloneqq \frac{\Gamma(M \smallsetminus \hat{z}, \xi) \times \mathrm{Aut}_\partial(\xi)}{\mathrm{Aut}_\partial(\xi, \hat{z})}, \tag{3.2}$$

where, for a subspace $S \subseteq M$, we write $\Gamma(S, \xi) = \{s \in \mathrm{Map}(S, E) \mid \xi \circ s = \mathrm{incl}\}$. The right-action of $\mathrm{Aut}_\partial(\xi, \hat{z})$ on $\Gamma(M \smallsetminus \hat{z}, \xi)$ is given by $(\varphi, g) \colon s \mapsto g^{-1} \circ s \circ \varphi$.

Lemma 3.7 *There is a bijection between the set defined in Definition 3.2 and the space* (3.2).

Proof. There is a continuous map

$$p \colon \mathrm{C\Gamma}_k(M; \xi) \longrightarrow C_k(\mathring{M}) \tag{3.3}$$

given by $[s, (\varphi, g)] \mapsto \varphi(z)$. This map is equivariant with respect to the continuous left-actions of $\mathrm{Aut}_\partial(\xi)$, and the action of $\mathrm{Aut}_\partial(\xi)$ on $C_k(\mathring{M})$ admits local sections by Lemma 3.5. Thus, by Proposition 2.5, the map (3.3) is a fibre bundle. The fibre of (3.3) over $z \in C_k(\mathring{M})$ is

$$p^{-1}(z) \;=\; \frac{\Gamma(M \smallsetminus \hat{z}, \xi) \times \mathrm{Aut}_\partial(\xi, \hat{z} \mapsto z)}{\mathrm{Aut}_\partial(\xi, \hat{z})}. \tag{3.4}$$

Note that $\mathrm{Aut}_\partial(\xi)$ acts *transitively* on $C_k(\mathring{M})$. (This can be seen as follows. Let $z, z' \in C_k(\mathring{M})$ and choose an embedded codimension-zero ball $B \subseteq \mathring{M}$ such that $z \cup z' \subseteq \mathring{B}$. Since $\mathrm{Diff}_c(\mathring{B})$ acts transitively on $C_k(\mathring{B})$, we may find a diffeomorphism φ of $\mathring{B}$ such that $\varphi(z) = z'$, lift it to an automorphism of $\xi|_{\mathring{B}}$ by a choice of trivialisation of $\xi|_{\mathring{B}}$ and then extend it by the identity to obtain an element of $\mathrm{Aut}_\partial(\xi)$ sending z to z'.) Thus the subspace $\mathrm{Aut}_\partial(\xi, \hat{z} \mapsto z)$ is a coset of $\mathrm{Aut}_\partial(\xi, \hat{z})$ in $\mathrm{Aut}_\partial(\xi)$, and may be identified with $\Gamma(M \smallsetminus z, \xi)$ via $[s, (\varphi, g)] \mapsto g \circ s \circ \varphi^{-1}$. Thus there is a natural bijection between the set defined in Definition 3.2 and the formal definition (3.2). ◻

Definition 3.8 (*The associated fibre bundle.*) From Definition 3.6 and the proof of Lemma 3.7, we have a fibre bundle

$$p \colon \mathrm{C\Gamma}_k(M; \xi) \longrightarrow C_k(\mathring{M}) \tag{3.5}$$

whose fibre over a configuration $z \in C_k(\mathring{M})$ is the space of sections $\Gamma(M \smallsetminus z, \xi)$.

Definition 3.9 (*Boundary conditions.*) If we fix a subset $D \subseteq \partial M$ and a section $s_D \in \Gamma(D, \xi)$, we have a restricted version of the configuration-section space:

$$\mathrm{C\Gamma}_k^D(M; \xi) = \{(z, s) \in \mathrm{C\Gamma}_k(M; \xi) \mid s|_D = s_D\}. \tag{3.6}$$

Moreover, since D is contained in the boundary of M, the subspace $\mathrm{C\Gamma}_k^D(M;\xi) \subseteq \mathrm{C\Gamma}_k(M;\xi)$ is invariant under the action of $\mathrm{Aut}_\partial(\xi)$, so Proposition 2.5 and Lemma 3.5 imply that the restriction

$$p\colon \mathrm{C\Gamma}_k^D(M;\xi) \longrightarrow C_k(\mathring{M}) \tag{3.7}$$

of the map (3.3) is a fibre bundle. The fibre $p^{-1}(z)$ over a configuration $z \in C_k(\mathring{M})$ is the space of sections $\{s \in \Gamma(M \smallsetminus z, \xi) \mid s|_D = s_D\}$ (generalising Definition 3.8, which corresponds to $D = \varnothing$).

We now wish to define *configuration-section spaces with prescribed monodromy*

$$\mathrm{C\Gamma}_k^c(M;\xi) \subseteq \mathrm{C\Gamma}_k(M;\xi).$$

In other words, we wish to specify a certain "local behaviour" c of the section $s \in \Gamma(M \smallsetminus z, \xi)$ near the "singularities" $z \subseteq \mathring{M}$. To do this, we first construct a covering space over the interior $\mathring{M}$ of M depending on the fibre bundle $\xi\colon E \to M$.

Definition 3.10 (*The covering space of local sections of ξ, informal.*) Informally, the covering space

$$\eta(\xi)\colon \Sigma(\xi) \longrightarrow \mathring{M}$$

is defined by prescribing that its fibre $\eta(\xi)^{-1}(p)$ over a point p in the interior of M consists of the set of all *germs* of sections of ξ defined on a small punctured-disc neighbourhood of p in M.

Definition 3.11 (*The covering space of local sections of ξ, formal.*) Fix a point $p \in \mathring{M}$. A *local section near p of ξ* is a pair (B, σ) consisting of a subset $B \subseteq M$ homeomorphic to a d-dimensional ball, containing p in its interior, together with a section σ of $\xi|_{\partial B}$. Let $\sim$ be the equivalence relation on such pairs generated by $(B,\sigma) \sim (B',\sigma')$ if B is contained in the interior of B' and the section $\sigma \sqcup \sigma'$ defined on $\partial B \sqcup \partial B'$ extends to a section over $B' \smallsetminus \mathrm{int}(B)$. Write $[B,\sigma]_p$ for the equivalence class containing (B,σ).

Let $\Sigma(\xi)$ be the set of all pairs $(p, [B,\sigma]_p)$ where $p \in \mathring{M}$ and $[B,\sigma]_p$ is an equivalence class of local sections near p of ξ. We define a topology on $\Sigma(\xi)$ as follows. Let $(p,[B,\sigma]_p) \in \Sigma(\xi)$ and choose a representative (B,σ) for the equivalence class $[B,\sigma]_p$. Define

$$\mathcal{N}_{p,B,\sigma} = \{(q,[B,\sigma]_q) \mid q \in \mathrm{int}(B)\} \subseteq \Sigma(\xi).$$

Then one may check that the collection $\mathcal{N}$ of the sets $\mathcal{N}_{p,B,\sigma}$ for all $(p,[B,\sigma]_p) \in \Sigma(\xi)$ is a basis for a topology $\mathcal{T}$ on $\Sigma(\xi)$ such that the map

$$\eta(\xi)\colon \Sigma(\xi) \longrightarrow \mathring{M}$$

given by $(p,[B,\sigma]) \mapsto p$ is a covering map.

Remark 3.12 Although the basis $\mathcal{N}$ depends on a choice, for each point $(p,[B,\sigma]_p) \in \Sigma(\xi)$, of a representative (B,σ) of the equivalence class $[B,\sigma]_p$, the topology $\mathcal{T}$ that it generates does not depend on these choices.

There is also a basepointed version of the covering space $\eta(\xi)$. As before, let M be a d-dimensional manifold and $\xi\colon E \to M$ a fibre bundle. Choose basepoints $* \in \partial M$ and $* \in E$ such that $\xi(*) = *$.

Definition 3.13 (*The covering space of pointed local sections.*) Fix a point $p \in \mathring{M}$. A *pointed local section near p of ξ* is a pair (B,σ) consisting of a subset $B \subseteq M$ homeomorphic to a d-dimensional ball, containing p in its interior and $*$ in its boundary, together with a section σ of $\xi|_{\partial B}$ such that $\sigma(*) = *$. Let $\sim$ be the equivalence relation on such pairs generated by $(B,\sigma) \sim (B',\sigma')$ if B is contained in $\mathrm{int}(B') \cup \{*\}$ and the section $\sigma \vee \sigma'$ defined on $\partial B \vee \partial B'$ extends to a section over $B' \smallsetminus \mathrm{int}(B)$. Write $[B,\sigma]_p^*$ for the equivalence class containing (B,σ).

Let $\Sigma_*(\xi)$ be the set of all pairs $(p,[B,\sigma]_p^*)$ where $p \in \mathring{M}$ and $[B,\sigma]_p^*$ is an equivalence class of pointed local sections near p of ξ. Analogously to Definition 3.11, we define a topology on $\Sigma_*(\xi)$ such that the map

$$\eta_*(\xi)\colon \Sigma_*(\xi) \longrightarrow \mathring{M}$$

given by $(p, [B, \sigma]_p^*) \mapsto p$ is a covering map. There is a map of covering spaces over $\mathring{M}$

$$\Sigma_*(\xi) \longrightarrow \Sigma(\xi) \tag{3.8}$$

given by $(p, [B, \sigma]_p^*) \mapsto (p, [B, \sigma]_p)$.

Definition 3.14 A *singularity condition* for $\xi \colon E \to M$ is a set of trivial components of the covering $\eta(\xi) \colon \Sigma(\xi) \to \mathring{M}$. In other words, it is a subset of $\Gamma(\eta(\xi))$, the set of sections of $\eta(\xi)$. Similarly, a *pointed singularity condition* is a subset of $\Gamma(\eta_*(\xi))$.

Remark 3.15 The map (3.8) of covering spaces induces a function

$$(3.8) \circ - \colon \Gamma(\eta_*(\xi)) \longrightarrow \Gamma(\eta(\xi)),$$

so a pointed singularity condition determines an (unpointed) singularity condition.

Remark 3.16 Over a point $p \in \mathring{M}$, the fibre of $\eta(\xi) \colon \Sigma(\xi) \to \mathring{M}$ may be identified with the set of (unbased) homotopy classes of maps $[S^{d-1}, F]$, where $F = \xi^{-1}(p)$. This identification is canonical once we have chosen a local orientation of $\mathring{M}$ at p. Similarly, the fibre of $\eta_*(\xi) \colon \Sigma_*(\xi) \to \mathring{M}$ may be identified with $\pi_{d-1}(F)$, after choosing a basepoint of F. Under these identifications, the map $(3.8)|_p$ is the quotient $\pi_{d-1}(F) \twoheadrightarrow \pi_{d-1}(F)/\pi_1(F) = [S^{d-1}, F]$.

In order to define configuration-section spaces with prescribed singularity conditions, we need to show that any configuration-section of the bundle $\xi \colon E \to M$ induces a section of the covering $\eta(\xi) \colon \Sigma(\xi) \to \mathring{M}$.

Construction 3.17 There is a locally-constant map

$$\mathrm{loc}_\xi \colon \mathrm{C\Gamma}_k(M; \xi) \longrightarrow \Gamma(\eta(\xi)) \tag{3.9}$$

that records the local behaviour of a given configuration-section (z, s) in a punctured neighbourhood of each $p \in \mathring{M}$. This is possible since, although s is not defined on all of $\mathring{M}$, it *is* defined on a punctured neighbourhood of every point of $\mathring{M}$, since it is only undefined on a discrete subset $z \subseteq \mathring{M}$.

Formally, the construction is as follows. Given $(z, s) \in \mathrm{C\Gamma}_k(M; \xi)$ and $p \in \mathring{M}$, we need to choose a point in the fibre $\eta(\xi)^{-1}(p)$. To do this, first modify (z, s) if necessary, staying within the same path-component of $\mathrm{C\Gamma}_k(M; \xi)$, so that $p \in z$. We pause the construction briefly to prove that this is always possible:

Lemma 3.18 *Given any $(z, s) \in \mathrm{C\Gamma}_k(M; \xi)$ and $p \in \mathring{M}$, there is a path in $\mathrm{C\Gamma}_k(M; \xi)$ from (z, s) to (z', s') such that $p \in z'$.*

Proof. By the connectivity of M, we may find an embedded $D^d \subseteq M$ containing p and a point $q \in z$ in its interior, and disjoint from $z \smallsetminus \{q\}$. Over this embedded disc, ξ is isomorphic to a trivial bundle $D^d \times X \to D^d$, so we have a map

$$\mathrm{CMap}_1^{\partial D^d}(D^d; X) \longrightarrow \mathrm{C\Gamma}_k(M; \xi)$$

that extends a 1-point configuration-mapping in D^d (agreeing with $s|_{\partial D^d}$ on ∂D^d) by $z \smallsetminus \{q\}$ and $s|_{M \smallsetminus (z \cup D^d)}$. It therefore suffices to find a path in $\mathrm{CMap}_1^{\partial D^d}(D^d; X)$ from $(\{q\}, s|_{D^d})$ to $(\{p\}, s')$ for some map $s' \colon D^d \smallsetminus \{p\} \to X$. Next, note that there is a natural left-action of $\mathrm{Homeo}(D^d; \partial D^d)$ on $\mathrm{CMap}_1^{\partial D^d}(D^d; X)$ given by pre-composition, so it suffices to find a path in $\mathrm{Homeo}(D^d; \partial D^d)$ from the identity to a homeomorphism φ such that $\varphi(p) = q$. But $\mathrm{Homeo}(D^d; \partial D^d)$ acts transitively on $\mathrm{int}(D^d)$ and is also path-connected (in fact contractible) by the Alexander trick. ☐

Continuing with the construction, we may assume that $p \in z$. Choose an embedded d-dimensional ball $B \subseteq M$ containing p in its interior and disjoint from $z \smallsetminus \{p\}$. Then define

$$\mathrm{loc}_\xi(z, s)(p) = (p, [B, s|_{\partial B}]_p).$$

One may then easily check that this gives a well-defined locally-constant map of the form (3.9).

Definition 3.19 (*Configuration-section spaces with prescribed monodromy.*) As in Definition 3.2, let M be a smooth, connected d-manifold and let $\xi\colon E \to M$ be a fibre bundle. Also choose a *singularity condition* $c \subseteq \Gamma(\eta(\xi))$ (*cf.* Definition 3.14). Then

$$\mathrm{C}\Gamma_k^c(M;\xi) \coloneqq \mathrm{loc}_\xi^{-1}(c) \subseteq \mathrm{C}\Gamma_k(M;\xi).$$

In other words, it is the subspace of $\mathrm{C}\Gamma_k(M;\xi)$ of configuration-sections (z,s) such that, if $B \subseteq M$ is a subset that is homeomorphic to a d-dimensional ball, contains one point p of z in its interior and is disjoint from $z \smallsetminus \{p\}$, then $(p,[B,s|_{\partial B}]_p) \in \Sigma(\xi)$ lies in the image of one of the sections in c.

If we fix a subset $D \subseteq \partial M$ and a section $s_D \in \Gamma(D,\xi)$, we define (just as in (3.6)):

$$\mathrm{C}\Gamma_k^{c,D}(M;\xi) = \{(z,s) \in \mathrm{C}\Gamma_k^c(M;\xi) \mid s|_D = s_D\}. \tag{3.10}$$

Remark 3.20 Since (3.9) is locally-constant, the subspace $\mathrm{C}\Gamma_k^c(M;\xi)$ of $\mathrm{C}\Gamma_k(M;\xi)$ is a union of path-components, and similarly for the restricted version (3.10) $\subseteq$ (3.6). (Compare Remark 2.12.)

Definition 3.21 (*The associated fibration.*) From Definition 3.9 we have a fibre bundle (3.7) with total space $\mathrm{C}\Gamma_k^D(M;\xi)$. By Remark 3.20, the subspace $\mathrm{C}\Gamma_k^{c,D}(M;\xi) \subseteq \mathrm{C}\Gamma_k^D(M;\xi)$ is a union of path-components. Thus, the restriction

$$p\colon \mathrm{C}\Gamma_k^{c,D}(M;\xi) \longrightarrow C_k(\mathring{M}) \tag{3.11}$$

of the fibre bundle (3.7) to the subspace $\mathrm{C}\Gamma_k^{c,D}(M;\xi)$ is a Hurewicz fibration.[1] This is the *associated fibration* of the configuration-section space $\mathrm{C}\Gamma_k^{c,D}(M;\xi)$. Its fibre over $z \in C_k(\mathring{M})$ is denoted by

$$\Gamma^{c,D}(M \smallsetminus z;\xi) = \{s \in \Gamma(M \smallsetminus z;\xi) \mid s|_D = s_D \text{ and } \mathrm{loc}_\xi(z,s) \in c\}.$$

As our first family of examples, we explain how to recover the notion of *configuration-mapping space* of §2 in the case of a trivial bundle over M.

Example 3.22 If ξ is the trivial bundle $M \times X \to M$ for a space X, then we clearly have an identification

$$\mathrm{C}\Gamma_k(M;\xi) = \mathrm{CMap}_k(M;X). \tag{3.12}$$

If M is orientable, then the covering space $\eta(\xi)\colon \Sigma(\xi) \to \mathring{M}$ is simply the disjoint union of copies of the trivial (identity) covering $\mathring{M} \to \mathring{M}$, one for each element of $[S^{d-1},X]$. In other words, $\eta(\xi)$ is isomorphic as a covering space to the projection $\mathring{M} \times [S^{d-1},X] \to \mathring{M}$. Similarly, the covering space $\eta_*(\xi)\colon \Sigma_*(\xi) \to \mathring{M}$ is isomorphic as a covering space to the projection $\mathring{M} \times \pi_{d-1}(X) \to \mathring{M}$. In each case, the isomorphism of covering spaces depends on a choice of orientation of M.

If M is non-orientable, then the covering space $\eta(\xi)\colon \Sigma(\xi) \to \mathring{M}$ is a disjoint union of a number of copies of the identity covering $\mathring{M} \to \mathring{M}$ and a number of copies of the orientation double covering $\mathring{M}^{\mathrm{or}} \to \mathring{M}$. More precisely, consider the involution on the set $[S^{d-1},X]$ given by precomposition by a reflection of S^{d-1}. There is one copy of the identity covering in $\eta(\xi)$ for each fixed point of this action and one copy of the orientation double covering in $\eta(\xi)$ for each orbit of size two. In other words, writing $\mathcal{O}_1 = [S^{d-1},X]^{\mathbb{Z}/2}$ for the set of orbits of size 1 and $\mathcal{O}_2$ for the set of orbits of size 2 in $[S^{d-1},X]$, we have that $\eta(\xi)$ is isomorphic as a covering space to

$$\mathrm{pr} \circ (\mathrm{id} \sqcup (\mathrm{or} \times \mathrm{id}))\colon (\mathring{M} \times \mathcal{O}_1) \sqcup (\mathring{M}^{\mathrm{or}} \times \mathcal{O}_2) \longrightarrow \mathring{M} \times (\mathcal{O}_1 \sqcup \mathcal{O}_2) \longrightarrow \mathring{M}, \tag{3.13}$$

where or is the orientation double covering of $\mathring{M}$. The isomorphism is canonical up to the action of the $2^{\mathcal{O}_2}$ deck transformations of (3.13) corresponding to the deck transformations of each of the copies of $\mathring{M}^{\mathrm{or}}$ (acting independently). There is an analogous involution on the set $\pi_{d-1}(X)$, and the covering space $\eta_*(\xi)\colon \Sigma_*(\xi) \to \mathring{M}$ is a disjoint union of one copy of the identity covering for each fixed point of $\pi_{d-1}(X)$ and one copy of the orientation double covering for each orbit of size two in $\pi_{d-1}(X)$. This may also be written in the form (3.13), where $\mathcal{O}_1$ and $\mathcal{O}_2$ now denote the orbits of size 1 and 2 of the involution on $\pi_{d-1}(X)$, and the isomorphism is again canonical up to the $2^{\mathcal{O}_2}$ deck transformations of (3.13) described above.

A singularity condition $c \subseteq \Gamma(\eta(\xi))$ therefore corresponds to (compare Remark 2.3):

[1] The base space of the fibre bundle (3.7) is paracompact, so it is a Hurewicz fibration. In general, if $f\colon E \to B$ is a Hurewicz fibration and $E_0 \subseteq E$ is a union of path-components, the composition $f \circ \mathrm{incl}\colon E_0 \hookrightarrow E \to B$ is also a Hurewicz fibration.

- (if M is orientable) a subset of $[S^{d-1}, X]$,
- (if M is non-orientable) a subset of $[S^{d-1}, X]^{\mathbb{Z}/2}$, the set of fixed points of $[S^{d-1}, X]$ under the involution given by pre-composition with a reflection of S^{d-1}.

In the orientable case, this correspondence depends on a choice of orientation of M; in the non-orientable case, it does not depend on any choice. Analogously, a pointed singularity condition corresponds either to a subset of $\pi_{d-1}(X)$ or a subset of $\pi_{d-1}(X)^{\mathbb{Z}/2}$, depending on whether M is orientable or not, respectively. Again, this correspondence depends on a choice of orientation of M if M is orientable.

Interpreting the singularity condition c in this way on the right-hand side, the identification (3.12) restricts to identifications:

$$\mathrm{C}\Gamma_k^c(M;\xi) = \mathrm{CMap}_k^c(M;X) \tag{3.14}$$

for each $c \subseteq \Gamma(\eta(\xi))$.

There is a natural variant of configuration-section spaces where configurations are equipped just with *homotopy-classes* of sections on their complements:

Definition 3.23 (*Configuration-section spaces up to homotopy.*) Define $h\mathrm{C}\Gamma_k(M;\xi)$ to be the quotient of $\mathrm{C}\Gamma_k(M;\xi)$ by the equivalence relation given by $(z,s) \sim (z',s')$ if $z = z'$ and the sections $s, s' \colon M \smallsetminus z \to E$ are homotopic through sections of $\xi|_{M \smallsetminus z}$, in other words, lie in the same path-component of the space $\Gamma(M \smallsetminus z, \xi)$. The locally-constant map (3.9) $= \mathrm{loc}_\xi$ factors into the quotient map $\mathrm{C}\Gamma_k(M;\xi) \to h\mathrm{C}\Gamma_k(M;\xi)$ and a locally-constant map

$$h\mathrm{loc}_\xi \colon h\mathrm{C}\Gamma_k(M;\xi) \longrightarrow \Gamma(\eta(\xi)), \tag{3.15}$$

since homotopic sections of $\xi|_{M \smallsetminus z}$ have the same local germs in punctured neighbourhoods of every point in M. We then define

$$h\mathrm{C}\Gamma_k^c(M;\xi) \coloneqq h\mathrm{loc}_\xi^{-1}(c)$$

for any singularity condition $c \subseteq \Gamma(\eta(\xi))$. We may equivalently define $h\mathrm{C}\Gamma_k^c(M;\xi)$ to be the quotient of $\mathrm{C}\Gamma_k^c(M;\xi)$ by the restriction of the equivalence relation above. More generally, if we fix a subset $D \subseteq \partial M$ and a section $s_D \in \Gamma(D,\xi)$, we may define $h\mathrm{C}\Gamma_k^{c,D}(M;\xi)$ to be the quotient of $\mathrm{C}\Gamma_k^{c,D}(M;\xi)$ by the equivalence relation given by $(z,s) \sim (z',s')$ if $z = z'$ and the sections s, s' lie in the same path-component of the space

$$\Gamma(M \smallsetminus z, \xi; s_D) = \{ s \in \Gamma(M \smallsetminus z, \xi) \mid s|_D = s_D \}.$$

Example 3.24 (*Configurations equipped with a cohomology class of the complement.*) As a special case, of course, we have *configuration-mapping spaces up to homotopy*

$$h\mathrm{CMap}_k^c(M;X) = h\mathrm{C}\Gamma_k^c(M; M \times X).$$

As a set, $h\mathrm{CMap}_k^c(M;X)$ consists of pairs (z,f), where $z \subseteq \mathring{M}$ has cardinality k and f is a homotopy class of maps $M \smallsetminus z \to X$ whose monodromy around each point of z lies in $c \subseteq [S^{d-1}, X]$. If we take X to be the Eilenberg-MacLane space $K(G, d-1)$ for an abelian group G, then c is a subset of $H^{d-1}(S^{d-1}; G) \cong G$ and a point in

$$h\mathrm{CMap}_k^c(M; K(G, d-1))$$

is a configuration $z \subseteq \mathring{M}$ equipped with a cohomology class $\alpha \in H^{d-1}(M \smallsetminus z; G)$ whose restriction to each embedded sphere $S^{d-1} \subseteq M \smallsetminus z$ that encloses exactly one point of z, lies in c.

Before describing the next example of configuration-mapping spaces up to homotopy, we note that, under certain conditions, configuration-mapping spaces up to homotopy have the same weak homotopy type as the corresponding configuration-mapping spaces (not up to homotopy).

Lemma 3.25 *Let M be a compact, connected d-manifold with basepoint $* \in \partial M$ and X a based, path-connected space with a choice of subset $c \subseteq [S^{d-1}, X]$. Assume that M is homotopy equivalent to a wedge of $(d-1)$-spheres and that X is d-coconnected, meaning that $\pi_i(X) = 0$ for all $i \geqslant d$. Then, for any $k \geqslant 0$, the quotient map*

$$\mathrm{CMap}_k^{c,*}(M;X) \longrightarrow h\mathrm{CMap}_k^{c,*}(M;X) \tag{3.16}$$

is a weak homotopy equivalence. In particular, this holds if $M = S$ is a compact, connected surface, $X = BG$ is the classifying space of a discrete group G and c is a set of conjugacy classes of G.

Proof. The quotient map fits into a map of fibration sequences:

$$\begin{tikzcd} \mathrm{Map}^{c,*}(M \smallsetminus z, X) \arrow[r, "(\star)"] \arrow[d] & \pi_0(\mathrm{Map}^{c,*}(M \smallsetminus z, X)) \arrow[d] \\ \mathrm{CMap}_k^{c,*}(M;X) \arrow[r, "(3.16)"] \arrow[dr] & h\mathrm{CMap}_k^{c,*}(M;X) \arrow[d] \\ & C_k(\mathring{M}) \end{tikzcd} \tag{3.17}$$

Since M is homotopy equivalent to a wedge of some number ℓ of $(d-1)$-spheres and has non-empty boundary, we have $M \smallsetminus z \simeq \bigvee^{|z|+\ell} S^{d-1}$ and thus

$$\mathrm{Map}^{c,*}(M \smallsetminus z, X) \simeq (\Omega^{d-1}X)^{\ell} \times (\Omega_c^{d-1}X)^{|z|}, \tag{3.18}$$

where $\Omega_c^{d-1}X \subseteq \Omega^{d-1}X$ is the union of path-components corresponding to $c \subseteq [S^{d-1}, X]$ under the identification of $[S^{d-1}, X]$ with $\pi_{d-1}(X)/\pi_1(X)$. Since X is d-coconnected, the space (3.18) is 1-coconnected, in other words weakly contractible, and so the map $(\star)$ of fibres in (3.17) is a weak homotopy equivalence, and therefore so is the map (3.16). ∎

Remark 3.26 Lemma 3.25 generalises in an analogous way to configuration-section spaces, for a bundle $\xi \colon E \to M$ equipped with a section over $\{*\} \subseteq \partial M$, i.e., a point $e_0 \in E$ with $\xi(e_0) = *$.

Example 3.27 (*Hurwitz spaces.*) Following on from Example 3.24, let S be a compact, connected surface with boundary and now take $X = K(G,1) = BG$ for a (not necessarily abelian) discrete group G. Note that $S \smallsetminus z$, for any finite subset $z \subseteq \mathring{S}$, is aspherical, so we have a natural bijection

$$[S \smallsetminus z, BG] \cong \mathrm{Hom}(\pi_1(S \smallsetminus z), G)/G,$$

where the quotient on the right-hand side is by the action of G given by post-composition by inner automorphisms. A point in

$$h\mathrm{CMap}_k^{c}(S;BG)$$

therefore consists of a configuration $z \subseteq \mathring{S}$ equipped with a homomorphism $\pi_1(S \smallsetminus z) \to G$ modulo inner automorphisms of G. If we now take $D = \{*\} \subseteq \partial S$ to be a point, we have a natural bijection

$$\langle S \smallsetminus z, BG \rangle \cong \mathrm{Hom}(\pi_1(S \smallsetminus z), G),$$

where $\langle -, - \rangle$ denotes based homotopy classes of based maps, and so a point in

$$h\mathrm{CMap}_k^{c,*}(S;BG)$$

consists of a configuration $z \subseteq \mathring{S}$ equipped with a homomorphism $\pi_1(S \smallsetminus z) \to G$. In particular, if $S = D^2$ we have a homeomorphism

$$h\mathrm{CMap}_k^{c,*}(D^2;BG) \cong \mathrm{Hur}_{G,k}^{c},$$

where the right-hand side is the corresponding Hurwitz space. Moreover, Lemma 3.25 implies that the quotient map

$$\mathrm{CMap}_k^{c,*}(S;BG) \longrightarrow h\mathrm{CMap}_k^{c,*}(S;BG) \tag{3.19}$$

is a weak homotopy equivalence for any compact, connected surface-with-boundary S. In the case $S = D^2$, we therefore have weak homotopy equivalences (compare Remark 2.14):

$$\mathrm{CMap}_k^{c,I}(D^2;BG) \simeq \mathrm{CMap}_k^{c,*}(D^2;BG) \simeq h\mathrm{CMap}_k^{c,*}(D^2;BG) \cong \mathrm{Hur}_{G,k}^{c}. \tag{3.20}$$

Example 3.28 (*Tangential structures on the complement.*) If $\theta\colon E \to BO(n)$ is a type of tangential structure, and τ_M denotes the homotopy class of maps $M \to BO(n)$ classifying the tangent bundle of M, then a tangential structure of type θ on M is a lift of τ_M up to homotopy to a map $M \to E$. Equivalently, we may pick a specific map T_M in the homotopy class τ_M, and a tangential structure is then a lift up to homotopy of T_M. In other words, it is a section up to homotopy of the pullback bundle $(T_M)^*(\theta)\colon (T_M)^*(E) \to M$. The configuration-section space

$$h\mathrm{C}\Gamma_k(M;(T_M)^*(\theta))$$

is therefore the moduli space of k-point configurations in M whose complement is equipped with a tangential structure of type θ, and its subspaces corresponding to a singularity condition c may be interpreted as moduli spaces of configurations whose complement is equipped with a tangential structure of type θ, whose monodromy around the configuration points is prescribed.

Example 3.29 (*Linearly independent vector fields.*) Let $TM \to M$ denote the tangent bundle of M and write $\xi_r\colon \mathrm{Lin}_r(TM) \to M$ for the associated fibre bundle whose fibre over $p \in M$ is the subspace of $(T_pM)^r$ consisting of linearly independent r-tuples of tangent vectors at p. The configuration-section space

$$\mathrm{C}\Gamma_k(M;\xi_r)$$

is then the moduli space of k-point configurations $z \subseteq \mathring{M}$ equipped with an r-tuple of linearly independent vector fields on $M \smallsetminus z$. In particular, when $r = 1$, this is the space of configurations equipped with a non-vanishing vector field on the complement.

Example 3.30 (*Magnetic monopoles.*) Going back to configuration-mapping spaces, we may consider $\mathrm{CMap}_k^*(D^d;K(\mathbb{Z},d))$. As remarked in the introduction, for a given configuration $z \subset \mathbb{R}^d \cong \mathring{D}^d$, the based map $f\colon D^d \smallsetminus z \to K(\mathbb{Z},d)$ may be thought of as a *phase* in $S^1 \simeq \Omega^{d-1}K(\mathbb{Z},d)$ associated to each particle $p \in z$, after choosing a deformation retraction of $D^d \smallsetminus z$ onto a wedge of k copies of S^{d-1}. However, this deformation retraction cannot be chosen consistently as the configuration z varies, so this description of an element of $\mathrm{CMap}_k^*(D^d;K(\mathbb{Z},d))$ as a configuration of particles equipped with phases in S^1 is valid only in small neighbourhoods of the configuration-mapping space. Globally, an element of $\mathrm{CMap}_k^*(D^d;K(\mathbb{Z},d))$ models a configuration equipped with some *non-local* data that does not decompose into pieces associated to each particle.

When $d = 3$, this space could be a model for an asymptotic part of the *moduli space of magnetic monopoles* $\mathcal{M}_k$ of total charge k in $\mathbb{R}^3$, namely the part consisting of "widely separated" monopoles (*cf.* [AH88, Proposition 3.12], [Seg97]). We note that it is not the case that $\mathrm{CMap}_k^*(D^3;K(\mathbb{Z},3))$ is the *whole* moduli space $\mathcal{M}_k$, since the fundamental group of $\mathcal{M}_k$ is known to be $\mathbb{Z}$ (combining [Don84] with [Seg79, Proposition 6.4]), whereas the fundamental group of $\mathrm{CMap}_k^*(D^3;K(\mathbb{Z},3))$ surjects onto Σ_k via the forgetful map to $C_k(\mathbb{R}^3)$. It is interesting to note that the full moduli space $\mathcal{M}_k$ is also homologically stable with respect to the "magnetic charge" k (combining [Don84] with [Seg79, Proposition 1.1]).

4. On E_m-modules over E_n-algebras

All of the structure that we will use in studying configuration-section spaces will arise from their structure as an E_0*-module over an* E_{d-1}*-algebra*, which will be defined explicitly, in appropriate models, in this section. Before this, we recall the notion of E_m*-module over an* E_n*-algebra* for any $0 \leqslant m \leqslant n$, and explain why, for n fixed, these notions coincide for all $m \in \{0, 1, \ldots, n-1\}$.

Remark 4.1 The last sentence above means that, when $d \geqslant 3$, we may equally well describe the structure that we construct as an "E_1-module" over an E_{d-1}-algebra. However, since we also consider the case of dimension $d = 2$, we prefer to use the term "E_0-module" throughout, for consistency. This diverges from the terminology of [Kra19], where the name "E_1-module" is used – this is (as we explain below) equivalent since that paper considers only E_n-algebras where $n \geqslant 2$.

We begin by recalling several different flavours of Swiss cheese operads. For an integer $n \geqslant 0$, let D^n denote the closed unit disc in $\mathbb{R}^n$. For a space X and an integer $k \geqslant 0$, let $\bar{F}_k(X)$ denote the ordered configuration space of k points in X labelled by positive real numbers:

$$\bar{F}_k(X) = \{((x_1, r_1), \ldots, (x_n, r_n)) \in (X \times (0, \infty))^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Fix an integer $n \geqslant 0$.

Definition 4.2 The *little n-discs operad* D_n has one colour a. For any integer $k \geqslant 0$ its space of operations $\mathsf{D}_n(\mathsf{a}^k; \mathsf{a})$ is the subspace of $\bar{F}_k(D^n)$ of configurations satisfying

(i) $|x_i| \leqslant 1 - r_i$
(ii) $|x_i - x_j| \geqslant r_i + r_j$ for $i \neq j$.

Interpreting such configurations as little n-discs in D^n whose interiors are disjoint, with centres x_i and radii r_i – see Figure 4.1(a) – the operadic composition is defined by embedding D^n into these smaller discs by translations and dilations. The symmetric action is given by the natural action of Σ_k on $\bar{F}_k(D^n)$.

Definition 4.3 A *D_n-module operad* is any operad O with two colours a and m, and whose space of operations $\mathsf{O}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{a})$, for any integers $k, l \geqslant 0$, is equal to $\mathsf{D}_n(\mathsf{a}^k; \mathsf{a})$ for $l = 0$ and empty for $l \geqslant 1$. Moreover, its operadic composition, restricted to the colour a, must be equal to that of D_n.

Fix two integers $n \geqslant m \geqslant 0$. We now define five different D_n-module operads, the *Swiss cheese operad* ($\mathsf{SC}_{m,n}$), as well as its *extended* ($\mathsf{ESC}_{m,n}$), *variant* ($\mathsf{VSC}_{m,n}$), *concentric* ($\mathsf{CSC}_{m,n}$) and *linear* (LSC_n) cousins. In each case it will suffice to specify the spaces of operations $\mathsf{O}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{m})$ for integers $k, l \geqslant 0$ and describe how to extend the operadic composition of D_n to the colour m.

Remark 4.4 The original Swiss cheese operad $\mathsf{SC}_{1,2}$ was introduced by Voronov [Vor99], inspired by constructions of Kontsevich [Kon94; Kon03]. The extended Swiss cheese operad $\mathsf{ESC}_{m,n}$ was introduced by Willwacher [Wil], who cites V. Turchin for its invention in codimension 1 (when $n = m + 1$). The variant Swiss cheese operad $\mathsf{VSC}_{m,n}$ was introduced by Idrissi [Idr]. The linear Swiss cheese operad LSC_n was introduced (under the name SC_n) by Krannich [Kra19, §2.1].

Definition 4.5 (*Extended, variant and original Swiss cheese operads.*) Let $\mathsf{ESC}_{m,n}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{m})$ be the subspace of $\bar{F}_{k+l}(D^n)$ of configurations satisfying (i) and (ii) above, and

(iii) $x_i \in D^m$ for $i \geqslant k + 1$.

The space $\mathsf{VSC}_{m,n}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{m})$ is the subspace of configurations additionally satisfying the condition

(iv) $\mathrm{dist}(x_i, D^m) > r_i$ for $i \leqslant k$.

The space $\mathsf{SC}_{m,n}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{m})$ is the subspace of configurations satisfying conditions (i)–(iii) and

(v) $x_i \in \mathbb{R}^m \times (r_i, \infty)^{n-m}$ for $i \leqslant k$.

(Note that condition (v) is stronger than condition (iv).) Interpreting such configurations again as little non-overlapping n-discs in D^n – see Figure 4.1(b–d) – we may extend the operadic composition of D_n to each of these 2-coloured operads by embedding D^n into these smaller discs by translations and dilations.

Definition 4.6 (*The concentric Swiss cheese operad.*) The space of operations $\mathsf{CSC}_{m,n}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{m})$ is empty unless $l = 1$, in which case it is the subspace of $\bar{F}_{k+1}(D^n)$ of configurations satisfying (i), (ii) and (v) above, as well as

(vi) $x_{k+1} = 0$.

Its operadic composition is defined as before, interpreting these configuration spaces as spaces of little non-overlapping discs – see Figure 4.1(e).

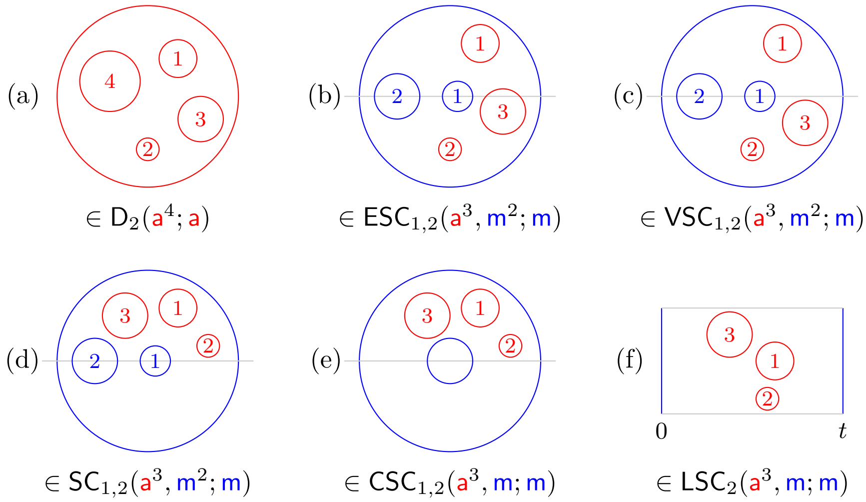


Figure 4.1 Some operations of the little discs operad and of different flavours of Swiss cheese operads in dimensions $(1, 2)$. Little blue (m colour) discs are always centred on the x-axis. Little red (a colour) discs are unrestricted in $\mathsf{ESC}_{1,2}$ (except that they must not overlap each other or the little blue discs, of course). In $\mathsf{VSC}_{1,2}$, little red discs must be disjoint from the x-axis (little red discs are called *aerial* and little blue discs *terrestrial* in [Idr]). In $\mathsf{SC}_{1,2}$, little red discs must lie in the upper half-disc (and we may choose to think of the little blue discs simply as little half-discs attached to the x-axis). In $\mathsf{CSC}_{1,2}$ the same conditions apply to little red discs, and there is now required to be exactly one little blue disc, centred at the origin. In LSC_2, little red discs may lie anywhere in the rectangle. See Definitions 4.2, 4.5, 4.6 and 4.7 for the precise definitions.

Definition 4.7 (*The linear Swiss cheese operad.*) The space of operations $\mathsf{LSC}_n(\mathsf{a}^k, \mathsf{m}^l; \mathsf{m})$ is empty unless $l = 1$, in which case it is the subspace of $\bar{F}_k([0,\infty) \times [-1,1]^{n-1}) \times [0,\infty)$ of configurations $((x_1, r_1), \ldots, (x_k, r_k))$ and $t \geqslant 0$ satisfying (ii) above, and

(vii) $x_i \in [r_i, t - r_i] \times [r_i - 1, 1 - r_i]^{n-1}$.

Its operadic composition is given by interpreting these configurations as configurations of little non-overlapping discs and embedding D^n by translations and dilations, as well as placing copies of the cuboid $[0,t] \times [-1,1]^{n-1}$ end-to-end in the first coordinate direction and adding the corresponding values of t. See Figure 4.1(f), and also [Kra19, Definition 2.1] for precise formulas for the operadic composition.

By definition, there are inclusions of operads

$$\mathsf{CSC}_{m,n} \hookrightarrow \mathsf{SC}_{m,n} \hookrightarrow \mathsf{VSC}_{m,n} \hookrightarrow \mathsf{ESC}_{m,n} \tag{4.1}$$

that restrict to the identity map of D_n on the a colour. The natural inclusions of discs $D^n \hookrightarrow D^{n+1}$ and cubes $[-1,1]^{n-1} \hookrightarrow [-1,1]^n$ induce dimension-increasing inclusions (see Figure 4.2)

$$\mathsf{LSC}_n \longrightarrow \mathsf{LSC}_{n+1} \qquad \text{and} \qquad \mathsf{XSC}_{m,n} \longrightarrow \mathsf{XSC}_{m,n+1} \tag{4.2}$$

(for $\mathsf{X} \in \{\mathsf{E}, \mathsf{V}, \varnothing, \mathsf{C}\}$) and similarly

$$\mathsf{XSC}_{m-1,n} \longrightarrow \mathsf{XSC}_{m,n} \tag{4.3}$$

(for $\mathsf{X} \in \{\mathsf{E}, \mathsf{V}, \varnothing, \mathsf{C}\}$), which commute with the "flavour-changing" inclusions (4.1). The connection between the four operads (4.1) and the linear Swiss cheese operads is given by the following lemma. Recall that a *weak equivalence of operads* is a map of operads $\mathsf{O} \to \mathsf{P}$ such that the maps of spaces $\mathsf{O}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{a}) \to \mathsf{P}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{a})$ and $\mathsf{O}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{m}) \to \mathsf{P}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{m})$ are all weak equivalences.

Lemma 4.8 *For any $0 \leqslant m \leqslant n$ there is a map of operads*

$$\iota_{m,n} \colon \mathsf{LSC}_n \longrightarrow \mathsf{CSC}_{m,n} \tag{4.4}$$

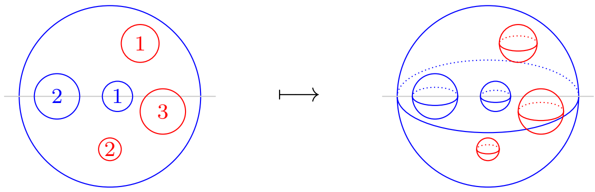


Figure 4.2 The dimension-increasing inclusion $\mathsf{ESC}_{1,2}(\mathsf{a}^3, \mathsf{m}^2; \mathsf{m}) \to \mathsf{ESC}_{1,3}(\mathsf{a}^3, \mathsf{m}^2; \mathsf{m})$. The numbering of the 3-discs has been omitted on the right-hand side to avoid overloading the diagram.

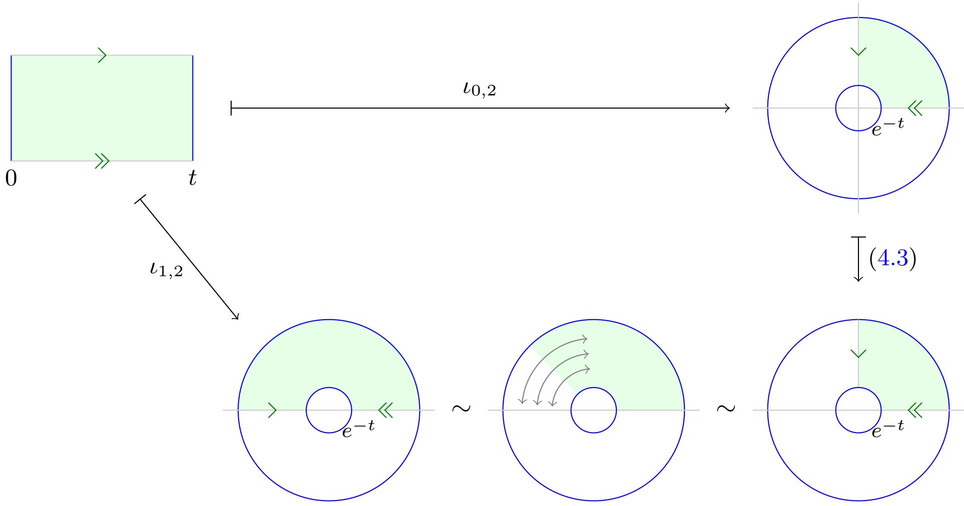


Figure 4.3 The map $\iota_{m,n}$ and the homotopy-commutativity of the right-hand triangle of (4.5).

commuting with (4.2) *and* (4.3) *in the sense that the following diagrams commute up to homotopy:*

$$\begin{tikzcd} \mathsf{LSC}_n \arrow[r, "\iota_{m,n}"] \arrow[d, "(4.2)"'] & \mathsf{CSC}_{m,n} \arrow[d, "(4.2)"] \\ \mathsf{LSC}_{n+1} \arrow[r, "\iota_{m,n+1}"'] & \mathsf{CSC}_{m,n+1} \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{CSC}_{m-1,n} \arrow[d, "(4.3)"] \\ \mathsf{LSC}_n \arrow[ur, "\iota_{m-1,n}"] \arrow[r, "\iota_{m,n}"'] & \mathsf{CSC}_{m,n} \end{tikzcd} \tag{4.5}$$

When $0 \leqslant m \leqslant n-1$*, the map* $\iota_{m,n}$ *is a weak equivalence of operads, so* (4.3)$\colon \mathsf{CSC}_{m-1,n} \to \mathsf{CSC}_{m,n}$ *is also a weak equivalence of operads in this range.*

Proof. The map $\iota_{m,n}$ is defined as pictured (for the cases $(m,n) = (1,2)$ and $(m,n) = (0,2)$) in Figure 4.3: a configuration in the rectangle is mapped via the indicated mapping of the rectangle onto a segment of the annulus, together with an appropriate rescaling of the labels in $(0,\infty)$ of the configuration points (interpreted as radii). The fact that $\iota_{m,n}$ is a weak equivalence of operads for $m \leqslant n-1$ follows from the fact that this mapping is a homeomorphism from the rectangle onto a proper segment of the annulus (when $m = n$ it is a quotient map from the rectangle onto the whole annulus). The left-hand square of (4.5) commutes on the nose, and the right-hand triangle of (4.5) commutes up to a "stretching" homotopy that is also pictured in Figure (4.5) (in the case $(m,n) = (1,2)$). $\square$

Remark 4.9 The map $\iota_{n,n} \colon \mathsf{LSC}_n \to \mathsf{CSC}_{n,n}$, on the other hand, is *not* a weak equivalence: for example, the space $\mathsf{LSC}_n(\mathsf{a}, \mathsf{m}; \mathsf{m})$ is contractible, whereas the space $\mathsf{CSC}_{n,n}(\mathsf{a}, \mathsf{m}; \mathsf{m})$ is homotopy equivalent to S^{n-1}.

Algebras. Recall that, in general, an *algebra* over a two-coloured operad O (with colours a and m) consists of a pair of spaces $(X_\mathsf{a}, X_\mathsf{m})$ together with maps

$$\mathsf{O}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{a}) \times (X_\mathsf{a})^k \times (X_\mathsf{m})^l \longrightarrow X_\mathsf{a} \qquad \text{and} \qquad \mathsf{O}(\mathsf{a}^k, \mathsf{m}^l; \mathsf{m}) \times (X_\mathsf{a})^k \times (X_\mathsf{m})^l \longrightarrow X_\mathsf{m}$$

satisfying appropriate associativity axioms. In the following, we will write $Y = X_{\mathsf{a}}$ (corresponding to the red colour in Figure 4.1) and $X = X_{\mathsf{m}}$ (corresponding to the blue colour in Figure 4.1).

Algebras over Swiss-cheese operads. Algebras over D_n are E_n-*algebras*, by definition. Now, the restriction of $\mathsf{SC}_{m,n}$ to the a colour is D_n and its restriction to the m colour is isomorphic to D_m, so an algebra over $\mathsf{SC}_{m,n}$ is a pair (X, Y) consisting of an E_n-algebra Y, an E_m-algebra X, together with an additional structure intertwining them. The same remarks apply also to $\mathsf{VSC}_{m,n}$ and $\mathsf{ESC}_{m,n}$, so algebras over each of these three operads consist of an E_n-algebra acting on an E_m-algebra, where the precise meaning of "acting" depends on the flavour.

The restriction of the *concentric* Swiss cheese operad $\mathsf{CSC}_{m,n}$ to the a colour is again D_n, but now its restriction to the m colour is trivial (it has no l-ary operations except when $l = 1$, and its space of 1-ary operations is homeomorphic to $(0, 1]$, so contractible). Algebras over $\mathsf{CSC}_{m,n}$ are thus E_m-*modules* over E_n-algebras, without any E_m-*algebra* structure on the module. By Lemma 4.8, we have $\mathsf{CSC}_{m,n} \simeq \mathsf{CSC}_{m',n}$ for any $m, m' \leqslant n-1$, so the notions of "*E_m-module over an E_n-algebra*", for fixed n, are equivalent for all $m \in \{0, 1, \ldots, n-1\}$, and are equivalently encoded by the *linear* Swiss cheese operad LSC_n. On the other hand, the notion of "*E_n-module over an E_n-algebra*" is stronger, and not encoded by the linear Swiss cheese operad, as pointed out in Remark 4.9.

Linear Swiss cheese structures on configuration-section spaces. Below, we define certain homotopy equivalent models $\dot{C}(M) \simeq C(\mathring{M})$ and $\dot{\mathrm{C}}\Gamma^{c,D}(M;\xi) \simeq \mathrm{C}\Gamma^{c,D}(M;\xi)$ for configuration spaces and configuration-section spaces (see Definitions 4.12 and 4.13, and Lemma 4.14). Definition 4.13 also explains how a singularity condition $c \subseteq \Gamma(\eta(\xi))$ determines a subset $c_D \subseteq [S^{d-1}, X]$. We use a slight abuse of notation by writing $\mathrm{CMap}^{c,\partial}(D^d;X)$ instead of $\mathrm{CMap}^{c_D,\partial}(D^d;X)$. We also abbreviate ∂D^d to ∂ and write $\frac{1}{2}\partial$ for one hemisphere of ∂D^d.

Proposition 4.10 *We have the following linear Swiss cheese structures on configuration and configuration-section spaces:*

(i). $(\dot{C}(M), C(\mathring{D}^d))$ *is an algebra over* LSC_d*,*
(ii). $(\dot{\mathrm{C}}\Gamma^{c,D}(M;\xi), \mathrm{CMap}^{c,\partial}(D^d;X))$ *is an algebra over* LSC_d*,*
(iii). $(\dot{\mathrm{C}}\Gamma^{c,D}(M;\xi), \mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X))$ *is an algebra over* LSC_{d-1}*.*

Moreover, the maps of pairs

$$(\dot{\mathrm{C}}\Gamma^{c,D}(M;\xi), \mathrm{CMap}^{c,\partial}(D^d;X)) \hookrightarrow (\dot{\mathrm{C}}\Gamma^{c,D}(M;\xi), \mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X)) \to (\dot{C}(M), C(\mathring{D}^d)) \tag{4.6}$$

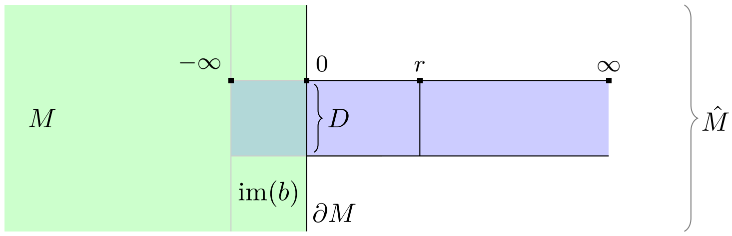
are maps of LSC_{d-1}*-algebras, and their composition is a map of* LSC_d*-algebras. Here, the first map is the inclusion and the second map sends a configuration-section to its underlying configuration, forgetting the section.*

Point (i) is essentially [Kra19, Lemma 5.1], and part of points (ii) and (iii) – the D_d-algebra structure, but not the LSC_d- and LSC_{d-1}-algebra structures – is [EVW, Propositions 2.6.1 and 2.6.2]. Before proving Proposition 4.10, we first define the appropriate models for configuration and configuration-section spaces referred to above.

Definition 4.11 Let M be a manifold equipped with an embedded codimension-zero disc $D \subseteq \partial M$ in its boundary and a collar neighbourhood of ∂M, namely an embedding $b\colon (-\infty, 0] \times \partial M \to M$ such that $b(0, x) = x$ for all $x \in \partial M$. Define:

$$\hat{M} = M \cup_b (\mathbb{R} \times D) \qquad \text{and} \qquad \hat{M}_r = M \cup_b ((-\infty, r] \times D)$$

for $r \in [0, \infty)$. Diagrammatically, $\hat{M}$ may be seen as follows, where M is green and $\mathbb{R} \times D$ is blue (and hence $(-\infty, 0] \times D$, which is identified with $b((-\infty, 0] \times D) \subseteq M$, is turquoise).



Definition 4.12 Let $\dot{C}_k(M)$ be the subspace of $C_k(\hat{M}) \times (0,\infty)$ of pairs (z,t) with $z \subseteq \mathrm{int}(\hat{M}_t)$, and define
$$\dot{C}(M) = \bigsqcup_{k\in\mathbb{N}} \dot{C}_k(M).$$

Definition 4.13 Let M, D and b be as in Definition 4.11. Also choose a bundle $\xi\colon E \to M$, a subset $c \subseteq \Gamma(\eta(\xi))$ (*cf.* Definition 3.14) and a section s_D of $\xi|_D$. Denote by $*$ the centre of the disc $D \subseteq \partial M$, let $X = \xi^{-1}(*)$ and choose a point $x_0 \in X$. Choose a trivialisation $\varphi\colon \xi|_D \cong D \times X$ such that s_D corresponds to the constant section of $D \times X$ at x_0. Using φ, glue ξ along $D = D \times \{0\}$ to the trivial X-bundle over $[0,\infty) \times D$ to obtain a bundle on $\hat{M}$, which we denote by $\hat{\xi}$. Since the construction $\eta(-)$ of Definition 3.11 commutes with restriction, we have restriction maps of sections
$$\Gamma(\eta(\hat{\xi})) \xrightarrow{-|_M} \Gamma(\eta(\xi)) \xrightarrow{-|_D} \Gamma(\eta(\xi|_D)) \cong [S^{d-1}, X],$$
where the last bijection is induced by the trivialisation φ of $\xi|_D$ and the fact that η of a trivial bundle over an orientable d-manifold with fibre X is the trivial covering space with fibre $[S^{d-1}, X]$ (see Example 3.22). Note that the first restriction map $-|_M$ is a bijection, since any section of $\eta(\xi)$ may be extended uniquely to a section of $\eta(\hat{\xi})$, due to the fact that $\eta(\hat{\xi})$ is trivial over $D \times [0,\infty)$. Thus the subset $c \subseteq \Gamma(\eta(\xi))$ determines subsets
$$\hat{c} \subseteq \Gamma(\eta(\hat{\xi})) \qquad \text{and} \qquad c_D \subseteq [S^{d-1}, X].$$
Define $\dot{\mathrm{C}\Gamma}{}^{c,D}_k(M;\xi)$ to be the subspace of $\mathrm{C}\Gamma^{\hat{c}}_k(\hat{M};\hat{\xi}) \times (0,\infty)$ of elements (z,s,t) consisting of a real number $t > 0$ and a configuration-section (z,s) such that

- $z \subseteq \mathrm{int}(\hat{M}_t)$,
- s is the constant section at x_0 on the subspace $[t,\infty) \times D \subseteq \hat{M}$,

and let
$$\dot{\mathrm{C}\Gamma}{}^{c,D}(M;\xi) = \bigsqcup_{k\in\mathbb{N}} \dot{\mathrm{C}\Gamma}{}^{c,D}_k(M;\xi).$$
We will also use the following slight abuse of notation:
$$\mathrm{CMap}^{c,\partial}(D^d;X) = \bigsqcup_{k\in\mathbb{N}} \mathrm{CMap}^{c_D,\partial D^d}_k(D^d;X) \quad , \quad \mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X) = \bigsqcup_{k\in\mathbb{N}} \mathrm{CMap}^{c_D,\partial_0 D^d}_k(D^d;X),$$
where $\partial_0 D^d = \partial D^d \cap \{x_d \leqslant 0\}$ is the southern hemisphere of ∂D^d.

Lemma 4.14 *For each k, there are natural embeddings*
$$C_k(M) \hookrightarrow \dot{C}_k(M) \qquad \text{and} \qquad \mathrm{C}\Gamma^{c,D}_k(M;\xi) \hookrightarrow \dot{\mathrm{C}\Gamma}{}^{c,D}_k(M;\xi)$$
admitting deformation retractions. In particular, $C(M) \simeq \dot{C}(M)$ and $\mathrm{C}\Gamma^{c,D}(M;\xi) \simeq \dot{\mathrm{C}\Gamma}{}^{c,D}(M;\xi)$.

Proof. We will prove this just for the second embedding (for configuration-section spaces), since the proof for the first embedding (for configuration spaces) is identical, forgetting the sections and considering just configurations of points. The embedding is defined by sending a configuration-section (z,s) of ξ to the element $(z,\hat{s},1)$ of $\dot{\mathrm{C}\Gamma}{}^{c,D}_k(M;\xi)$, where $\hat{s}$ is the section of $\hat{\xi}|_{\hat{M}\smallsetminus z}$ given by extending s by the constant section at x_0 on $D \times [0,\infty)$.

We now construct a deformation retraction of $\dot{\mathrm{C}\Gamma}{}^{c,D}_k(M;\xi)$ onto the image of this embedding. First we choose a family of diffeomorphisms $\varphi\colon [0,1] \times [0,\infty) \to \mathrm{Diff}(\hat{M})$ with the properties that

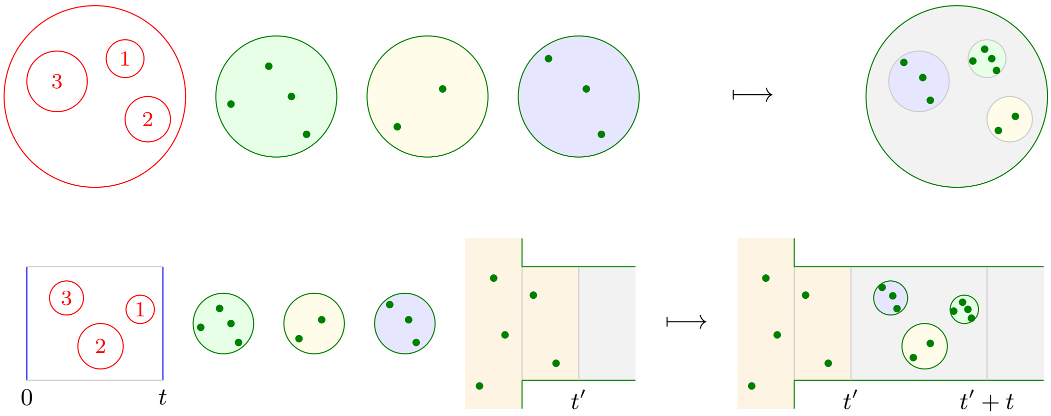


Figure 4.4 The maps (4.7) defining the LSC_d action on $(\dot{\Gamma}^{c,D}(M;\xi), \mathrm{CMap}^{c,\partial}(D^d;X))$, in dimension $d=2$ and for $k=3$. The light green, yellow, blue and orange colours represent sections defined on the complement of each configuration. Light grey indicates regions where the section is constant at the basepoint of X (note that ξ is trivial with fibre X over the grey regions, so this makes sense).

(i) $\varphi(0,t) = \mathrm{id}$ and (ii) $\varphi(1,t)(\hat{M}_t) \subseteq M$. Namely, we define $\varphi(u,t)$ to be the identity outside of $N = (\partial M \times (-\infty,0]) \cup (D \times [0,\infty))$, and for a point (x,v) of N, define $\varphi(u,t)(x,v) = (x, v-ut)$. This may be lifted to a family of automorphisms $\widetilde{\varphi}\colon [0,1]\times[0,\infty) \to \mathrm{Aut}(\hat{\xi})$ such that $\widetilde{\varphi}(u,t)$ covers $\varphi(u,t)$, using the observation that the obvious projection $p\colon N \to \partial M$ is a homotopy equivalence, so we may identify $\hat{\xi}|_N$ with $p^*(\hat{\xi}|_{\partial M})$. The deformation retraction is defined by sending (z,s,t), at time $u\in[0,1]$, to:

$$\big(\varphi(u,t)(z)\,,\ \widetilde{\varphi}(u,t)\circ s\circ\varphi(u,t)^{-1}\,,\ (1-u)t+u\big). \qquad \square$$

Proof of Proposition 4.10. The LSC_d action on $(\dot{\Gamma}^{c,D}(M;\xi), \mathrm{CMap}^{c,\partial}(D^d;X))$ is determined by maps

$$\begin{aligned} \mathsf{LSC}_d(\mathsf{a}^k;\mathsf{a}) \times (\mathrm{CMap}^{c,\partial}(D^d;X))^k &\longrightarrow \mathrm{CMap}^{c,\partial}(D^d;X) \\ \mathsf{LSC}_d(\mathsf{a}^k,\mathsf{m};\mathsf{m}) \times (\mathrm{CMap}^{c,\partial}(D^d;X))^k \times \dot{\Gamma}^{c,D}(M;\xi) &\longrightarrow \dot{\Gamma}^{c,D}(M;\xi), \end{aligned} \tag{4.7}$$

which are defined by picture in Figure 4.4. The LSC_{d-1} action on $(\dot{\Gamma}^{c,D}(M;\xi), \mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X))$ is determined by maps

$$\begin{aligned} \mathsf{LSC}_{d-1}(\mathsf{a}^k;\mathsf{a}) \times (\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X))^k &\longrightarrow \mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X) \\ \mathsf{LSC}_{d-1}(\mathsf{a}^k,\mathsf{m};\mathsf{m}) \times (\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X))^k \times \dot{\Gamma}^{c,D}(M;\xi) &\longrightarrow \dot{\Gamma}^{c,D}(M;\xi), \end{aligned} \tag{4.8}$$

which are defined by picture in Figure 4.5. The fact that these are well-defined actions of linear Swiss cheese operads may be verified easily from the construction. This gives the action of LSC_d for case (ii) and of LSC_{d-1} for case (iii). The action of LSC_d for case (i) is identical to that for case (ii), forgetting all sections and remembering just the configurations (compare the bottom line of Figure 4.4 with [Kra19, Figure 4]). The statements that the maps (4.6) respect the LSC_{d-1} structures (and that their composition respects the LSC_d structures) is also clear from the construction. $\square$

5. Monodromy actions

Definition 5.1 Let $f\colon E \to B$ be a Serre fibration and $F = f^{-1}(b)$ for a point $b\in B$. Assume either that F is a CW-complex or that f is a Hurewicz fibration. Then the *monodromy action* of f is the action-up-to-homotopy

$$\mathrm{mon}_f\colon \pi_1(B,b) \longrightarrow \pi_0(\mathrm{hAut}(F)) \tag{5.1}$$

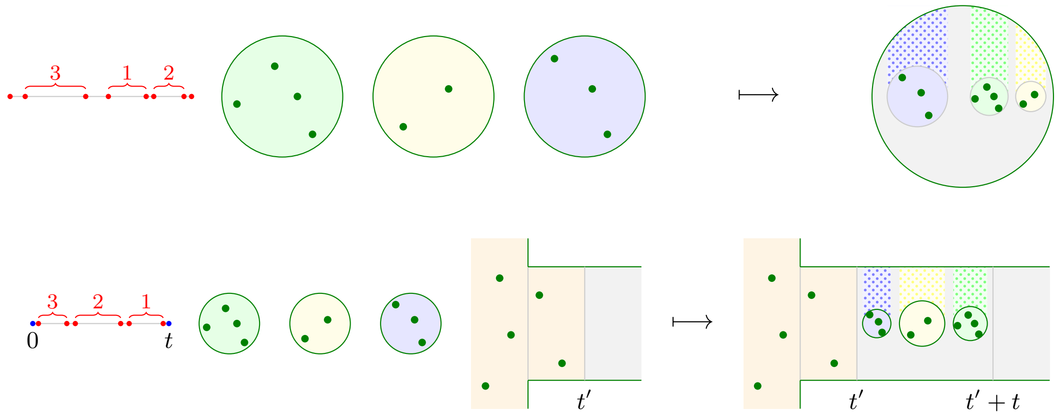


Figure 4.5 The maps (4.8) defining the LSC_{d-1} action on $(\dot{\mathrm{C}}\Gamma^{c,D}(M;\xi), \mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X))$, also in dimension $d = 2$ and for $k = 3$. The light green, yellow, blue and orange colours represent sections defined on the complement of each configuration. Light grey indicates regions where the section is constant at the basepoint of X (note that ξ is trivial with fibre X over the grey regions, so this makes sense). Dotted regions indicate that the map to X is extended into this region by defining it to be independent of the vertical direction in this region. (*Cf.* Figure 7.1 for a 3-dimensional picture.)

of $\pi_1(B, b)$ on the fibre F defined as follows. Given an element $[\gamma] \in \pi_1(B, b)$ and a representative loop $\gamma\colon [0,1] \to B$, let $g\colon F \times [0,1] \to E$ be a choice of lift in the diagram:

$$\begin{tikzcd} F \arrow[rr, "\mathrm{incl}"] \arrow[d, "{(-,0)}"'] & & E \arrow[d, "f"] \\ F \times [0,1] \arrow[r, two heads] \arrow[urr, dashed] & {[0,1]} \arrow[r, "\gamma"'] & B \end{tikzcd} \tag{5.2}$$

and define

$$\mathrm{mon}_f([\gamma]) = [g(-,1)].$$

Remark 5.2 In fact, all one needs for Definition 5.1 is a continuous map $f\colon E \to B$ and a point $b \in B$ such that f satisfies the homotopy lifting property with respect to F and $F \times [0,1]$ (*cf.* the proof of Lemma 5.3 below).

Lemma 5.3 *The construction of Definition 5.1 using the lifting diagram* (5.2) *gives a well-defined group homomorphism* (5.1).

Proof. Suppose that γ' is another representative of $[\gamma]$ and that g' is a lift of γ' in the diagram (5.2) (with γ replaced by γ'). Let $k\colon F \times [0,1]^2 \to E$ be a choice of lift in the diagram:

$$\begin{tikzcd} F \times (\square) \arrow[rr, "g_1 \cup \mathrm{const}_{\mathrm{incl}} \cup g_2"] \arrow[d, "\mathrm{incl}"'] & & E \arrow[d, "f"] \\ F \times [0,1]^2 \arrow[r, two heads] \arrow[urr, dashed] & {[0,1]^2} \arrow[r, "h"'] & B \end{tikzcd} \tag{5.3}$$

where h is a homotopy $\gamma \simeq \gamma'$ relative to endpoints. Then $k|_{F \times \{1\} \times [0,1]}$ is a homotopy $g(-,1) \simeq g'(-,1)$ of self-maps of F. This implies that the construction of Definition 5.1 gives a well-defined function $\mathrm{mon}_f\colon \pi_1(B,b) \to \pi_0(\mathrm{Map}(F,F))$. It remains to prove that mon_f is a homomorphism of monoids, since it will then follow that it has image contained in the underlying group $\pi_0(\mathrm{hAut}(F))$ of $\pi_0(\mathrm{Map}(F,F))$. It is clear that mon_f takes the constant loop to the identity map of F, since in this case we may take the lift in (5.2) to be the projection $F \times [0,1] \twoheadrightarrow F$ followed by the inclusion $F \hookrightarrow E$. We therefore just have to prove that

$$\mathrm{mon}_f([\gamma_2.\gamma_1]) = \mathrm{mon}_f([\gamma_2]) \circ \mathrm{mon}_f([\gamma_1])$$

for elements $[\gamma_1], [\gamma_2] \in \pi_1(B, b)$. Choose lifts g_1 and g_2 in the diagrams:

$$\begin{tikzcd} F \arrow[r, "\mathrm{incl}"] \arrow[d, "{(-,0)}"'] & E \arrow[d, "f"] \\ F \times [0,1] \arrow[r, two heads] \arrow[ur, dashed, "g_1"] & {[0,1]} \xrightarrow[\gamma_1]{} B \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\mathrm{incl}"] \arrow[d, "{(-,1)}"'] & E \arrow[d, "f"] \\ F \times [1,2] \arrow[r, two heads] \arrow[ur, dashed, "g_2"] & {[1,2]} \xrightarrow[\gamma_2]{} B \end{tikzcd} \tag{5.4}$$

so we have $\mathrm{mon}_f([\gamma_2]) \circ \mathrm{mon}_f([\gamma_1]) = [g_2(-,2) \circ g_1(-,1)]$. We may now define a lift $F \times [0,2] \to E$ of the diagram:

$$\begin{tikzcd} F \arrow[r, "\mathrm{incl}"] \arrow[d, "{(-,0)}"'] & E \arrow[d, "f"] \\ F \times [0,2] \arrow[r, two heads] \arrow[ur, dashed] & {[0,2]} \xrightarrow[\gamma_2 . \gamma_1]{} B \end{tikzcd} \tag{5.5}$$

by:

$$(x,t) \longmapsto \begin{cases} g_1(x,t) & t \in [0,1] \\ g_2(g_1(x,1),t) & t \in [1,2]. \end{cases}$$

By definition, it follows that $\mathrm{mon}_f([\gamma_2 . \gamma_1]) = [g_2(g_1(-,1),2)] = [g_2(-,2) \circ g_1(-,1)]$. □

Notation 5.4 From now on, we fix, once and for all, choices of the objects of Definitions 4.11 and 4.13, namely:

- a manifold M equipped with an embedded codimension-zero disc $D \subseteq \partial M$ with centre $*$,
- a collar neighbourhood of ∂M, namely an embedding $b\colon (-\infty,0] \times \partial M \hookrightarrow M$ so that $b(0,-)$ is the inclusion $\partial M \subset M$,
- ▷ This determines the manifold $\hat{M}$ and its submanifolds $\hat{M}_r$ ($r \in [0,\infty)$) as in Definition 4.11.
- a fibre bundle $\xi\colon E \to M$, with basepoint $x_0 \in X \coloneqq \xi^{-1}(*)$,
- a subset $c \subseteq \Gamma(\eta(\xi))$ (*cf.* Definition 3.14),
- a trivialisation $\theta\colon \xi|_D \cong D \times X$.
- ▷ We write s_D for the section of $\xi|_D$ corresponding to the constant section of $D \times X$ at x_0.
- ▷ Using the trivialisation θ, we extend ξ by a trivial X-bundle to obtain a bundle $\hat{\xi}$ over $\hat{M}$.

Recall the homotopy-equivalent models $\dot{C}_k(M) \simeq C_k(\mathring{M})$ and $\dot{C}\Gamma_k^{c,D}(M;\xi) \simeq C\Gamma_k^{c,D}(M;\xi)$ for configuration spaces and configuration-section spaces from §4 (Definitions 4.12 and 4.13).

Lemma 5.5 *There is a Hurewicz fibration*

$$\dot{C}\Gamma_k^{c,D}(M;\xi) \longrightarrow \dot{C}_k(M) \tag{5.6}$$

given by forgetting the section data of a configuration-section.

Proof. The forgetful map (5.6) is defined by $(z,s,t) \mapsto (z,t)$, where $t > 0$ is a real number, z is a configuration in the interior of $\hat{M}_t$ and s is a section of $\hat{\xi}$ over $\hat{M}_t \smallsetminus z$. There are homeomorphisms

$$\dot{C}_k(M) \cong C_k(\mathring{M}) \times (0,\infty) \qquad \text{and} \qquad \dot{C}\Gamma_k^{c,D}(M;\xi) \cong C\Gamma_k^{c,D}(M;\xi) \times (0,\infty) \tag{5.7}$$

under which (5.6) corresponds to the map $p \times \mathrm{id}_{(0,\infty)}$, where p is the Hurewicz fibration (3.11). Hence (5.6) is also a Hurewicz fibration. The homeomorphisms above may be defined as follows. Let D' be an open codimension-zero disc in ∂M containing D in its interior. Choose an identification of $b((-\infty,0] \times D') \subseteq M$ with $(-\infty,0] \times D$ so that $b(\{0\} \times D)$ corresponds to $\{0\} \times D$ (Figure 5.1). This induces an embedding $D \times \mathbb{R} \hookrightarrow \hat{M}$, and we obtain a homeomorphism $\psi_r\colon \hat{M} \to \hat{M}$ for each $r \in \mathbb{R}$ by defining $\psi_r(x,t) = (x,t+r)$ for $(x,t) \in D \times \mathbb{R}$ and $\psi_r(y) = y$ for $y \in \hat{M} \smallsetminus (D \times \mathbb{R})$. The left-hand homeomorphism of (5.7) may then be defined by

$$(z,t) \longmapsto (\psi_{-t}(z), t).$$

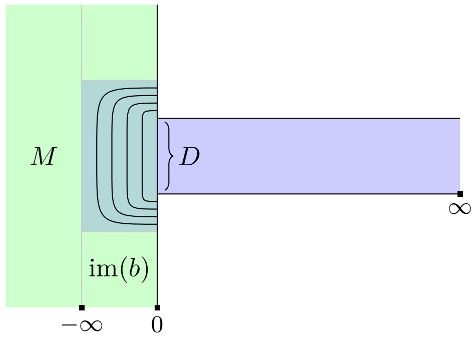


Figure 5.1 The turquoise region is $b((-\infty,0]\times D')$ and is identified with $(-\infty,0]\times D$ so that $b(\{0\}\times D)$ corresponds to $\{0\}\times D$. For illustration, four other slices $\{t\}\times D$ under this identification are drawn.

Choosing a trivialisation of $\hat{\xi}|_{D\times\mathbb{R}}$ (extending the identity trivialisation of $\hat{\xi}|_{D\times[0,\infty)}$, which is trivial by construction), we may lift $\psi_r\colon \hat{M}\to\hat{M}$ to a bundle-homeomorphism $\widetilde{\psi}_r\colon\hat{\xi}\to\hat{\xi}$. The right-hand homeomorphism of (5.7) may then be defined by

$$(z,s,t)\longmapsto(\psi_{-t}(z),\widetilde{\psi}_t\circ s\circ\psi_{-t},t).$$

□

Remark 5.6 The homeomorphisms (5.7) constructed in the proof of Lemma 5.5 give an alternative proof of Lemma 4.14, although the construction of (5.7) is a little more ad hoc.

Definition 5.7 For a real number $r\geqslant 1$, let $p_r=(*,r-\frac{1}{2})\in D\times[0,\infty)\subseteq\hat{M}$, where $*$ denotes the centre of the disc D. For an integer $k\geqslant 1$, the k-th "standard configuration" in $\hat{M}$ is defined to be $z_k=\{p_1,p_2,\ldots,p_k\}$, and the basepoint of $\dot{C}_k(M)$ is defined to be (z_k,k).

Remark 5.8 Stabilisation maps for configuration spaces and configuration-section spaces will be defined in §7, using the E_0-module structure of §4. However, at the level of configuration spaces, it is already clear that the stabilisation map $\dot{C}_k(M)\to\dot{C}_{k+1}(M)$ should be defined by

$$(z,t)\longmapsto(z\sqcup\{p_{t+1}\},t+1).$$

Lemma 5.9 *The fibre of* (5.6) *over* $(z_k,k)\in\dot{C}_k(M)$ *is the space*

$$\Gamma_k^{c,D}(M;\xi)\coloneqq\Gamma^{c_D,D\times\{k\}}\bigl(\hat{M}_k\smallsetminus z_k;\hat{\xi}\bigr)\tag{5.8}$$

of sections s of $\hat{\xi}$ defined over $\hat{M}_k\smallsetminus z_k\subseteq\hat{M}$ such that

- *the restriction of s to a small punctured neighbourhood of p_i lies in $c_D\subseteq[S^{d-1},X]$ for each $i\in\{1,\ldots,k\}$, where c_D is as in Definition 4.13,*
- *the restriction of s to $D\times\{k\}\subseteq\partial\hat{M}_k$ is constant at the basepoint x_0 of X.*

Corollary 5.10 *We have a Hurewicz fibration sequence of the form*

$$\Gamma_k^{c,D}(M;\xi)\longrightarrow\dot{C}\Gamma_k^{c,D}(M;\xi)\longrightarrow\dot{C}_k(M).\tag{5.9}$$

Proof of Corollary 5.10. This follows immediately from Lemmas 5.5 and 5.9. □

Proof of Lemma 5.9. Directly from the definitions, the fibre of (5.6) over (z_k,k) may be described as written, except that the first condition says that s must satisfy the singularity condition $\hat{c}\subseteq\Gamma(\eta(\hat{\xi}))$, where $\hat{c}$ is determined by $c\subseteq\Gamma(\eta(\xi))$ as explained in Definition 4.13. But all of the points of z_k lie in $D\times[0,\infty)\subseteq\hat{M}$, over which the bundle $\hat{\xi}$ is trivial with fibre X, so the singularity conditions around these points are equivalent to the conditions written in the lemma. □

Definition 5.11 Let

$$\mathrm{Br}(M)=\bigl[\pi_1(\dot{C}_k(M))\bigr]_{k\in\mathbb{N}}$$

denote the groupoid whose objects are $\mathbb{N}$, whose automorphism group of $k\in\mathbb{N}$ is $\pi_1(\dot{C}_k(M))$ and which has no morphisms between distinct objects.

Definition 5.12 Fix $(M, D, *, b, \xi, x_0, c, \theta)$ as in Notation 5.4. The *associated monodromy functor*

$$\mathrm{Mon}^{c,D}(M,\xi)\colon \mathrm{Br}(M) \longrightarrow \mathrm{Ho}(\mathrm{Top}) \tag{5.10}$$

takes the object $k \in \mathbb{N}$ to the fibre $\Gamma_k^{c,D}(M;\xi)$ of (5.6). On automorphisms of k, it is defined by the monodromy action (5.1), with $f =$ (5.6).

6. Braid categories

Definition 6.1 ([Gra76, p. 219]) The *Quillen bracket construction* $\langle \mathcal{D}, \mathcal{C} \rangle$ of a category $\mathcal{C}$ equipped with a left-action of a monoidal category $\mathcal{D}$ is the category with the same objects as $\mathcal{C}$, and with morphisms given by $\langle \mathcal{D}, \mathcal{C} \rangle(c, c') = \mathrm{colim}_{\mathcal{D}}(\mathcal{C}(- \oplus c, c'))$, where $\oplus$ is the action of $\mathcal{D}$ on $\mathcal{C}$. In other words, a morphism $c \to c'$ in $\langle \mathcal{D}, \mathcal{C} \rangle$ is an equivalence class of morphisms $\varphi\colon d \oplus c \to c'$ in $\mathcal{C}$ with d in $\mathcal{D}$, and where $(d_1, \varphi_1) \sim (d_2, \varphi_2)$ if there exists $\theta\colon d_1 \to d_2$ in $\mathcal{D}$ such that $\varphi_1 = \varphi_2 \circ (\theta \oplus \mathrm{id}_c)$. This comes equipped with a canonical functor

$$\mathcal{C} \longrightarrow \langle \mathcal{D}, \mathcal{C} \rangle$$

given by the identity on objects and by $\varphi \mapsto 0 \oplus \varphi$ on morphisms, where 0 is the unit object of the monoidal structure on $\mathcal{D}$.

Lemma 6.2 *The groupoid* $\mathrm{Br}(D^d)$ *has a monoidal structure given by taking the boundary connected sum of two discs. If $d \geqslant 2$ it is braided and if $d \geqslant 3$ it is symmetric. Now let M be a connected d-manifold with non-empty boundary, and let $D \subseteq \partial M$ be an embedded $(d-1)$-dimensional disc. There is a well-defined action of* $\mathrm{Br}(D^d)$ *on* $\mathrm{Br}(M)$ *given by boundary connected sum along D.*

Proof. Let us write $\mathrm{Br}(M) = \pi_1(\dot{C}(M), \{z_k\}_{k\in\mathbb{N}})$, the fundamental groupoid of $\dot{C}(M)$ with respect to the set of basepoints $\{z_k \in \dot{C}_k(M) \mid k \in \mathbb{N}\}$. This is a skeleton for, hence equivalent to, the full fundamental groupoid $\pi_1(\dot{C}(M))$, with objects *all* points of $\dot{C}(M)$. When $M = D^d$ we have $\pi_1(\dot{C}(D^d)) \simeq \pi_1(C(\mathring{D}^d))$, since $\dot{C}(D^d)$ and $C(\mathring{D}^d)$ are homotopy equivalent. By Proposition 4.10(i), $C(\mathring{D}^d)$ is an E_d-algebra and $\dot{C}(M)$ is an E_0-module over it. Passing to fundamental groupoids and pulling back along the equivalences, this gives rise to the structure on $\mathrm{Br}(D^d)$ and $\mathrm{Br}(M)$ claimed in the lemma. ∎

Remark and Notation 6.3 Note that $\mathrm{Br}(D^1)$ is the free monoidal category on one object, $\mathrm{Br}(D^2)$ is the free *braided* monoidal category on one object and $\mathrm{Br}(D^3)$ is the free *symmetric* monoidal category on one object. We will therefore abbreviate these groupoids by $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{S}$ respectively. The standard inclusions $D^1 \hookrightarrow D^2 \hookrightarrow D^3$ induce monoidal functors $\mathcal{M} \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{S}$ (the second one is also braided monoidal), and for $d \geqslant 3$ the standard inclusion $D^3 \hookrightarrow D^d$ induces an isomorphism $\mathcal{S} \cong \mathrm{Br}(D^d)$.

Definition 6.4 ([Kra19, §5.2]) Let M be a connected d-manifold with non-empty boundary, with $d \geqslant 2$, and let $D \subseteq \partial M$ be an embedded $(d-1)$-dimensional disc. By Lemma 6.2, there is a well-defined action of the braided monoidal category $\mathrm{Br}(D^d)$ on $\mathrm{Br}(M)$. The standard inclusion $D^2 \hookrightarrow D^d$ induces a braided monoidal functor $\mathcal{B} \to \mathrm{Br}(D^d)$, and hence an action of $\mathcal{B}$ on $\mathrm{Br}(M)$. Using Definition 6.1, we may therefore define the category $\mathcal{C}(M) = \langle \mathcal{B}, \mathrm{Br}(M) \rangle$, which is equipped with a canonical functor

$$\mathrm{Br}(M) \longrightarrow \mathcal{C}(M). \tag{6.1}$$

Lemma 6.5 *The functor* (6.1) *is the inclusion of the underlying groupoid of* $\mathcal{C}(M)$.

Proof. This is an immediate generalisation of the proof of [RW17, Proposition 1.7]. ∎

Remark 6.6 Our notation is the reverse of that of [Kra19], since we are using the opposite convention of considering *left*-actions of monoidal categories.

Definition 6.7 ([Pal18, §2.3 and §3.1]) The categories $\mathcal{B}(M)$ and $\mathcal{B}_\sharp(M)$ both have $\mathbb{N}$ as their set of objects. A morphism $k \to \ell$ in $\mathcal{B}_\sharp(M)$ is a path γ, up to endpoint-preserving homotopy, in the space $C_r(\mathring{M})$ for some r, satisfying $\gamma(0) \subseteq \{p_1, \ldots, p_k\}$ and $\gamma(1) \subseteq \{p_1, \ldots, p_\ell\}$. Composition is defined by analogy with composition of partially-defined functions: given morphisms $\gamma\colon k \to \ell$ and $\delta\colon \ell \to m$, let $r = |\gamma(1) \cap \delta(0)|$ and define $\delta \circ \gamma$ to be the path in $C_r(\mathring{M})$ obtained by concatenating the corresponding restrictions of γ and δ. A morphism $k \to \ell$ in the subcategory $\mathcal{B}(M) \subseteq \mathcal{B}_\sharp(M)$ is a path γ as above, with $r = k$.

Remark 6.8 The categories $\mathcal{B}(M)$ and $\mathcal{B}_\sharp(M)$ were denoted by $\mathcal{B}_\mathfrak{f}(M)$ and $\mathcal{B}(M)$ respectively in [Pal18]. We have modified that notation to fit more naturally with the standard notation FI and $\mathrm{FI}_\sharp$ for the categories of *finite sets and injections*, and of *finite sets and partially-defined injections*, respectively, as well as with the notation of [Kra19]. We note (*cf.* [Kra19, Remark 5.10]) that, if $\dim(M) \geqslant 3$ and M is simply-connected, then $\mathcal{B}(M) \cong \mathrm{FI}$ and $\mathcal{B}_\sharp(M) \cong \mathrm{FI}_\sharp$.

Lemma 6.9 *Under the conditions of Definition 6.4, there is a canonical functor*

$$\mathcal{C}(M) \longrightarrow \mathcal{B}(M),$$

which is the identity on objects and is also:

- *full, for any M,*
- *faithful (and hence an isomorphism) if and only if $\dim(M) \geqslant 3$ and M is simply-connected.*

Proof. The set of morphisms $m \to n$ of $\mathcal{C}(M)$ is empty if $m > n$, and if $m \leqslant n$ it is naturally identified with the orbit set $B_n(M)/B_m$, where $B_n(M) = \pi_1(C_n(M), \{p_1, \ldots, p_n\})$ and B_m acts on $B_n(M)$ via the homomorphism $v^m_{n-m}\colon B_m \to B_n(M)$ of Definition 7.1 below followed by right-multiplication of $B_n(M)$ on itself. On the other hand, the set of morphisms $m \to n$ of $\mathcal{B}(M)$ is also empty if $m > n$, and if $m \leqslant n$ it is a homotopy class of paths in $C_m(M)$ from the basepoint configuration $\{p_1, \ldots, p_m\}$ to a subconfiguration of $\{p_1, \ldots, p_n\}$.

The functor $\mathcal{C}(M) \to \mathcal{B}(M)$ is defined on morphisms as follows. Given a morphism $m \to n$ of $\mathcal{C}(M)$, represented by a loop of configurations γ in $C_n(M)$ based at $\{p_1, \ldots, p_n\}$, forget all strands of γ that start at p_i for $m+1 \leqslant i \leqslant n$. The result is a path of configurations in $C_m(M)$ representing a morphism $m \to n$ of $\mathcal{B}(M)$.

Given any path of configurations δ in $C_m(M)$ from $\{p_1, \ldots, p_m\}$ to a subconfiguration of $\{p_1, \ldots, p_n\}$, it is always possible to "adjoin strands" to δ to extend it to a loop of configurations γ in $C_n(M)$. This implies that the functor $\mathcal{C}(M) \to \mathcal{B}(M)$ is full.

The fact that the functor $\mathcal{C}(M) \to \mathcal{B}(M)$ is faithful if $\dim(M) \geqslant 3$ and M is simply-connected is stated in Remark 5.10 of [Kra19]. It also follows from Lemma 4.1 of [Til16], which implies that the functor is given by $\Sigma_n/\Sigma_m \to \mathrm{Inj}(m,n)$ on morphism-sets (from m to n); we already know that this is surjective, so injectivity follows from a simple counting argument.

If M is 2-dimensional or $\pi_1(M)$ is non-trivial, it is easy to construct pairs of distinct morphisms $1 \to 2$ in $\mathcal{C}(M)$ that become equal in $\mathcal{B}(M)$, i.e., after forgetting the second strand of a 2-strand braid on M. Thus the conditions that $\dim(M) \geqslant 3$ and M is simply-connected are necessary for $\mathcal{C}(M) \to \mathcal{B}(M)$ to be a faithful functor. ∎

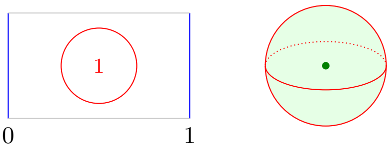
Summary 6.10 Let M be a connected d-manifold with non-empty boundary, with $d \geqslant 2$, and let $D \subseteq \partial M$ be an embedded $(d-1)$-dimensional disc. There are then canonical functors

$$\mathrm{Br}(M) \hookrightarrow \mathcal{C}(M) \twoheadrightarrow \mathcal{B}(M) \hookrightarrow \mathcal{B}_\sharp(M) \tag{6.2}$$

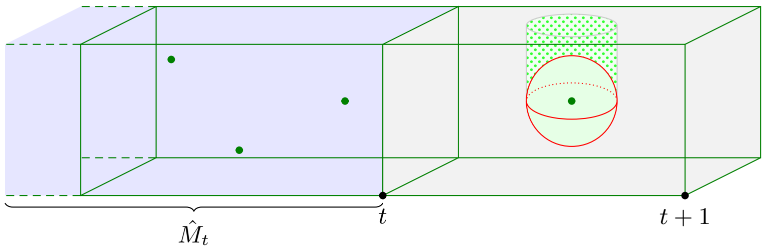
that all act by the identity on their common set of objects (which is $\mathbb{N}$). The first and third functors are faithful, and the second functor is full. The second functor is also faithful (and therefore an isomorphism) if and only if M is simply-connected and has dimension at least 3.

7. Stabilisation maps and extension to $\mathcal{C}(M)$

As in §5, we fix the data $(M, D, *, b, \xi, x_0, c, \theta)$ of Notation 5.4, namely a bundle $\xi\colon E \to M$ over a d-manifold, a disc $D \subseteq \partial M$, etc.



(a) Choices of elements of $\mathsf{LSC}_{d-1}(\mathsf{a},\mathsf{m};\mathsf{m})$ and of $\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X)$. The light green region represents a map $D^d \smallsetminus \{0\} \to X$ in one of the homotopy classes $c_D \subseteq [S^{d-1},X]$, where c_D is determined by c as in Definition 4.13, sending the southern hemisphere of ∂D^d to $\{x_0\} \subseteq X$.



(b) The stabilisation maps of (7.1). Given a (blue) configuration-section (z,s) lying in $\hat{M}_t$ with $s|_{D\times\{t\}} = \mathrm{const}_{x_0}$, add a new point at $p_{t+1} = (*, t+\frac{1}{2})$ and extend the section as illustrated: grey indicates the constant map to x_0, green indicates the configuration-section in D^d chosen above and dotted-green indicates that the green region is extended, as in Figure 4.5, by defining it to be independent of the vertical direction in this region.

Figure 7.1 Stabilisation maps for configuration-section spaces.

Stabilisation maps. Let us choose, once and for all, an element of $\mathsf{LSC}_{d-1}(\mathsf{a},\mathsf{m};\mathsf{m})$ consisting of one $(d-1)$-disc in a $(d-1)$-rectangle of width 1, as well as an element of $\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X)$ where the configuration has exactly one point. See Figure 7.1(a).

By Proposition 4.10, the pair $(\dot{\mathrm{C}}\Gamma^{c,D}(M;\xi), \mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X))$ is an algebra over the linear Swiss cheese operad LSC_{d-1} (*i.e.* $\dot{\mathrm{C}}\Gamma^{c,D}(M;\xi)$ *is an* E_0*-module over the* E_{d-1}*-algebra* $\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X)$). Moreover, the pair $(\dot{C}(M), C(\mathring{D}^d))$ is an algebra over LSC_d, and hence also over LSC_{d-1} by restriction, and the maps

$$(\dot{\mathrm{C}}\Gamma^{c,D}(M;\xi), \mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X)) \longrightarrow (\dot{C}(M), C(\mathring{D}^d))$$

that send a configuration-section to its underlying configuration (forgetting the section) are maps of LSC_{d-1}-algebras. This structure induces (horizontal) *stabilisation maps*

$$\begin{tikzcd}
\dot{\mathrm{C}}\Gamma^{c,D}_k(M;\xi) \arrow[r] \arrow[d] & \dot{\mathrm{C}}\Gamma^{c,D}_{k+1}(M;\xi) \arrow[d] \\
\dot{C}_k(M) \arrow[r] & \dot{C}_{k+1}(M)
\end{tikzcd} \tag{7.1}$$

commuting with the (vertical) forgetful maps, for all $k \in \mathbb{N}$. Concretely, the top horizontal map is given by the second line of (4.8), plugging in our choices of elements above. The bottom horizontal map is defined similarly, ignoring sections and considering just configurations. See Figure 7.1(b). Note that the bottom horizontal map of (7.1) is exactly as already described in Remark 5.8.

Definition 7.1 The map

$$\pi_1(\dot{C}_k(M)) \longrightarrow \pi_1(\dot{C}_{k+1}(M))$$

of fundamental groups induced by the stabilisation map (7.1) will be denoted by σ_k. Identifying the interior of the $(d-1)$-disc $D \subseteq \partial M$ with $(-1,1)^{d-1}$, we also have inclusions

$$(-1,1) \times \{0\}^{d-2} \times (k, k+\ell) \longhookrightarrow \mathrm{int}(\hat{M}_{k+\ell})$$

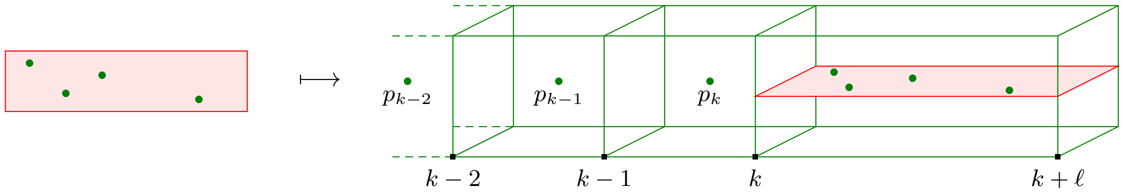


Figure 7.2 The map (7.2) inducing $v_k^\ell \colon B_\ell \to \pi_1(\dot{C}_{k+\ell}(M))$.

for integers $k \geqslant 0$ and $\ell \geqslant 1$, which induce maps

$$C_\ell((-1,1) \times \{0\}^{d-2} \times (k, k+\ell)) \longrightarrow \dot{C}_{k+\ell}(M) \tag{7.2}$$

given by $S \mapsto (S \sqcup \{p_1, \ldots, p_k\}, k+\ell)$ as illustrated in Figure 7.2. The induced map

$$B_\ell \cong \pi_1(C_\ell((-1,1) \times \{0\}^{d-2} \times (k, k+\ell))) \longrightarrow \pi_1(\dot{C}_{k+\ell}(M)) \tag{7.3}$$

of fundamental groups is denoted by v_k^ℓ.

The maps $\sigma_k \colon \pi_1(\dot{C}_k(M)) \to \pi_1(\dot{C}_{k+1}(M))$ and $v_k^\ell \colon B_\ell \to \pi_1(\dot{C}_{k+\ell}(M))$ of Definition 7.1 may be used to characterise extensions of functors along the inclusion $\mathrm{Br}(M) \subset \mathcal{C}(M)$:

Proposition 7.2 ([Kra19, §5.2]) *Let M be a connected d-manifold, for $d \geqslant 2$, and let $D \subsetneq \partial M$ be an embedded $(d-1)$-dimensional disc. Choose a collar neighbourhood b as in Notation 5.4 and take the basepoint $* \in \partial M$ to be the centre of D. Let $F \colon \mathrm{Br}(M) \to \mathcal{D}$ be a functor. An extension of F to the larger category $\mathcal{C}(M) \supseteq \mathrm{Br}(M)$ is equivalent to a choice of morphism $s_k \colon F(k) \to F(k+1)$ of $\mathcal{D}$ for each $k \in \mathbb{N}$ such that, for any $k \geqslant 0$, $\ell \geqslant 1$, $\alpha \in \pi_1(\dot{C}_k(M))$ and $\beta \in B_\ell$, the following two diagrams commute, where $s_k^\ell = s_{k+\ell-1} \circ \cdots \circ s_{k+1} \circ s_k$.*

$$\begin{tikzcd} F(k) \arrow[r, "s_k"] \arrow[d, "F(\alpha)"'] & F(k+1) \arrow[d, "F(\sigma_k(\alpha))"] \\ F(k) \arrow[r, "s_k"] & F(k+1) \end{tikzcd} \qquad \begin{tikzcd} & F(k+\ell) \arrow[d, "F(v_k^\ell(\beta))"] \\ F(k) \arrow[ur, "s_k^\ell"] \arrow[r, "s_k^\ell"'] & F(k+\ell) \end{tikzcd} \tag{7.4}$$

Proposition 7.3 *The stabilisation maps (7.1) determine an extension of the monodromy functor (5.10) $= \mathrm{Mon}^{c,D}(M,\xi) \colon \mathrm{Br}(M) \to \mathrm{Ho}(\mathrm{Top})$ to a functor*

$$\widetilde{\mathrm{Mon}}^{c,D}(M,\xi) \colon \mathcal{C}(M) \longrightarrow \mathrm{Ho}(\mathrm{Top}). \tag{7.5}$$

Proof. We will apply Proposition 7.2 with $\mathcal{D} = \mathrm{Ho}(\mathrm{Top})$ and $F = (5.10) = \mathrm{Mon}^{c,D}(M,\xi)$. Recall that $\mathrm{Mon}^{c,D}(M,\xi)$ sends k to the fibre $\Gamma_k^{c,D}(M,\xi)$ of the Hurewicz fibration (5.6) over the basepoint $(z_k, k) \in \dot{C}_k(M)$. The bottom horizontal map of the map of Hurewicz fibrations (7.1) preserves basepoints, so its top horizontal map restricts to a map of fibres

$$F(k) = \Gamma_k^{c,D}(M,\xi) \longrightarrow \Gamma_{k+1}^{c,D}(M,\xi) = F(k+1),$$

which we define to be s_k. It remains to check the two conditions (7.4) of Proposition 7.2.

The element $\alpha \in \pi_1(\dot{C}_k(M))$ acts on $F(k) = \Gamma_k^{c,D}(M,\xi)$ through a "point-pushing" diffeomorphism θ_α of $\hat{M}_k \smallsetminus z_k$. (Only the isotopy class of θ_α is important, since the diagrams (7.4) live in the homotopy category.) The element $\sigma_k(\alpha) \in \pi_1(\dot{C}_{k+1}(M))$ is the extension of α to a loop of $k+1$ points in $\hat{M}_{k+1}$ given by leaving the point p_{k+1} fixed. Hence we may choose the point-pushing diffeomorphism $\theta_{\sigma_k(\alpha)}$ of $\hat{M}_{k+1} \smallsetminus z_{k+1}$ (through which $\sigma_k(\alpha)$ acts on $F(k+1) = \Gamma_{k+1}^{c,D}(M,\xi)$) to be the extension of θ_α by the identity on $(D \times [k,k+1]) \smallsetminus \{p_{k+1}\}$. This implies that the left-hand square of (7.4) commutes up to homotopy.

Now consider any element $s \in F(k)$, a section of $\hat{\xi}$ defined on $\hat{M}_k \smallsetminus z_k$. Extend it in a standard way ℓ times, as shown in Figure 7.1(b), to obtain a section $\bar{s}$ of $\hat{\xi}$ defined on $\hat{M}_{k+\ell} \smallsetminus z_{k+\ell}$. The restriction of $\bar{s}$ to $P = (D \times [k, k+\ell]) \smallsetminus \{p_{k+1}, \ldots, p_{k+\ell}\}$ may be thought of as a map $P \to X$, since $\hat{\xi}$ is trivial over P. Note that this map $P \to X$ does not in fact depend on s: it just consists of ℓ concatenated copies of the standard map to X illustrated on the right-hand side of Figure 7.1(b). We denote it by $f_\ell \colon P \to X$. Any element $\beta \in B_\ell$ determines a self-diffeomorphism (up to isotopy) $\beta \colon P \to P$ by point-pushing.

Claim: $f_\ell \circ \beta$ is homotopic to f_ℓ through maps sending the two ends $D \times \{k\}$ and $D \times \{k+\ell\}$ to the basepoint $x_0 \in X$.

This claim implies that $\bar{s} \cdot \beta$ is homotopic to $\bar{s}$ through maps sending $D \times \{k+\ell\}$ to x_0, in other words, there is a path $\bar{s} \cdot \beta \sim \bar{s}$ in $\Gamma^{c,D}_{k+\ell}(M, \xi) = F(k+\ell)$, and moreover these paths may be chosen continuously in s. In other words, the right-hand triangle of (7.4) commutes up to homotopy.

It therefore remains to prove the claim above, and it will suffice to prove it when β is the standard generator of B_ℓ that interchanges the two punctures p_i and p_{i+1} for $k+1 \leqslant i \leqslant \ell - 1$. We will do this diagrammatically for dimensions $d \geqslant 3$ and by considering fundamental groups for dimension $d = 2$.

First assume that $d \geqslant 3$. In this case, Figure 7.3 illustrates a "bird's eye" view of the map $f_\ell \colon P \to X$, where $P = (D \times [k, k+\ell]) \smallsetminus \{p_{k+1}, \ldots, p_{k+\ell}\}$, by collapsing the "vertical" direction of Figures 7.1(b) and 7.2 (corresponding to the last copy of $[-1,1]$ in our identification of $D \subseteq \partial M$ with $[-1,1]^{d-1}$). Since $d \geqslant 3$ the resulting bird's-eye-view picture still has $d - 1 \geqslant 2$ dimensions (only two are pictured in Figure 7.3, of course). Now consider the point-pushing diffeomorphism $\beta \colon P \to P$ corresponding to the generator of the braid group that interchanges the punctures p_i and p_{i+1}. From Figure 7.3, and due to the fact that (the bird's eye view of) f_ℓ is *the same* in a small neighbourhood of each puncture $p_{k+1}, \ldots, p_{k+\ell}$, it is clear that the homotopy class of $f_\ell \circ \beta$ is the same as that of f_ℓ. Moreover, a homotopy connecting them may be chosen to be supported in a small neighbourhood of an arc connecting p_i and p_{i+1}, so it does not affect the ends $D \times \{k\}$ and $D \times \{k+\ell\}$ of P, which are therefore still mapped to the basepoint $x_0 \in X$ during the homotopy. This proves the claim when $d \geqslant 3$.

When $d = 2$ we do not collapse the vertical direction, and in this case Figure 7.4(a) illustrates the map $f_\ell \colon P \to X$ without any dimension reduction, and Figure 7.4(b) illustrates the effect of the point-pushing diffeomorphism $\beta \colon P \to P$ that interchanges the punctures p_i and p_{i+1}. We will argue that 7.4(a) $\simeq$ 7.4(b) relative to the left, bottom and right edges of the rectangle (and hence in particular relative to the left and right edges of the rectangle, which is the statement of the claim above).

A map $P \to X$ sending the left, bottom and right edges of the rectangle to the basepoint $x_0 \in X$ corresponds, up to relative homotopy, to an ordered ℓ-tuple of elements of $\pi_1(X, x_0)$. Recall that, when defining the stabilisation maps, we chose an element of $\mathrm{CMap}_1^{c,\frac{1}{2}\partial}(D^2; X)$ (see Figure 7.1(a)), which is, up to homotopy, a choice of element $\kappa \in \pi_1(X, x_0)$ lying in one of the conjugacy classes $c_D \subseteq [S^1, X] = \mathrm{Conj}(\pi_1(X, x_0))$, where c_D is determined by c as explained in Definition 4.13. The map 7.4(a)$\colon P \to X$ therefore corresponds to the ordered ℓ-tuple $(\kappa, \kappa, \ldots, \kappa)$. Now the effect of the point-pushing diffeomorphism $\beta \colon P \to P$, under this correspondence, is to interchange the i-th and $(i+1)$-st elements of an ordered tuple while conjugating one by the other; in symbols: $(\ldots, \lambda_i, \lambda_{i+1}, \ldots) \mapsto (\ldots, \lambda_i \lambda_{i+1} \lambda_i^{-1}, \lambda_i, \ldots)$. But the ordered ℓ-tuple $(\kappa, \kappa, \ldots, \kappa)$ corresponding to 7.4(a) is clearly fixed under this action, so 7.4(b) $=$ 7.4(a) $\circ\, \beta$ also corresponds to $(\kappa, \kappa, \ldots, \kappa)$, and therefore 7.4(a) $\simeq$ 7.4(b) relative to the left, bottom and right edges of the rectangle. □

Remark 7.4 The general setting of [Kra19] is for E_0-modules over E_2-algebras; the theorem stated in the next section (Theorem 8.7) in terms of the category $\mathcal{C}(M)$ is a rephrasing of this in a special case. If $d \geqslant 3$, the fact that the configuration-section spaces on M form an E_0-module over an E_{d-1}-algebra, hence in particular over an E_2-algebra, *automatically* implies that we have an extension to $\mathcal{C}(M)$. On the other hand, when $d = 2$, it is not tautological that we have an extension to $\mathcal{C}(M)$, although it is still true, by Proposition 7.3.

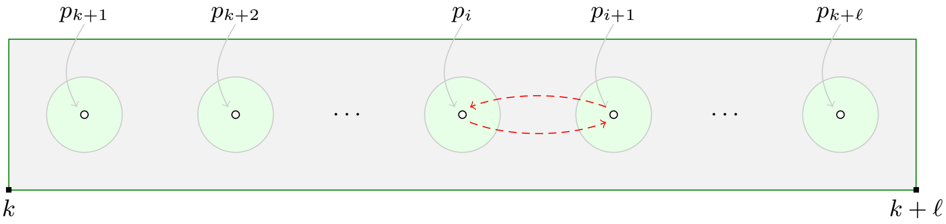


Figure 7.3 A bird's eye view of the map $f_\ell \colon P \to X$, where $P = (D \times [k, k+\ell]) \smallsetminus \{p_{k+1}, \ldots, p_{k+\ell}\}$. Grey indicates the constant map to the basepoint $x_0 \in X$ and the green regions are mapped to X as in Figure 7.1(b), depending in particular on our choices in Figure 7.1(a). The red arrows illustrate the effect of the point-pushing diffeomorphism $\beta \colon P \to P$.

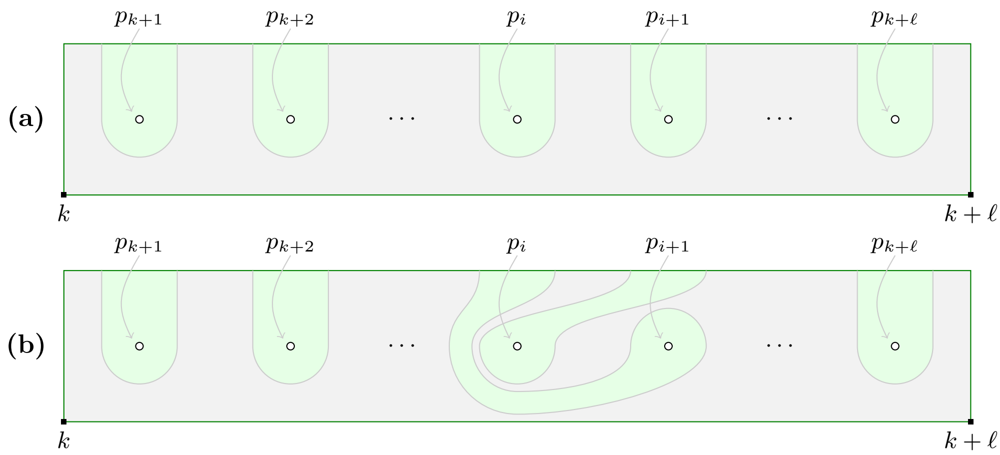


Figure 7.4 **(a)** The map $f_\ell \colon P \to X$ when $d = 2$, with colour-coding as in Figure 7.3. **(b)** The map $f_\ell \circ \beta \colon P \to X$, where β is the point-pushing diffeomorphism interchanging the punctures p_i and p_{i+1}.

8. Polynomiality and stability

In this section we complete the proof of our main homological stability result for configuration-section spaces. First, we show that the composition of (7.5) with the functor $H_i(-;K) \colon \mathrm{Ho}(\mathrm{Top}) \to \mathrm{Vect}_K \to \mathrm{Ab}$ is "of degree $\leqslant i$" for all $i \geqslant 0$ and all fields K. Via a result of [Kra19] (recalled below as Theorem 8.7), this implies twisted homological stability for configuration spaces with coefficients in $H_i((7.5); K)$, which then implies the desired result by a spectral sequence comparison argument.

Polynomial functors. The category $\mathcal{C}(M)$ has a canonical endofunctor

$$s \colon \mathcal{C}(M) \longrightarrow \mathcal{C}(M)$$

defined as follows. The maps $\sigma_k \colon \pi_1(\dot{C}_k(M)) \to \pi_1(\dot{C}_{k+1}(M))$ induced by the stabilisation maps (*cf.* Definition 7.1) induce an endofunctor of $\mathrm{Br}(M)$ given by $k \mapsto k+1$ on objects. Similarly, the standard inclusions of braid groups $B_k \to B_{k+1}$ induce an endofunctor of $\mathcal{B}$ given by $k \mapsto k+1$ on objects. The endofunctor $\mathcal{B} \to \mathcal{B}$ is braided monoidal and the two endofunctors $\mathcal{B} \to \mathcal{B}$ and $\mathrm{Br}(M) \to \mathrm{Br}(M)$ are compatible with the left-action of $\mathcal{B}$ on $\mathrm{Br}(M)$, so they induce an endofunctor of $\mathcal{C}(M) = \langle \mathcal{B}, \mathrm{Br}(M) \rangle$, which we denote by s. There is moreover a natural transformation

$$\iota \colon \mathrm{id}_{\mathcal{C}(M)} \longrightarrow s$$

given by the morphisms $\iota_k = (1, \mathrm{id}_{k+1}) \colon k \to k+1$ of $\mathcal{C}(M)$ for $k \in \mathbb{N}$. (Recall that a morphism $a \to b$ in $\mathcal{C}(M)$ is determined by an object c of $\mathcal{B}$ and a morphism $a + c \to b$ of $\mathrm{Br}(M)$.) We note that $s(\iota_k) = (1, v_k^2(\tau_1)) = v_k^2(\tau_1) \circ \iota_{k+1}$, where $\tau_1 \in B_2$ is the standard generator and $v_k^2 \colon B_2 \to \pi_1(\dot{C}_{k+2}(M))$ is the homomorphism defined in Definition 7.1.

Definition 8.1 For a category $\mathcal{C}$ equipped with an endofunctor $s\colon \mathcal{C} \to \mathcal{C}$ and natural transformation $\iota\colon \mathrm{id}_{\mathcal{C}} \to s$, and an abelian category $\mathcal{A}$, the *degree* of a functor $T\colon \mathcal{C} \to \mathcal{A}$ takes values in $\{-1\} \cup \mathbb{N} \cup \{\infty\}$ and is defined recursively as follows. The only functor T of degree -1 is $T = 0$. If functors of degree $\leqslant d$ have been defined, we say that T has degree $d+1$ if and only if the natural transformation $T\iota\colon T \to Ts$ is split injective in the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ and the functor

$$\Delta T = \mathrm{coker}(T\iota\colon T \to Ts)\colon \mathcal{C} \to \mathcal{A}$$

has degree d. Once all functors of finite degree have been defined, all remaining functors are said to have degree ∞.

Remark 8.2 For $\mathcal{C} = \mathcal{C}(M)$, equipped with the endofunctor and natural transformation described above, this corresponds to the notion of "*split degree at* 0" of [Kra19, Definition 4.6]. There are an analogous endofunctor s and natural transformation ι on the category $\mathcal{B}_\sharp(M)$ (which commute with the functor $\mathcal{C}(M) \to \mathcal{B}_\sharp(M)$ from Summary 6.10), and when $\mathcal{C} = \mathcal{B}_\sharp(M)$ equipped with these, the definition above corresponds to the *degree* of [Pal18, Definition 3.1]. In this setting, the degree of T has an alternative characterisation in terms of *cross-effects* of T (Definition 3.15 and Lemma 3.16 of [Pal18]). See also [Pal] for a more general overview of notions of *degree* of a functor via recursion (as above) or via cross-effects.

Lemma 8.3 *Let $\mathcal{C}$ be a category as in Definition 8.1, $\mathcal{A}$ an abelian category, T_1 and $T_2\colon \mathcal{C} \to \mathcal{A}$ functors and $A \in \mathrm{ob}(\mathcal{A})$. Then we have*

- $\deg(T_1 \oplus T_2) = \max\{\deg(T_1), \deg(T_2)\}$*,*
- $\deg(T_1 \otimes A) \leqslant \deg(T_1)$*, and, more generally,*
- $\deg(T_1 \otimes T_2) \leqslant \deg(T_1) + \deg(T_2)$ *whenever* $\deg(T_1)$ *and* $\deg(T_2)$ *are non-negative,*

where we assume that $(\mathcal{A}, \otimes)$ is an abelian monoidal category for the second and third points.

Proof. This is a direct generalisation of [Pal18, Lemma 3.18], whose proof also generalises directly. ∎

Definition 8.4 Let $\mathcal{C}$ be as in Definition 8.1. We say that a functor

$$T\colon \mathcal{C} \longrightarrow \mathrm{Ho}(\mathrm{Top})$$

has *slope* $\leqslant \lambda$, for $\lambda \in (0, \infty)$, if for every field $\mathbb{K}$ and for each integer $i \geqslant 0$, the composite functor

$$H_i(-;\mathbb{K}) \circ T\colon \mathcal{C} \longrightarrow \mathrm{Vect}_{\mathbb{K}}$$

has degree $\leqslant \lambda i$ in the sense of Definition 8.1.

Our main homological stability result is the following. Suppose that we have chosen a bundle over a manifold $\xi\colon E \to M$, a disc $D \subseteq \partial M$, a singularity condition $c \subseteq \Gamma(\eta(\xi))$, etc, as described in Notation 5.4. In particular, the singularity condition c determines a subset $c_D \subseteq [S^{d-1}, X]$, where X is the fibre of ξ over the basepoint of M, as explained in Definition 4.13, and thus a subset $\widetilde{c}_D \subseteq \pi_{d-1}(X)$, defined to be the preimage of c_D under the canonical projection

$$\pi_{d-1}(X) \longrightarrow \pi_{d-1}(X)/\pi_1(X) \cong [S^{d-1}, X].$$

Theorem 8.5 *Suppose that the subset $\widetilde{c}_D \subseteq \pi_{d-1}(X)$ has size 1. Then the stabilisation maps*

$$\mathrm{C}\Gamma_k^{c,D}(M;\xi) \longrightarrow \mathrm{C}\Gamma_{k+1}^{c,D}(M;\xi)$$

induce isomorphisms on integral homology up to degree $\frac{k}{2} - 2$ and surjections up to degree $\frac{k}{2} - 1$. With coefficients in a field, both of these ranges may be improved by one.

In particular, this implies Theorem A of the introduction. The main technical input for this is the following.

Proposition 8.6 *If $|\widetilde{c}_D| = 1$, the functor (7.5) $= \widetilde{\mathrm{Mon}}^{c,D}(M, \xi)$ has slope $\leqslant 1$.*

We will also use part of Theorem D of [Kra19], which we recall in the following:

Theorem 8.7 ([Kra19, part of Theorem D]) *Let $(M, D, b, *)$ be as in Proposition 7.2 and let*

$$G\colon \mathcal{C}(M) \longrightarrow \mathrm{Ab}$$

be a functor to the category of abelian groups. If $\deg(G) \leqslant r$, then the maps

$$H_i(\dot{C}_k(M); G(k)) \longrightarrow H_i(\dot{C}_{k+1}(M); G(k+1)),$$

induced by the stabilisation maps (7.1) together with the functor G, are isomorphisms in the range of degrees $2i \leqslant k - r - 2$ and surjections in the range of degrees $2i \leqslant k - r$.

Proof of Theorem 8.5 (homological stability for configuration-section spaces). From the fibration sequences (5.9) and the stabilisation maps (7.1), we have a map of fibration sequences of the form:

$$\begin{tikzcd}
\Gamma_k^{c,D}(M,\xi) \arrow[r] \arrow[d] & \Gamma_{k+1}^{c,D}(M,\xi) \arrow[d] \\
\dot{\mathrm{C}}\Gamma_k^{c,D}(M;\xi) \arrow[r] \arrow[d] & \dot{\mathrm{C}}\Gamma_{k+1}^{c,D}(M;\xi) \arrow[d] \\
\dot{C}_k(M) \arrow[r] & \dot{C}_{k+1}(M),
\end{tikzcd} \tag{8.1}$$

which has an associated map of Serre spectral sequences

$$\begin{array}{ccc}
H_p(\dot{C}_k(M); H_q(\Gamma_k^{c,D}(M,\xi);\mathbb{K})) & \longrightarrow & H_p(\dot{C}_{k+1}(M); H_q(\Gamma_{k+1}^{c,D}(M,\xi);\mathbb{K})) \\
\Downarrow & & \Downarrow \\
H_{p+q}(\dot{\mathrm{C}}\Gamma_k^{c,D}(M;\xi);\mathbb{K}) & \longrightarrow & H_{p+q}(\dot{\mathrm{C}}\Gamma_{k+1}^{c,D}(M;\xi);\mathbb{K})
\end{array} \tag{8.2}$$

for any field $\mathbb{K}$. By Proposition 8.6, the functor $H_q(-;\mathbb{K}) \circ \widetilde{\mathrm{Mon}}^{c,D}(M,\xi)$ has degree $\leqslant q$ for each $q \geqslant 0$, and hence Theorem 8.7 implies that (8.2) is an isomorphism on E^2 pages in the range of bidegrees $2p \leqslant k - q - 2$, and a surjection for $2p \leqslant k - q$. In particular, it is an isomorphism for total degree $p + q \leqslant \frac{k}{2} - 1$ and a surjection for $p + q \leqslant \frac{k}{2}$. By a spectral sequence comparison argument (see [Zee57, Theorem 1] or [CDG13, Remarque 2.10]), the same statements hold also in the limit. Composing with the homotopy equivalences of Lemma 4.14, we conclude that the stabilisation maps

$$\mathrm{C}\Gamma_k^{c,D}(M;\xi) \simeq \dot{\mathrm{C}}\Gamma_k^{c,D}(M;\xi) \longrightarrow \dot{\mathrm{C}}\Gamma_{k+1}^{c,D}(M;\xi) \simeq \mathrm{C}\Gamma_{k+1}^{c,D}(M;\xi) \tag{8.3}$$

induce isomorphisms on $\mathbb{K}$-homology up to degree $\frac{k}{2} - 1$ and surjections up to degree $\frac{k}{2}$. Applying this for $\mathbb{K} = \mathbb{F}_p$ and using the maps of long exact sequences induced by the short exact sequences of coefficients $0 \to \mathbb{Z}/p^r \to \mathbb{Z}/p^{r+1} \to \mathbb{Z}/p \to 0$, we deduce the same statements for homology with coefficients in $\mathbb{Z}(p^\infty)$, where $\mathbb{Z}(p^\infty)$ is the direct limit of $\mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p^3 \to \cdots$. Then using the short exact sequence of coefficients

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}(p^\infty) \to 0,$$

we conclude that the stabilisation maps (8.3) induce isomorphisms on integral homology up to degree $\frac{k}{2} - 2$ and surjections up to degree $\frac{k}{2} - 1$. ∎

Proof of Proposition 8.6. Let $\mathbb{K}$ be a field and $i \geqslant 0$ an integer. Write

$$F_i = H_i(-;\mathbb{K}) \circ \widetilde{\mathrm{Mon}}^{c,D}(M,\xi)\colon \mathcal{C}(M) \longrightarrow \mathrm{Vect}_{\mathbb{K}}.$$

In this notation, we need to show that $\deg(F_i) \leqslant i$, where $\deg(-)$ is as in Definition 8.1.

Recall that we have assumed that the subset $\widetilde{c}_D \subseteq \pi_{d-1}(X)$ has cardinality 1, and denote by Z the corresponding path-component of $\Omega^{d-1}X$. Also write $Y = \Gamma^D(M,\xi)$ for the space of sections of $\xi \colon E \to M$ that restrict to the fixed section s_D (*cf.* Notation 5.4) on $D \subseteq \partial M$. There are natural maps

$$e_k \colon Y \times Z^k \longrightarrow \Gamma_k^{c,D}(M,\xi) = \widetilde{\mathrm{Mon}}^{c,D}(M,\xi)(k), \tag{8.4}$$

defined in Figure 8.1, such that the square

$$\begin{array}{ccc} Y \times Z^k & \longrightarrow & Y \times Z^{k+1} \\ {\scriptstyle e_k}\downarrow & & \downarrow{\scriptstyle e_{k+1}} \\ \Gamma_k^{c,D}(M,\xi) & \longrightarrow & \Gamma_{k+1}^{c,D}(M,\xi) \end{array}$$

commutes up to homotopy, where the bottom horizontal map is the stabilisation map (namely the top horizontal map of (8.1), which is also $\widetilde{\mathrm{Mon}}^{c,D}(M,\xi)(\iota_k)$) and the top horizontal map is the obvious inclusion $(s,f_1,\ldots,f_k) \mapsto (s,f_1,\ldots,f_k,*)$, where $* \in Z$ is any basepoint (exactly which basepoint does not matter since Z is path-connected). Moreover, the map (8.4) is a topological embedding, and it is not hard to define a deformation retraction of $\Gamma_k^{c,D}(M,\xi)$ onto its image – hence (8.4) is a homotopy equivalence.

Now consider an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{C}(M)}(k) = \pi_1(\dot{C}_k(M))$, which, via the endofunctor s of $\mathcal{C}(M)$, induces an automorphism $s(\alpha) = \sigma_k(\alpha) \in \pi_1(\dot{C}_{k+1}(M))$. (Recall that the notation $\sigma_k(-)$ was introduced in Definition 7.1.) One may check that, under the identifications (8.4), we have an equality

$$\widetilde{\mathrm{Mon}}^{c,D}(M,\xi)(s(\alpha)) = \widetilde{\mathrm{Mon}}^{c,D}(M,\xi)(\alpha) \times \mathrm{id}_Z \tag{8.5}$$

in the group of homotopy automorphisms up to homotopy $\pi_0(\mathrm{hAut}(Y \times Z^{k+1}))$.

Next, consider the morphism $\iota_k \colon k \to k+1$ of $\mathcal{C}(M)$. As observed earlier in this section, we have the identity $s(\iota_k) = v_k^2(\tau_1) \circ \iota_{k+1}$, where $\tau_1 \in B_2$ is the standard generator and the homomorphism $v_k^2 \colon B_2 \to \pi_1(\dot{C}_{k+2}(M))$ is as in Definition 7.1. We also noted above that, under the identifications (8.4), the map $\widetilde{\mathrm{Mon}}^{c,D}(M,\xi)(\iota_k)$ corresponds to the obvious inclusion of $Y \times Z^k$ into $Y \times Z^{k+1}$ using the fixed basepoint $*$ of Z. Using this, one may check that, under the identifications (8.4), we have an equality

$$\widetilde{\mathrm{Mon}}^{c,D}(M,\xi)(s(\iota_k)) = \widetilde{\mathrm{Mon}}^{c,D}(M,\xi)(\iota_k) \times c_Z \tag{8.6}$$

in the homotopy set $\pi_0(\mathrm{Map}(Y \times Z^{k+1}, Y \times Z^{k+2})) = [Y \times Z^{k+1}, Y \times Z^{k+2}]$, where $c_Z \colon Z \to Z$ is the identity map if $d \geqslant 3$, and if $d = 2$ it is the homotopy automorphism of $Z \subseteq \Omega X$ given by conjugating a given loop in the path-component Z of ΩX by the fixed loop $* \in Z$.

The morphisms of $\mathcal{C}(M)$ are generated by its automorphisms together with the morphisms ι_k for $k \in \mathbb{N}$, so the identifications (8.4), (8.5) and (8.6) imply that we have a natural isomorphism

$$\widetilde{\mathrm{Mon}}^{c,D}(M,\xi) \circ s \;\cong\; \widetilde{\mathrm{Mon}}^{c,D}(M,\xi) \times \mathrm{C}(c_Z) \tag{8.7}$$

of functors $\mathcal{C}(M) \to \mathrm{Ho(Top)}$, where, for an endomorphism $f \colon A \to A$ in $\mathrm{Ho(Top)}$, the functor $\mathrm{C}(f) \colon \mathcal{C}(M) \to \mathrm{Ho(Top)}$ sends each object to A, each automorphism to id_A and each morphism ι_k to f. Applying $H_i(-;\mathbb{K})$ and using the Künneth theorem (including the *naturality* of the Künneth isomorphism), the decomposition (8.7) induces an isomorphism of functors

$$F_i \circ s \;\cong\; \bigoplus_{j=0}^{i} F_{i-j} \otimes H_j(\mathrm{C}(c_Z);\mathbb{K}), \tag{8.8}$$

such that the natural transformation $F_i\,\iota \colon F_i \to F_i \circ s$ corresponds, under (8.8), to the inclusion of the $j=0$ summand, where we are using the fact that Z is path-connected to identify $H_0(\mathrm{C}(c_Z);\mathbb{K})$ with the constant functor at $\mathbb{K}$.

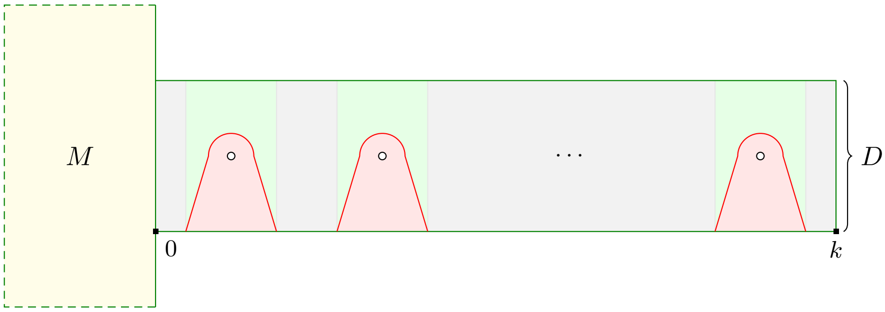


Figure 8.1 The map $e_k\colon Y \times Z^k \longrightarrow \Gamma_k^{c,D}(M,\xi)$ from the proof of Proposition 8.6, where Z is a given path-component of $\Omega^{d-1}X = \mathrm{Map}((D^{d-1},\partial D^{d-1}),(X,\{x_0\}))$ and $Y = \Gamma^D(M,\xi)$.
Given inputs $(s,f_1,\ldots,f_k)$, the section $e_k(s,f_1,\ldots,f_k)$ of $\hat{\xi}$ over $\hat{M}_k \smallsetminus z_k$ is given by s in the yellow region (namely M) and by the maps $f_1,\ldots,f_k$ on the red arcs (representing embedded $(d-1)$-discs with their boundary on the bottom face of $D\times[0,k]$). Recall that, over $(D\times[0,k])\smallsetminus z_k$, the bundle is trivial with fibre X, so we may think of sections as maps $(D\times[0,k])\smallsetminus z_k \to X$. The bottom face of $D\times[0,k]$ is sent to the basepoint x_0 of X. We then extend the map in the red regions by defining it to be constant along radii centred at the punctures z_k, and we extend it in the green regions by defining it to be constant in the vertical direction.

Using (8.8) and the fact that $F_i\,\iota$ corresponds to the inclusion of the $j=0$ summand under this identification, we deduce (i) that $F_i\,\iota$ is split-injective in the functor category $\mathrm{Fun}(\mathcal{C}(M),\mathrm{Vect}_{\mathbb{K}})$ and (ii) that we have an isomorphism of functors

$$\Delta F_i \;\cong\; \bigoplus_{j=1}^{i} F_{i-j}\otimes H_j(\mathrm{C}(c_Z);\mathbb{K}). \tag{8.9}$$

The fact that c_Z is a homotopy automorphism, i.e., invertible in $\mathrm{Ho}(\mathrm{Top})$, implies that the functor $H_j(\mathrm{C}(c_Z);\mathbb{K})$ sends each ι_k to an isomorphism, which implies, by definition, that

$$\deg(H_j(\mathrm{C}(c_Z);\mathbb{K})) \leqslant 0. \tag{8.10}$$

We now prove that $\deg(F_i)\leqslant i$ by induction on $i\geqslant 0$. For $i=0$, the identification (8.9) says that $\Delta F_0 = 0$. Together with the fact that $F_0\,\iota$ is split-injective, this implies that $\deg(F_0)\leqslant 0$. For $i\geqslant 1$, using the identification (8.9), Lemma 8.3, the fact (8.10) and the inductive hypothesis, we see that

$$\deg(\Delta F_i) = \deg\Bigl(\bigoplus_{j=1}^{i} F_{i-j}\otimes H_j(\mathrm{C}(c_Z);\mathbb{K})\Bigr) \leqslant \max_{j=1}^{i}\{\deg(F_{i-j})\} \leqslant i-1.$$

Together with the fact that $F_i\,\iota$ is split-injective, this implies by definition that $\deg(F_i)\leqslant i$. $\square$

Remark 8.8 (*The hypothesis of Proposition 8.6.*) The assumption $|\widetilde{c}_D| = 1$ (where $\widetilde{c}_D \subseteq \pi_{d-1}(X)$ is the subset induced by the "singularity condition" $c\subseteq \Gamma(\eta(\xi))$) of Proposition 8.6 means that the corresponding subspace $Z\subseteq \Omega^{d-1}X$ is a single path-component, rather than a union of several path-components. The path-connectedness of Z is used in a key way for the identification (8.9) of ΔF_i in terms of $F_{i-1},F_{i-2},\ldots$. If Z were disconnected, we would have $H_0(\mathrm{C}(c_Z);\mathbb{K}) = \mathbb{K}^{\pi_0(Z)}$ and, denoting by Z_0 the path-component of Z containing its basepoint, the identification (8.9) would become

$$\Delta F_i \;\cong\; F_i\otimes \mathbb{K}^{\pi_0(Z)\smallsetminus\{Z_0\}} \oplus \bigoplus_{j=1}^{i} F_{i-j}\otimes H_j(\mathrm{C}(c_Z);\mathbb{K}),$$

which would break the inductive argument, since this decomposition involves F_i itself.

Remark 8.9 (*Naturality of the Künneth theorem.*) In order to transform the identification (8.7) of functors $\mathcal{C}(M)\to \mathrm{Ho}(\mathrm{Top})$ into the identification (8.8) of functors $\mathcal{C}(M)\to\mathrm{Vect}_{\mathbb{K}}$, we used the Künneth theorem for field coefficients, which does not involve any Tor terms. If we had worked instead with $\mathbb{Z}$ coefficients, we would have obtained a decomposition similar to (8.8) – *at the level of objects* – including also some Tor terms. The appearance of the Tor terms themselves is no problem, since one may prove an analogue of the second and third points of Lemma 8.3 for $\mathrm{Tor}(-)$ instead of $\otimes$, so they behave as desired with respect to degree. The problem instead is that the Künneth short exact sequences are split, but not *naturally* split (unless – of course – the Tor terms vanish). Thus, we would not have been able to obtain a decomposition *of functors* analogous to (8.8), since the naturality of the Künneth theorem, in the case where the Tor terms vanish, was key to upgrading (8.8) from an isomorphism at the level of objects to an isomorphism of functors. See also Remark 4.4 of [Pal18] for a similar comment about the (non-)naturality of the splitting of the Künneth short exact sequence.

Remark 8.10 (*Improving the range of stability.*) Our main homological stability result states that, on homology with field coefficients, the stabilisation maps (8.3) induce isomorphisms up to degree $\frac{k}{2}-1$ and surjections up to degree $\frac{k}{2}$. This then implies the analogous statements for homology with integral coefficients – but only in a range of degrees that is smaller by one, i.e., isomorphisms up to degree $\frac{k}{2}-2$ and surjections up to degree $\frac{k}{2}-1$.

However, under certain hypotheses, this loss of one in the range of degrees for integral homology may be avoided: namely, if the integral homology groups $H_*(\Omega^{d-1}X;\mathbb{Z})$ and $H_*(\Gamma^D(M,\xi);\mathbb{Z})$ are torsion-free in all degrees. Under this assumption, one may run the proof of Proposition 8.6 using $\mathbb{Z}$ in place of $\mathbb{K}$, since the torsion-freeness assumption implies that one may apply the Künneth theorem for $\mathbb{Z}$ coefficients without the appearance of any Tor terms (see Remark 8.9 above for the importance of the vanishing of the Tor terms, and see Remark 4.4 of [Pal18] for the analogous remark in a similar setting). We could then also run the proof of Theorem A directly with $\mathbb{Z}$ coefficients, without any need to pass from field coefficients to $\mathbb{Z}$ coefficients at the end, which is where we lose 1 from the range of stability.

Note that, in the case where $M=D^2$ and $X=BG$, this hypothesis amounts to requiring that $H_*(G;\mathbb{Z})$ is torsion-free in all degrees. (In particular, this means that the abelianisation of G must be torsion-free, which is not the case for G finite cyclic or $G=A\rtimes\mathbb{Z}/2$, where A is abelian and $\mathbb{Z}/2$ acts by inversion.)

9. Extension to $\mathcal{B}_\sharp(M)$ and split-injectivity

In this brief section we prove a split-injectivity result for the homology of configuration-mapping spaces, under certain conditions on the underlying manifold M. Let us fix a connected manifold M, an embedded disc $D\subseteq\partial M$, a based space X and a subset $c\subseteq[S^{d-1},X]$. Recall that the stabilisation maps $\mathrm{CMap}_k^{c,D}(M;X)\to\mathrm{CMap}_{k+1}^{c,D}(M;X)$ fit into a map of fibre sequences of the form (8.1), inducing a map of their associated Serre spectral sequences:

$$\begin{array}{ccc} H_p(\dot{C}_k(M);H_q(\mathrm{Map}^{c,D}(\hat{M}_k\smallsetminus z_k,X);\mathbb{Z})) & \longrightarrow & H_p(\dot{C}_{k+1}(M);H_q(\mathrm{Map}^{c,D}(\hat{M}_{k+1}\smallsetminus z_{k+1},X);\mathbb{Z})) \\ \Downarrow & & \Downarrow \\ H_{p+q}(\mathrm{CMap}_k^{c,D}(M;X);\mathbb{Z}) & \longrightarrow & H_{p+q}(\mathrm{CMap}_{k+1}^{c,D}(M;X);\mathbb{Z}) \end{array} \tag{9.1}$$

(Note that above we take coefficients in $\mathbb{Z}$ rather than in a field, as was the case in (8.2).)

The main result of this section is the following.

Theorem 9.1 *Let M have dimension at least 3 and assume that either $\pi_1(M)=0$ or the handle-dimension of M is at most $\dim(M)-2$. Then the map of Serre spectral sequences (9.1) above is split-injective on each entry of the E^2 pages.*

Remark 9.2 Note that, since M is connected and has non-empty boundary, its handle-dimension is at most $\dim(M)-1$. Thus the only manifolds excluded by the additional hypotheses of Theorem 9.1 are (i) surfaces and (ii) non-simply-connected manifolds of maximal handle dimension.

Recall from §7 that we have a *monodromy functor*

$$\mathcal{C}(M)\longrightarrow\mathrm{Ho}(\mathrm{Top}) \tag{9.2}$$

encoding both the stabilisation maps and the monodromy action of $\pi_1(\dot{C}_k(M), z_k)$ on the fibre $\mathrm{Map}^{c,D}(\hat{M}_k \smallsetminus z_k, X)$ of the fibration $\mathrm{CMap}_k^{c,D}(M;X) \to \dot{C}_k(M)$ for each k (see Proposition 7.3). Recall also from §6 that we have a functor of braid categories

$$\mathcal{C}(M) \longrightarrow \mathcal{B}_\sharp(M) \tag{9.3}$$

(see Summary 6.10).

Proposition 9.3 *Let* $\dim(M) \geqslant 3$ *and assume that either* $\pi_1(M) = 0$ *or the handle-dimension of* M *is at most* $\dim(M) - 2$*. Then the monodromy functor* (9.2) *extends to* $\mathcal{B}_\sharp(M)$*.*

Proof. The fibre $\mathrm{Map}^{c,D}(\hat{M}_k \smallsetminus z_k, X)$ of the fibration $\mathrm{CMap}_k^{c,D}(M;X) \to \dot{C}_k(M)$ decomposes up to homotopy as $\mathrm{Map}_*(M,X) \times (\Omega_c^{d-1}X)^k$, where $\Omega_c^{d-1}X$ denotes the union of path-components of the loopspace $\Omega^{d-1}X$ corresponding to the subset $c \subseteq [S^{d-1}, X]$. Since M has dimension at least 3, the fundamental group $\pi_1(\dot{C}_k(M))$ decomposes as $\pi_1(M)^k \rtimes \Sigma_k$ [Til16, Lemma 4.1]. Under these identifications, and under the given assumptions on M, [PT, Corollary 7.2] implies that the monodromy action is given, for $\alpha_i \in \pi_1(M)$, $\sigma \in \Sigma_k$, $f \in \mathrm{Map}_*(M,X)$ and $g_i \in \Omega_c^{d-1}X$, by the following formula:

$$(\alpha_1, \ldots, \alpha_k; \sigma) \cdot (f, g_1, \ldots, g_k) = (f, \bar{g}_1, \ldots, \bar{g}_k), \tag{9.4}$$

where $\bar{g}_i = f_*(\alpha_i).g_{\sigma(i)}.\mathrm{sgn}(\alpha_i)$. The element $f_*(\alpha_i) \in \pi_1(X)$ acts on the point $g_{\sigma(i)} \in \Omega_c^{d-1}X$ via the usual action-up-to-homotopy of $\pi_1(X)$ on the iterated loopspaces of X. The sign $\mathrm{sgn}(\alpha_i) \in \{\pm 1\}$ depends on whether or not the loop α_i lifts to a loop in the orientation double cover of M, and -1 acts on $g_{\sigma(i)} \in \Omega_c^{d-1}X$ via a reflection of S^{d-1}.

The formula (9.4) may be visualised as follows (*cf.* Figure 7.1 of [PT]). The element $(\alpha_1, \ldots, \alpha_k; \sigma)$ of $\pi_1(M)^k \rtimes \Sigma_k$ is a k-strand braid, where the strands may pass through each other, and where each strand is labelled by an element of $\pi_1(M)$. The element $(f, g_1, \ldots, g_k)$ is acted upon by pushing the i-th element g_i backwards along the i-th strand of the braid while acting on it by $f_*(\;\;)$ of the label of that strand as well as the involution $\mathrm{sgn}(\;\;)$ of the label of the strand.

This description of the monodromy action immediately suggests how to extend the monodromy functor (9.2) to $\mathcal{B}_\sharp(M)$, since the morphisms of $\mathcal{B}_\sharp(M)$ have a similar combinatorial description when M has dimension at least 3. Namely, a morphism $m \to n$ in $\mathcal{B}_\sharp(M)$ may be viewed as a braid (whose strands may cross each other) from a subset of $\{1, \ldots, m\}$ to a subset of $\{1, \ldots, n\}$, where each strand is labelled by an element of $\pi_1(M)$ (*cf.* [Kra19, Remark 5.10]). In fact, we will define an extension of the monodromy functor to $\mathcal{B}_\sharp(M)^{\mathrm{op}}$, but this will finish the proof since $\mathcal{B}_\sharp(M)$ is canonically isomorphic to its opposite category.

To define the extension of the monodromy functor to $\mathcal{B}_\sharp(M)^{\mathrm{op}}$, we describe how a morphism $m \to n$ of $\mathcal{B}_\sharp(M)$ acts on $(f, g_1, \ldots, g_n)$ for $f \in \mathrm{Map}_*(M,X)$ and $g_i \in \Omega_c^{d-1}X$. First, fix a basepoint $* \in \Omega_c^{d-1}X$. If there is a strand ending at position i, we push the i-th element g_i backwards along this strand, acting on it by $f_*(\;\;)$ and $\mathrm{sgn}(\;\;)$ of the strand's label. We then fill in any blanks in the resulting partial m-tuple of elements of $\Omega_c^{d-1}X$ with the basepoint $*$.

In formulas, this is written as follows. A morphism $m \to n$ of $\mathcal{B}_\sharp(M)$ is given by a partially-defined injective function σ from a subset of $\{1, \ldots, m\}$ to a subset of $\{1, \ldots, n\}$ and an element $\alpha_i \in \pi_1(M)$ for each $i \in \mathrm{dom}(\sigma)$. This morphism acts by

$$(f, g_1, \ldots, g_n) \longmapsto (f, \bar{g}_1, \ldots, \bar{g}_m),$$

where $\bar{g}_i = f_*(\alpha_i).g_{\sigma(i)}.\mathrm{sgn}(\alpha_i)$ if $i \in \mathrm{dom}(\sigma)$ and $\bar{g}_i = *$ otherwise. ◻

Remark 9.4 The assumption that either $\pi_1(M) = 0$ or the handle-dimension of M is at most $\dim(M) - 2$ is essential in the proof above. Indeed, the monodromy action is in general given by a more complicated formula than (9.4) without these assumptions (*cf.* [PT, Remark 7.3]). In particular, there is a non-trivial action on the component $f \in \mathrm{Map}_*(M,X)$ of the tuple.

Proof of Theorem 9.1. By [Pal18, Theorem A and Remark 1.3], for any abelian group-valued functor $T \colon \mathcal{B}_\sharp(M) \to \mathrm{Ab}$, the induced stabilisation maps on T-twisted homology

$$H_*(C_k(M); T(k)) \longrightarrow H_*(C_{k+1}(M); T(k+1))$$

are split-injective in all degrees. This implies the statement of Theorem 9.1 by applying it to the extended monodromy functor $\mathcal{B}_\sharp(M) \to \mathrm{Ho}(\mathrm{Top})$ of Proposition 9.3 composed with the homology functor $H_q(-;\mathbb{Z})\colon \mathrm{Ho}(\mathrm{Top}) \to \mathrm{Ab}$. ◻

Remark 9.5 Split-injectivity of a map of Serre spectral sequences on each entry of the E^2 pages does not, however, automatically imply split-injectivity on any of the further pages, since we do not know whether the splittings commute with the differentials.

10. Group-completion and stable homology

The identification of the stable homology follows a relatively well-established path. For configuration spaces of manifolds with labels in a fixed space this is semi-classical [May72] [McD75], [Seg73], [Böd87], [Sal01]. In the case of configuration-mapping spaces this has been done in [EVW]. Here we also generalise the arguments to configuration-section spaces.[2]

In the classical setting an important role is played by the scanning map from the configuration spaces to certain mapping or section spaces. The part that differs from one situation to the other, is the identification of the 'local data', i.e. what is seen in a small disk modulo its boundary. The remaining arguments are nearly formal and – somewhat surprisingly – remain so also in our case where the data associated to a configuration depends on global information rather than the local information in the neighborhood of the configuration points themselves.

In outline, we will first consider configuration-section spaces of any number of particles where the particles can 'vanish' at the boundary or in some other subspace of the underlying manifold M and establish a quasi-fibration sequence for them. Then we identify the 'local data'. Using an induction on the handlebody decomposition of M we can then 'integrate' the local data to the whole manifold. Finally, using the group-completion theorem we relate this to the finite configuration-section spaces.

10.1. Relative configuration-section spaces and quasi-fibrations. Let M be a compact, connected manifold of dimension d and $N \subset M$ be a co-dimension zero closed submanifold such that $M \smallsetminus N$ is open. As before, let $\xi\colon E \to M$ be a fibre bundle and let S be a subspace of M. We assume that ξ has a canonical section s_0 defined over all of M and the sections that we will consider agree with s_0 on S. (In previous sections, we imposed "boundary conditions" on subsets $D \subseteq \partial M$; the definition generalises straightforwardly to arbitrary subsets $S \subseteq M$.) Let $c \subseteq \Gamma(\eta(\xi))$ be a singularity condition (*cf.* Definition 3.14).

Definition 10.1 Let $\mathrm{C}\Gamma^{c,S}(M,N;\xi)$ be the quotient of $\mathrm{C}\Gamma^{c,S}(M;\xi) = \coprod_k \mathrm{C}\Gamma^{c,S}_k(M;\xi)$ by the equivalence relation $(z,s) \sim (z',s')$ whenever

$$z \cap \overline{M \smallsetminus N} = z' \cap \overline{M \smallsetminus N} \text{ and } s|_{\overline{M \smallsetminus N}} = s'|_{\overline{M \smallsetminus N}}.$$

So (z,s) and (z',s') agree outside the interior of N.

Thus points can disappear as they move into the submanifold N. It will be convenient to also allow points to disappear on the boundary of M. In that case we will write

$$\mathrm{C}\Gamma^{c,S}(M,\partial M;\xi)$$

were we interpret this to mean that ∂M is thickened to a small collar $b : [-\epsilon,0] \times \partial M \subset M$ to fit the above definition.

Proposition 10.2 *Let M' be a compact co-dimension 0 submanifold of M, such that $(M', N \cap M')$ is connected. Then the canonical quotient map*

$$\mathrm{C}\Gamma^{c,S}(M,N;\xi) \xrightarrow{\pi} \mathrm{C}\Gamma^{c,S}(M,M' \cup N;\xi)$$

[2] We also correct the arguments in [EVW] in several places: this includes the statement of the quasi-fibration sequence in Proposition 10.2; the proof of Theorem 10.12 and in particular the correction of diagram (10.2).

is a quasi-fibration with fibre

$$\mathrm{C}\Gamma^{c',S'\cup\partial'}(M', N\cap M';\xi');$$

here $S' = S\cap M'$; $\partial' = \partial M'\cap\partial K$ *where* $K = \overline{(M\setminus M')}$ *is the closure of the complement of* M' *in* M; $\xi' := \xi|_{M'}$ *and* $c' \subseteq \Gamma(\eta(\xi'))$ *are the restrictions of* ξ *and* c *to* M'.

Proof. We need to identify the fibre $\pi^{-1}((z,s))$ where $(z,s)\in\mathrm{C}\Gamma^{c,S}(M,N;\xi)$ is a representative of $(z,s)\in\mathrm{C}\Gamma^{c,S}(M,M'\cup N;\xi)$; here $s\in\Gamma(M\setminus z,\xi)$ is a section satisfying the restrictions imposed by c and S. By definition, this fibre consists of all those (z',s') that agree with (z,s) when restricted to K. Thus $z'\setminus(K\cap z')$ can be an arbitrary subset of the interior of M' and s' is a section of ξ' which agrees with s on the interface $\partial' = \partial M'\cap\partial K$. Taking the relation introduced by N into account, we see that

$$\pi^{-1}((z,s)) = \{(z',s')\in\mathrm{C}\Gamma^{c',S\cap M'}(M',N\cap M';\xi')\mid s'|_{\partial'} = s|_{\partial'}\}.$$

Thus the fibre is independent of z and in case that $s|_{\partial'} = s_0|_{\partial'}$ it is precisely given by the fibre of the proposition.

To show π is a quasi-fibration in the sense of Dold-Thom [DT58] we use the natural filtrations by number of points in the configuration of the base space:

$$B_k := \{(z,s)\in\mathrm{C}\Gamma^{c,S}(M,M'\cup N;\xi)\ \text{ with } |z|\leqslant k\}.$$

Step 1: We will prove that π restricted to $B_k\setminus B_{k-1}$ is a fibration.[3] For this note that the map $B_k\setminus B_{k-1}\to\mathrm{C}\Gamma_k^{c'',S\cap K}(K;\xi'')$ sending (z,s) to $(z\cap(M\setminus M'), s|_K)$ is a homeomorphism onto its image of points (z'',s'') where s'' extends to a section over all of M while restricting to s_0 over S. Let $\Gamma(E|_{\partial'})$ denote the space of sections of E restricted to ∂' and consider the homeomorphism

$$\phi : (B_k\setminus B_{k-1})\times_{\Gamma(E|_{\partial M'})}\mathrm{C}\Gamma^{c',S\cap M'}(M',N\cap M';\xi)\longrightarrow\pi^{-1}(B_k\setminus B_{k-1})$$

defined by

$$\phi((z'',s''),(z',s')) = (z''\cup z', s''\cup_{\partial'} s').$$

Here the fibre product is taken using the natural maps that take (a configuration and) a section to its restriction over ∂'. As these are fibrations, the projection of the source of ϕ to $B_k\setminus B_{k-1}$ is a fibration, thus proving (1).

Step 2: Let U be a tubular neighbourhood of M' in M and let J_t be an isotopy of M such that

$$J_0 = id_M \quad\text{ and }\quad J_1(U)\subset M' \quad\text{ while for all } t\in[0,1]\quad J_t(S)\subset S \quad\text{ and }\quad J_t(N\cap U)\subset N.$$

We can now define $U_k\subset B_k$ with at most $k-1$ point in $M\setminus U$. So in particular $B_{k-1}\subset U_k$. Define a homotopy H_t by the formula

$$H_t(z,s) = (J_t(z), s\circ J_t^{-1}).$$

By definition this commutes with π and hence is a fiberwise homotopy. We want to show that $H_1:\pi^{-1}(z,s)\to\pi^{-1}(H_1(z,s))$ induces a homotopy equivalence on fibres over points $(z,s)\in U_k$. But this follows as $(M', N\cap M')$ is connected and both fibres can be identified with

$$\mathrm{C}\Gamma^{c',S'}(M',N\cap M';\xi').$$

Together *Step 1* and *Step 2* prove that π satisfies the Dold-Thom criteria for a quasi-fibration. ◻

[3] Up to homeomorphism we may assume that none of the k points are in $[-\epsilon,0]\times\partial M'\subset K$.

10.2. Local data. We will now identify the 'local data' for the scanning process. By definition, 'local data' is the restriction of the configuration section spaces to a disk relative to its boundary. In other words, it is the configuration section space on the disk where points can disappear at the boundary. As the fibre bundle ξ is locally trivial, the 'local data' in the case of configuration-section spaces does not see the global twisting of ξ. We can thus assume we have a trivial bundle with fixed, based fibre X and a fixed subset $c \subseteq \pi_{d-1}(X)$ of monodromies, and thus reduce to the case of configuration-mapping spaces.

Let $(X, *)$ be a pointed space and let $c \subseteq \pi_{d-1}(X)$ be a subset of monodromies.

Definition 10.3 Let $A_d(X, c)$ be the pushout of the diagram

$$D^d \times \mathrm{Map}^c(S^{d-1}, X) \xleftarrow{i} S^{d-1} \times \mathrm{Map}^c(S^{d-1}, X) \xrightarrow{ev} X,$$

where the left arrow is the inclusion and the right arrow the evaluation map $ev(t, f) := f(t)$. Let $A'_d(X, c)$ be the homotopy fibre of the natural inclusion $X \to A_d(X, c)$. When $c = \pi_{d-1}(X)$ we drop c from the notation and write $A_d(X)$ and $A'_d(X)$.

Lemma 10.4 $A_d(X, c) \simeq \mathrm{CMap}^c(D^d, \partial D^d; X)$.

Proof. We expand on the arguments given in [EVW].

Let C_{01} denote the subspace of $\mathrm{CMap}^c(D^d, \partial D^d; X)$[4] of pairs (z, f) where $z \cap \mathrm{int}(D^d)$ has size either zero or one. By radial expansion, we have a deformation retraction

$$\mathrm{CMap}^c(D^d, \partial D^d; X) \simeq C_{01}.$$

The radial expansion varies continuously with the configuration z: If $z_1 \in z$ is the closest point to zero, and $z_2 \in z$ the second most close (it could be as close as z_1) then the radial expansion proceeds at rate $1/|z_2|$.

Note that

$$C_{01} = C_0 \cup C_1,$$

where

- C_0 is the subspace of C_{01} of pairs (z, f) where $z \subseteq N(\partial)$,
- C_1 is the subspace of C_{01} of pairs (z, f) where $z \subseteq \mathrm{int}(D^d)$ has cardinality one,

and hence

- $C_0 \cap C_1$ is the subspace of C_{01} of pairs (z, f) where $z \subseteq N(\partial) \cap \mathrm{int}(D^d)$ has cardinality one.

Note that the inclusion $C_0 \cap C_1 \hookrightarrow C_1$ is a (closed) cofibration, since we may exhibit $(C_1, C_0 \cap C_1)$ as an NDR-pair by using a closed 2ϵ-neighbourhood of ∂D^d. Hence we have

$$\begin{aligned} \mathrm{CMap}^c(D^d, \partial D^d; X) \simeq C_{01} = C_0 \cup C_1 &= \mathrm{Pushout}(C_1 \hookleftarrow C_0 \cap C_1 \hookrightarrow C_0) \\ &\simeq \mathrm{hPushout}(C_1 \hookleftarrow C_0 \cap C_1 \hookrightarrow C_0). \end{aligned}$$

It therefore suffices to identify the two inclusion maps $C_0 \cap C_1 \hookrightarrow C_0$ and $C_0 \cap C_1 \hookrightarrow C_1$ up to homotopy with the two maps in the diagram of Definition 10.3, since $A_d(X, c)$ is the pushout of this diagram, and thus also the homotopy pushout, since the map i of the diagram is a cofibration.

There are homotopy equivalences

$$C_1 \xrightarrow{g_1} D^d \times \mathrm{Map}^c(S^{d-1}, X) \qquad C_0 \cap C_1 \xrightarrow{g_{01}} S^{d-1} \times \mathrm{Map}^c(S^{d-1}, X) \qquad C_0 \xrightarrow{g_0} X$$

given, respectively, by

$$(z, f) \mapsto (z, f|_{S^{d-1}}) \qquad (z, f) \mapsto \left(-\tfrac{z}{|z|}, f|_{S^{d-1}}\right) \qquad (z, f) \mapsto f(0).$$

[4] Strictly speaking, we extend both D^d and ∂D^d by a small collar $[1, 1+\epsilon] \times \partial D^d$.

(The reason for the negative sign in the middle formula will become clear below.) Moreover, one may easily write down a homotopy

$$g_1 \circ \text{inclusion} \simeq \text{inclusion} \circ g_{01} \colon C_0 \cap C_1 \longrightarrow D^d \times \mathrm{Map}^c(S^{d-1}, X),$$

using the straight line in D^d between a given point $z \in N(\partial) \cap \mathrm{int}(D^d)$ and $-\frac{z}{|z|}$. This identifies the inclusion $C_0 \cap C_1 \hookrightarrow C_1$ with the inclusion map i of Definition 10.3.

Similarly, we may define a homotopy

$$g_0 \circ \text{inclusion} \simeq ev \circ g_{01} \colon C_0 \cap C_1 \longrightarrow X,$$

by $(z, f) \mapsto f(-\frac{tz}{|z|})$ for $t \in [0,1]$. This is well-defined, since, for all $z \in N(\partial) \cap \mathrm{int}(D^d)$, the line segment in D^d between 0 and $-\frac{z}{|z|}$ does not pass through z, and hence f is defined on all of this line segment. (This is the reason for defining the homotopy equivalence g_{01} with the negative sign in the formula above; without the negative sign, we would be forced to consider the line segment between 0 and $\frac{z}{|z|}$ instead, which passes through z, meaning that f is not defined on the whole line segment.) This identifies the inclusion $C_0 \cap C_1 \hookrightarrow C_0$ with the map ev of Definition 10.3. ◻

We will now identify multiple deloopings of configuration-mapping spaces associated to D^d. For this it is easier to work with cubes rather than with disks. Thus we fix an identification of the d-disk with the d-cube via a homeomorphism that takes the southern hemisphere of the boundary of the cube less one of its faces:

$$D^d \equiv I^d \quad \text{and} \quad \frac{1}{2}\partial \equiv \partial I^d \smallsetminus (I^{d-1} \times \{1\}); \tag{10.1}$$

here $I = [0,1]$. Recall from Proposition 4.10 that the disjoint union of the configuration mapping spaces $\mathrm{CMap}_k^{c,S}(D^d; X), k \geqslant 0$, is an E_d-algebra when $S = \partial$ and an E_{d-1}-algebra when $S = \frac{1}{2}\partial$. Thus these two spaces have d and respectively $d-1$ fold classifying spaces. The homotopy commuting product structures correspond to stacking the cubes along different pairs of opposite faces that are mapped to the basepoint in X. As in the case of configuration spaces with labels and other similar cases, it turns out that taking the classifying space with respect to multiplication corresponding to one such pair of opposite faces is equivalent to allowing the configurations to vanish on those faces which we will now make precise. We introduce the notation

$$B_k(\mathrm{CMap}^{c,S}(D^d; X)) := \mathrm{CMap}^{c,S}(I^d, \partial I^k \times I^{d-k}; X).$$

Lemma 10.5 *There are homotopy equivalences*

(i) $s_1 : B_1(\mathrm{CMap}^{c,\partial}(D^d; X)) \xrightarrow{\simeq} B(\mathrm{CMap}^{c,\partial}(D^d; X))$;

(ii) $s_1 : B_1(\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d; X)) \xrightarrow{\simeq} B(\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d; X))$.

Proof. For $c = \pi_{d-1}(X)$, this is Lemma 3.3.1 of [EVW]. The proof follows standard arguments, compare [May72], [McD75], [Sal01], and automatically extends to an arbitrary set c of monodromies. ◻

Lemma 10.6 *There are homotopy equivalences*

(i) $s_k : B_{k-1}(\mathrm{CMap}^{c,\partial}(D^d; X)) \xrightarrow{\simeq} \Omega B_k(\mathrm{CMap}^{c,\partial}(D^d; X))$ *for* $1 < k \leqslant d$;

(ii) $s_k : B_{k-1}(\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d; X)) \xrightarrow{\simeq} \Omega B_k(\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d; X))$ *for* $1 < k < d$.

Proof. For $c = \pi_{d-1}(X)$, this is Lemma 3.5.1 of [EVW]. The proof follows standard arguments, compare [May72], [McD75], [Sal01], and automatically extends to an arbitrary set c of monodromies. ◻

Lemma 10.7 *There are homotopy equivalences*

$$A_d(X,c) \simeq B_d(\mathrm{CMap}^{c,\partial}(D^d;X)) \quad \text{and} \quad A'_d(X,c) \simeq B_{d-1}(\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X)).$$

Proof. For $c = \pi_{d-1}(X)$, this is Lemma 3.5.2 of [EVW] and the proof generalises for an arbitrary set c of monodromies. Indeed, the first homotopy equivalence holds by Lemma 10.4, the definition of B_k, and because $\mathrm{CMap}^{c,\partial}(I^d,\partial I^d;X) \simeq \mathrm{CMap}^c(I^d,\partial I^d;X)$ as the restriction on the maps to be constant on the boundary imposes no additional restriction in the quotient configuration space where all sections are identified that agree on the complement of a collar of the boundary. □

Corollary 10.8 *There are homotopy equivalences*

(i) $\Omega B\mathrm{CMap}^{c,\partial}(D^d;X) \simeq \Omega^d A(X,d)$ *for* $d \geqslant 1$*;*

(ii) $\Omega B\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X) \simeq \Omega^{d-1}A'_d(X,c) \simeq \mathrm{Map}_*((D^d,S^{d-1});(A_d(X,c);X))$ *for* $d \geqslant 2$*.*

Proof. To prove part (ii), note that there is a string of homotopy equivalences

$$\begin{aligned}\Omega B\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X) &\simeq \Omega B_1(\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X))\\ &\simeq \Omega^{d-1}B_{d-1}(\mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d;X))\\ &\simeq \Omega^{d-1}A'_d(X,c).\end{aligned}$$

The first and second homotopy equivalence follow from Lemma 10.5 and repeated application of Lemma 10.6. The last homotopy equivalence follows from Lemma 10.7. The second homotopy equivalence in part (ii) follows as by definition $A'(X,c)$ is the homotopy fibre of the canonical inclusion of X into $A(X,c)$. An entirely analogous argument proves part (i). □

10.3. Integrating local data for general manifolds. As in the classical case for configuration spaces with labels [McD75], the configuration-section spaces are describing section spaces of certain bundles that depend on the tangent bundle of the underlying manifold. Thus we need the following fiberwise generalisation of $A_d(X,c)$.

As before, let M be compact and connected, and fix a metric on M. Denote by D^dM and $S^{d-1}M$ the associated disk and sphere bundles of the tangent bundle TM.

Definition 10.9 Let $E_d(\xi,c)$ be the fibre-wise pushout of the diagram

$$D^dM \times_M \mathrm{Map}^c_M(S^{d-1}M,\xi) \xleftarrow{\;i\;} S^{d-1}M \times_M \mathrm{Map}^c_M(S^{d-1}M,\xi) \xrightarrow{\;ev\;} \xi,$$

where Map_M denotes the space of fiberwise maps between bundles over M and ev is the fiberwise evaluation. Thus $E_d(\xi,c)$ is a fibre bundle over M with fibres $A_d(X,c)$ for X a typical fibre of ξ. Write $E'_d(\xi,c)$ for the fibrewise homotopy fibre of the map of fibre bundles $\xi \to E_d(\xi,c)$ over M.

Example 10.10 If ξ is the trivial fibration with fibre X and M is parallelisable then $E_d(\xi,c)$ is the trivial fibration with fibre $A_d(X,c)$, and $E'_d(\xi,c)$ is the trivial fibration with fibre $A'_d(X,c)$, the homotopy fibre of the natural inclusion $X \to A_d(X,c)$.

Example 10.11 One reason to consider non-trivial bundles ξ over M is because it allows us to 'untwist' the tangent bundle. Thus if ξ is the sphere bundle of the cotangent bundle T^*M then $E_d(\xi,c)$ is the trivial fibration with fibre $A_d(S^{d-1},c)$.

Let M be pointed and have non-empty boundary and let $L \subsetneq \partial M$ be a $(d-1)$-dimensional closed proper submanifold embedded in the boundary ∂M. Write

$$\Gamma^S((M,L);E_d(\xi,c))$$

for the sections of $E_d(\xi,c) \to M$ which restrict on $\{*\} \cup S$ to s_0 and on $\overline{\partial M \setminus L}$ take values in $\xi|_{\partial M} \subset E_d(\xi,c)|_{\partial M}$.

Theorem 10.12 *There is a weak homotopy equivalence*

$$\mathrm{C}\Gamma^{c,*}(M, L; \xi) \longrightarrow \Gamma((M, L); E_d(\xi, c)).$$

Proof. One proceeds by induction on a handle decomposition of M using the quasi-fibration sequence of Proposition 10.2.

As M is of dimension d and has boundary, we can choose a handle decomposition of M that only contains handles of index $k < d$. In the initial stage of our induction, the case when $M = D^d$, both spaces are contractible: Since M retracts via isotopies into (a collar neighborhood of) L, all configurations can be contracted to the empty configuration and the maps in the fibre are pointed maps from the disk D^d to X. Similarly, the section space consist of pointed maps to X.

Now assume that M is obtained from M' by attaching a k-handle $D^k \times D^{d-k}$ for $0 < k < d$ along $S^{k-1} \times D^{d-k}$ to the boundary of M'. As before in (10.1) we identify the d-disk with the d-cube. Then

$$M = M' \sqcup_{\partial'} I^d \quad \text{where} \quad \partial' = \partial I^k \times I^{d-k}.$$

We may assume that $L \subsetneq \partial M'$ does not intersect ∂'. Consider the following diagram:

$$\begin{array}{ccccc} \mathrm{C}\Gamma^{c',*\cup\partial'}(M', L; \xi') & \longrightarrow & \mathrm{C}\Gamma^{c,*}(M, L; \xi) & \longrightarrow & \mathrm{C}\Gamma^{c,*}(M, L \cup M'; \xi'') \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma^{\partial'}((M', L); E_d(\xi', c')) & \longrightarrow & \Gamma((M, L); E_d(\xi, c)) & \longrightarrow & \Gamma((I^d, \partial'); E_d(\xi'', c''))_{[\xi]}. \end{array} \tag{10.2}$$

Here ξ' and ξ'' denote the bundle ξ restricted to appropriate submanifolds of M, and similarly c' and c'' are restrictions of c. The upper row is a quasi-fibration by Proposition 10.2. Note that (M', L) is indeed connected.

In the lower row of diagram (10.2), the subscript $[\xi]$ on the right denotes the subspace of those

$$s'' \in \Gamma((I^d, \partial'); E_d(\xi'', c''))$$

that are restrictions of sections in $\Gamma((M, L); E_d(\xi, c))$. This selects entire connected components of the section space determined by the homotopy class in $\pi_0(\Gamma(\partial'; E_d(\xi'', c'')))$ defined by the restriction of s'' to $\partial' \simeq S^{k-1}$. In this case however, since $s''|_{\partial'}$ also has to extend over I^d, it must be nullhomotopic in the first place as a map to $A(X, d)$ and can thus be extended to a section over M. Thus

$$\Gamma((I^d, \partial'); E_d(\xi'', c))_{[\xi]} = \Gamma((I^d, \partial'); E_d(\xi'', c)).$$

If $s_0|_{I^d}$ denotes the base point in $\Gamma((I^d, \partial'); E_d(\xi'', c))$, then the fibre of the bottom right restriction map is simply given by those sections defined on M' that agree with s_0 on ∂'. This describes the section space on the left if we also remember the restriction imposed by L. Thus, also the lower row of diagram (10.2) is a fibration sequence.

The vertical maps of diagram (10.2) are the scanning maps and the diagram commutes. Consider the right down arrow. By Lemmas 10.6 and 10.7 we have

$$\begin{aligned} \Gamma((I^d, \partial'); E_d(\xi'', c'')) &\simeq \mathrm{Map}_*((I^d, I^k \times \partial I^{d-k}); (A_d(X, c), X)) \\ &\simeq \Omega^{d-k-1} A'(X, c) \\ &\simeq B_k(\mathrm{CMap}^{c,*}(I^d, \partial I^k \times I^{d-k}; X)) \\ &\simeq \mathrm{C}\Gamma^{c'',*}(I^d, \partial'; \xi''). \end{aligned}$$

Restricting both sets to those components with sections s'' that can be extended to all of M and noting

$$\mathrm{C}\Gamma^{c'',*}(I^d, \partial'; \xi'')_{[\xi]} \simeq \mathrm{C}\Gamma^{c,*}(M, L \cup M'; \xi)$$

shows that the right down arrow of diagram (10.2) is a weak homotopy equivalence.

Now consider the left down arrow of diagram (10.2). It is also the left down arrow of the following commutative diagram:

$$\begin{tikzcd}
\mathrm{C}\Gamma^{c',*\cup\partial'}(M',L;\xi') \arrow[r] \arrow[d] & \mathrm{C}\Gamma^{c',*}(M',L\cup\partial';\xi') \arrow[r, "|_{\partial'}"] \arrow[d, "\simeq"'] & \Gamma(\partial';\xi_{\partial'})_{[\xi']} \arrow[d, "="'] \\
\Gamma^{\partial'}((M',L);E_d(\xi',c')) \arrow[r] & \Gamma((M',L\cup\partial');E_d(\xi',c')) \arrow[r, "|_{\partial'}"] & \Gamma(\partial';\xi_{\partial'})_{[\xi']}.
\end{tikzcd}$$

The space on the right is the space of based sections of $\xi|_{\partial'} = \xi'|_{\partial'}$ that can be extended to sections of ξ' on $M' \smallsetminus z$ for some configuration z' in M' or, equivalently, to sections of $E_d(\xi', c')$. This is exactly the image of the maps $|_{\partial'}$ that restricts the sections s' to ∂' and have as fibers (over the constant map) the spaces on the left. The middle arrow is a weak homotopy equivalence by induction hypothesis, and hence so is the left down arrow of this diagram and of diagram (10.2).

Finally, invoking the Five Lemma, we see that also the middle arrow of diagram (10.2) is a weak homotopy equivalence. $\square$

10.4. Group completions and the stable homology. We now relate the homology of the spaces in Theorem 10.12 to the homology of the configuration section spaces for finite configurations.

We first recall the group completion theorem from [Qui94] [MS76].

Theorem 10.13 *Let $\mathbb{A}$ be a well-pointed topological monoid. The canonical map of a monoid into its (derived) group completion induces an isomorphism*

$$H_*(\mathbb{A})[\pi_0\mathbb{A}^{-1}] \simeq H_*(\Omega B\mathbb{A})$$

whenever the localisation can be constructed by left fractions, and in particular whenever $\pi_0(\mathbb{A})$ is in the centre of $H_*(\mathbb{A})$.

Let $\pi_0(\mathbb{A})$ be generated by $s_1, \dots, s_n$ and define $s := s_1 \dots s_n$. Then

$$H_*(\mathbb{A})[\pi_0(\mathbb{A})^{-1}] \simeq H_*(\mathbb{A})[s^{-1}] \simeq H_*(\mathbb{A}_\infty)$$

where $\mathbb{A}_\infty := \mathrm{Tel}(\mathbb{A} \xrightarrow{s} \mathbb{A} \xrightarrow{s} \dots)$ is the telescope (or homotopy colimit) on left multiplication by a representative of s in $\mathbb{A}$. One can construct a comparison map $\alpha : \mathbb{A}_\infty \to \Omega B\mathbb{A}$ following [MS76]: Consider the map $p : \mathbb{A}_\infty \times_{\mathbb{A}} E\mathbb{A} \to B\mathbb{A}$. The source space is the telescope of $\mathbb{A} \times_{\mathbb{A}} E\mathbb{A} \simeq *$, and hence is contractible itself. Under the conditions of the theorem, McDuff and Segal show that the map p is a homology fibration and the canonical map from the fibre to the homotopy fibre

$$\alpha : \mathbb{A}_\infty \longrightarrow \Omega B\mathbb{A}$$

is hence an H_*-isomorphism. If $\mathbb{A}$ is homotopy commutative [Ran13a] [MP15], or satisfies a somewhat weaker condition [Gri], then the map p is a homology fibration for all abelian coefficients and the map α is therefore acyclic. In particular, the fundamental groups of all components of $\mathbb{A}_\infty$ are perfect, and α induces a weak homotopy equivalence on the plus construction.

Example 10.14 Consider the monoids

$$\mathbb{A}_\square := \mathrm{CMap}^{c,\partial}(D^d; X) \quad \text{and} \quad \mathbb{A}_\sqcup := \mathrm{CMap}^{c,\frac{1}{2}\partial}(D^d; X).$$

For $d \geqslant 2$ the monoid $\mathbb{A}_\square$, and for $d \geqslant 3$ the monoid $\mathbb{A}_\sqcup$ are homotopy commutative by Proposition 4.10 and hence satisfy the conditions of the group completion theorem. Assuming π_0 is finitely generated, we have that α is acyclic and, by Corollary 10.8, for any (abelian) local coefficient system $\mathcal{L}$ the following are isomorphisms

(i) $H_*(\mathbb{A}_{\square,\infty}; \mathcal{L}) \simeq H_*(\Omega^d A(X,d); \mathcal{L})$ for $d \geqslant 2$;

(ii) $H_*(\mathbb{A}_{\sqcup,\infty}; \mathcal{L}) \simeq H_*(\Omega^{d-1} A'(X,d); \mathcal{L})$ for $d \geqslant 3$.

We will now generalise the above discussion from monoids $\mathbb{A}$ to modules $\mathbb{M}$ over $\mathbb{A}$, or in our main example from the disk D^d to more general manifolds M. Let $\mathbb{A}$ be homotopy commutative, s be a product of generators of $\mathbb{A}$ and let $\mathbb{M}$ be an $\mathbb{A}$-module. Using the $\mathbb{A}$-module structure define the telescope

$$\mathbb{M}_\infty := \mathrm{Tel}(\mathbb{M} \xrightarrow{s} \mathbb{M} \xrightarrow{s} \dots).$$

Then arguing as before, the map $p : \mathbb{M}_\infty \times_{\mathbb{A}} E\mathbb{A} \to B\mathbb{A}$ is a homology-fibration. Thus the canoncial map

$$\mathbb{M}_\infty \longrightarrow \mathrm{hofib}(p : \mathbb{M}_\infty \times_{\mathbb{A}} E\mathbb{A} \to B\mathbb{A}) \tag{10.3}$$

from the fibre to the homotopy fibre of p is a homology isomorphism, compare [MS76] [MP15].

Returning to our main example of configuration-section spaces, let $\xi\colon E \to M$ be a fibre bundle over a connected manifold M of dimension $d \geqslant 2$ with path-connected fibre X. Let D denote a $(d-1)$-dimensional disc in the boundary of M. Define

$$\mathbb{M} := \mathrm{C}\Gamma^{c,D}(M;\xi) = \bigsqcup_{k\geqslant 0} \mathrm{C}\Gamma_k^{c,D}(M;\xi).$$

Theorem 10.15 *For $d \geqslant 3$, there are isomorphisms*

$$H_*(\mathbb{M}_\infty) \simeq H_*(\Gamma(M; E_d(\xi, c))).$$

By Proposition 4.10, $\mathbb{M}$ is a module over both $\mathbb{A}_\square$ and $\mathbb{A}_\sqcup$. To identify the homotopy fibre of p we will consider the $\mathbb{A}_\sqcup$ module structure, and hence have to assume $d \geqslant 3$ to ensure that $\mathbb{A}_\sqcup$ is homotopy commutative.

Proof. Using diagram (10.4) below we will identify the homotopy fiber of p as the space in (10.5) below. This holds for $d \geqslant 2$. When $d \geqslant 3$, p is a homology fibration and the homology equivalence (10.3) implies the theorem.

Let $M_1 = M \cup_D I^d$ be the manifold from Definition 4.11 and write D_1 for the copy $\{1\} \times I^{d-1}$ of $D = \{0\} \times I^{d-1}$ in the boundary of M_1. Using similar notation as in diagram (10.2), we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{M}_\infty \times_{\mathbb{A}_\sqcup} E\mathbb{A}_\sqcup \arrow[r, "\simeq"] \arrow[d, "p"] & \mathrm{C}\Gamma^{c_1,*}((M_1, D_1);\xi_1) \arrow[r, "\simeq"] \arrow[d, "\pi"] & \Gamma((M_1, D_1); E_d(\xi_1, c_1)) \arrow[d, "res"] \\
B\mathbb{A}_\sqcup \arrow[r, "\simeq"] & \mathrm{C}\Gamma^{c',*}((I^d, D \cup D_1);\xi') \arrow[r, "\simeq"] & \Gamma((I^d, D \cup D_1); E_d(\xi', c')).
\end{tikzcd} \tag{10.4}$$

The vertical map π is the quotient map of Proposition 10.2 which lets configuration points in M disappear. The right vertical map restricts sections on M_1 to I^d. Reasoning as for diagram (10.2), res is surjective and has fibre

$$\Gamma^D(M, D; E(\xi, c)) \simeq \Gamma(M, D; E(\xi, c)) \qquad (\text{as } * \in D \simeq *). \tag{10.5}$$

By Theorem 10.12 the two horizontal maps on the right are weak homotopy equivalences and the right square commutes up to homotopy. Indeed, let $S = \frac{1}{2}\partial \setminus D$ with $* \in S$ and replace the configuration-section spaces and the section spaces by those decorated by S. With these replacements the square commutes.

By by part (ii) of Lemma 10.5 we have homotopy equivalences

$$B\mathbb{A}_\sqcup \simeq \mathrm{C}\Gamma^{c',\frac{1}{2}\partial}((I^d, D \cup D_1);\xi') \simeq \mathrm{C}\Gamma^{c',*}((I^d, D \cup D_1);\xi').$$

The proof of part (ii) of Lemma 10.5 generalises to give also a homotopy equivalence

$$\mathbb{M} \times_{\mathbb{A}_\sqcup} E\mathbb{A}_\sqcup \simeq \mathrm{C}\Gamma^{c,\frac{1}{2}\partial}(M^+, D^+;\xi) \simeq \mathrm{C}\Gamma^{c,*}(M^+, D^+;\xi).$$

The stabilisation maps induce a homotopy equivalence on these spaces, and hence also the top left horizontal map is a homotopy equivalence. The left square clearly commutes. ◻

Note that we did not need to make any restriction on c to apply the group completion theorem. We also note that diagram (10.4) holds for all $d \geqslant 2$ but we need homotopy commutativity of $\mathbb{A}_{\sqcup}$ in order to deduce that p is a homology fibration. However, if we know independently that p is a homology fibration, as for example when we know that the components of $\mathbb{M}$ satisfy homology stability, the conclusion of the theorem still holds. Thus combining the above with Theorem 8.5 we can identify the stable homology of the finite configuration-section spaces in the following cases.

Corollary 10.16 *Suppose that $d \geqslant 2$ and the subset $\widetilde{c}_D \subseteq \pi_{d-1}(X)$ has size 1. Then the scanning maps*

$$\mathrm{C}\Gamma_k^{c,D}(M;\xi) \longrightarrow \Gamma(M;E_d(\xi,c))_{[k]}$$

induce isomorphisms on integral homology up to degree $\frac{k}{2}-2$ and surjections up to degree $\frac{k}{2}-1$. With coefficients in a field, both of these ranges may be improved by one. Here the subscript $[k]$ on the right indicates those components that intersect non-trivially with the image.

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