A unified functorial construction of homological representations of families of groups

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Abstract

The families of braid groups, surface braid groups, mapping class groups and loop braid groups have a representation theory of "wild type", so it is very useful to be able to construct such representations *topologically*, so that they may be understood by topological or geometric methods. For the braid groups $\mathbf{B}_n$, Lawrence and Bigelow have constructed families of representations starting from actions of $\mathbf{B}_n$ on the twisted homology of configuration spaces. These were then used by Bigelow and Krammer to prove that the braid groups are linear.

We develop a general underlying procedure to build homological representations of families of groups, encompassing all of the above-mentioned families and in principle many more, such as families of general motion groups. Moreover, these families of representations are *coherent*, in the sense that they extend to a functor on a larger category, whose automorphism groups are the family of groups under consideration and whose richer structure may be used (i) to organise the representation theory of the family of groups and (ii) to prove twisted homological stability results — both via the notion of *polynomiality*. We prove polynomiality for many such *homological functors*, including those (which we construct) extending the Lawrence-Bigelow representations.

This helps to unify previously-known constructions and to produce new families of representations — we do this for the loop braid groups, surface braid groups and mapping class groups. In particular, for the loop braid groups, we construct three analogues of the Lawrence-Bigelow representations (of the classical braid groups), which appear to be new.

Contents

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1 Introduction

Homological representations of families of groups.

A family of groups, namely a set of groups $\{G_n\}_{n\in\mathbb{N}}$ equipped with injections

$$\gamma_n\colon G_n \hookrightarrow G_{n+1},$$

is said to have *wild* representation theory if the indecomposable representations occur in families with at least two parameters. There is no classification schema in these cases. Such families of groups naturally arise in many situations in connection with topology: for instance the families of *braid groups*, *mapping class groups* and *loop braid groups* all have wild representation theory. Finding geometric or topological constructions of linear representations for these groups thus helps to understand and organise their representation theory, since one then has the tools of topology and geometry available.

For the family of braid groups $\mathbf{B}_n$ (where the inclusions are induced by adding a strand on the left), Lawrence [Law90] and Bigelow [Big01] constructed well-known families of linear representations, called the *Lawrence-Bigelow representations*, following different methods. They may be defined via actions on twisted homology groups of configuration spaces of unordered points in a punctured 2-disc. The most famous families among them are the family of (reduced) *Burau representations* originally introduced in [Bur35] and the family of *Lawrence-Bigelow representations* that Bigelow [Big01] and Krammer [Kra02] independently proved to be faithful (thus proving that the braid groups are *linear*). The keystone for constructing the Lawrence-Bigelow representations is actually based on a much more general underlying method. The first aim of this paper is to develop this general method in a larger context, as a procedure for constructing *homological representations* for any family of groups.

Coherent families of representations. A natural goal is to extend these constructions so as to deal with representations satisfying some compatibility, or coherence, conditions. More precisely, rather than considering a representation just for one group G_n, we are interested in collections of linear representations

$$\{\varrho_n : G_n \to GL_R(M_n)\}_{n\in\mathbb{N}}$$

satisfying the compatibility condition that the restriction of ϱ_{n+1} to G_n, with respect to some preferred maps $m_n : M_n \to M_{n+1}$, is ϱ_n. Then we say that the representations $\{M_n\}_{n\in\mathbb{N}}$ form a *family* of linear representations of the groups $\{G_n\}_{n\in\mathbb{N}}$. This notion can be encoded in a functorial way. Let $\mathcal{G}$ be the groupoid with objects indexed by natural numbers, with the groups $\{G_n\}_{n\in\mathbb{N}}$ as automorphism groups and with no morphisms between distinct objects. For instance, we consider the braid groupoid $\boldsymbol{\beta}$ to deal with braid groups and the decorated surfaces groupoid $\mathcal{M}_2$ for the mapping class groups of surfaces (see §3). We assume the existence of a category $\mathcal{C}_{\mathcal{G}}$ containing $\mathcal{G}$ as its underlying groupoid and with a preferred morphism $n \to n+1$ for each object n. In all the examples addressed in this paper, such a category $\mathcal{C}_{\mathcal{G}}$ is constructed through *Quillen's bracket construction* using a monoidal structure on the groupoid $\mathcal{G}$ (see §3). Then, denoting by R-Mod the category of R-modules (for R a commutative ring), considering a functor from $\mathcal{C}_{\mathcal{G}}$ to R-Mod is equivalent to considering a family of representations of the family of groups $\{G_n\}_{n\in\mathbb{N}}$.

Functorial homological constructions via topological categories. Our general procedure for defining homological representations is summarised in the diagram below.

$$\begin{tikzcd} \mathcal{C}^{\mathrm{t}}_{\mathcal{G}} \arrow[r, "F"] \arrow[d, "\pi_0"'] & \mathrm{Cov}_Q \arrow[r, "\mathrm{Lift}"] & \mathrm{Top}_{k[Q]} \arrow[r, "\otimes M"] & \mathrm{Top}_R \arrow[r, "H_i"] & R\text{-Mod} \\ \mathcal{C}_{\mathcal{G}} \arrow[urrrr, dashed, "L_i(F;M)"'] & & & & \end{tikzcd} \tag{1.1}$$

The desired output is the diagonal functor $\mathcal{C}_{\mathcal{G}} \to R$-Mod, a (coherent) family of representations of $\{G_n\}_{n\in\mathbb{N}}$. This is constructed in five steps:

- Constructing a topological category $\mathcal{C}^{\mathrm{t}}_{\mathcal{G}}$ whose π_0 recovers $\mathcal{C}_{\mathcal{G}}$. The category $\mathcal{C}_{\mathcal{G}}$ is typically constructed via Quillen's bracket construction from a braided monoidal groupoid, and we explain in §3.2 how this construction may be lifted to topological categories to produce an appropriate $\mathcal{C}^{\mathrm{t}}_{\mathcal{G}}$. Its morphism spaces are typically embedding spaces between manifolds. This makes the next step very natural to define:
- The key geometric input for the construction is a choice of functor F, defined on $\mathcal{C}^{\mathrm{t}}_{\mathcal{G}}$ and taking values in Cov_Q, the category of topological spaces equipped with regular coverings with fixed deck transformation group Q.
- The remaining steps encode in a general setting the idea of taking twisted homology of covering spaces: the functor Lift takes a regular covering with deck transformation group Q to the corresponding bundle of $k[Q]$-modules, the functor $\otimes M$ takes the fibrewise tensor product with a $(k[Q], R)$-bimodule M ("specialising the coefficients"), producing a bundle of R-modules, and finally H_i is simply the twisted homology functor in degree i.

There are also variants of this construction for *reduced homology* (where we work with categories of pairs of spaces) and for *Borel-Moore homology* (where we restrict to categories of spaces and proper maps). In particular, using Borel-Moore homology is especially interesting when the image of the functor F consists of *configuration spaces of points in a surface*: a general result then gives free generating sets for the resulting twisted Borel-Moore homology groups and a fortiori a better understanding of the constructed representations (see §6).

Many of these representations have been defined and studied before (at least at the level of individual groups, i.e., when restricted to the individual automorphism groups of $\mathcal{C}_{\mathcal{G}}$) — and indeed one purpose of describing this general procedure for constructing homological representations is to give a *unified* description for various different representations appearing in the literature, as well as suggesting new constructions by comparing representations coming from different settings in this unified context.

In §5 we discuss the representations that are recovered as part of this general homological construction. This includes the *Lawrence-Bigelow representations* for braid groups, the *Moriyama representations* [Mor07] of the mapping class groups of the smooth connected compact orientable surface of genus g and with one boundary component $\{\Sigma_{g,1}\}_{g\in\mathbb{N}}$, and the *Long-Moody construction* [Lon94]. We prove:

Theorem 1.1 *Let $m \geqslant 1$ be a fixed non-negative integer. There exist homological functors:*

- $\mathfrak{LB}_m^{BM}$ *encoding the* m*-th Lawrence-Bigelow representations, defined on a category having* $\boldsymbol{\beta}$ *as its underlying groupoid;*
- $\mathfrak{Mor}_m$ *encoding the* m*-th Moriyama representations, defined on a category having* $\mathcal{M}_2$ *as its underlying groupoid;*
- *encoding each generalised Long-Moody functor introduced in* [Sou18].

We also construct *new* families of representations for the surface braid groups and the loop braid groups analogous to those of Lawrence-Bigelow for classical braid groups, and new families of representations of mapping class groups of surfaces (see §4). We highlight here three analogues of the Lawrence-Bigelow representations for the loop braid groups (see §4.5 for more details).

Theorem 1.2 (see Theorems 4.24, 4.25 and 4.26) *For any integers* $m \geqslant 1$ *and* $i \geqslant 0$*, we construct homological functors*

$$L_i(F_m^{\alpha})\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta} \longrightarrow \mathbb{Z}[Q_m^{\alpha}]-\mathrm{Mod},$$
$$L_i(F_m^{\beta})\colon \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}}) \longrightarrow \mathbb{Z}[Q_m^{\beta}]-\mathrm{Mod},$$
$$L_i(F_m^{\gamma})\colon \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}}) \longrightarrow \mathbb{Z}[Q_m^{\gamma}]-\mathrm{Mod},$$

defined over the group-rings of:

- $Q_1^{\alpha} = \mathbb{Z}$, $Q_1^{\beta} = (\mathbb{Z}/2\mathbb{Z})^2$, $Q_1^{\gamma} = \mathbb{Z}/2\mathbb{Z}$, *and*
- $Q_m^{\alpha} = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$, $Q_m^{\beta} = (\mathbb{Z}/2\mathbb{Z})^4$, $Q_m^{\gamma} = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$, *for* $m \geqslant 2$.

In particular, these give coherent families of representations, over rings of Laurent polynomials, of the loop braid groups $\{\mathbf{LB}_n\}_{n\geqslant 1}$ *(for the first) and of the extended loop braid groups* $\{\mathbf{LB}_n^{\mathrm{ext}}\}_{n\geqslant 1}$ *(for the second and third).*

Polynomiality. On another note, under some assumptions on $\mathcal{C}_{\mathcal{G}}$, notions of *polynomiality* on the objects in the category $\mathbf{Fct}\,(\mathcal{C}_{\mathcal{G}}, R\text{-}\mathfrak{Mod})$ are introduced and of particular importance. The first notions of polynomial functors date back to Eilenberg and Mac Lane in [EM54]. Djament and Vespa [DV19] introduce *strong* and *weak* polynomial functors for symmetric monoidal categories in which the monoidal unit is initial. These definitions are extended to *pre-braided* monoidal categories in which the monoidal unit is initial in [Sou19; Sou18] and the notion of *very strong* polynomial functor in this context is introduced there. It is equivalent to the notion of *coefficient systems of finite degree* of [RW17]. All these notions of polynomiality straightforwardly extend to the slightly more general context of the present paper (see §7); various instances of these notions are studied in [Pal17].

Applications. The main motivation for our interest in very strong polynomial functors is their homological stability properties: Randal-Williams and Wahl [RW17] prove homological stability results for certain families of groups with twisted coefficients given by very strong polynomial functors (see §8.3.1). In particular their results hold for surface braid groups, mapping class groups of surfaces and loop braid groups. On the other hand, a first matter of interest in weak polynomial functors is that weak polynomiality reflects more accurately the stable behaviour of a given functor, than (very strong polynomiality). Also, contrary to (very) strong polynomial functors, weak polynomial functors of degree less or equal to some $d \in \mathbb{N}$ form a category $\mathcal{P}ol_d\,(\mathcal{C}_{\mathcal{G}})$ that is localizing in $\mathcal{P}ol_{d+1}\,(\mathcal{C}_{\mathcal{G}})$: this allows one to define quotient categories, which provide an organizing tool for families of representations (see §8.3.2). The representation theories of the families of groups that we study in this paper are wild and an active research topic (see for instance [BB05, Section 4.6], [Fun99], [Kor02] or [MR12]). Therefore the strong, very strong and weak polynomial functors associated with these groups are not well-understood. Hence, Theorem 1.3 offers a better understanding of these kinds functors and extends the scope of twisted homological stability to more sophisticated sequences of representations.

Theorem 1.3 *The* m*-th Lawrence-Bigelow functor* $\mathfrak{LB}_m^{BM}$ *is both very strong and weak polynomial of degree* m *if* $m \geqslant 2$*, and* $\mathfrak{LB}_1^{BM}$ *is strong polynomial of degree* 2 *but weak polynomial of degree* 1*.*

The Moriyama functor $\mathfrak{Mor}_m$ *is both very strong and weak polynomial of degree* m*.*

Outline. The paper is organised as follows. In §2, we introduce in detail the various tools to define homological functors, as summarised in Diagram (1.1). In §3, we recall Quillen's bracket construction, introduce its generalisation to deal with topological categories and then construct the various categories associated with the families of groups which will be used to apply our construction. We then apply the general construction of homological functors to the families of *classical and surface braid groups*, *mapping class groups of surfaces* and *loop braid groups* in §4. The next section §5 is devoted to recovering previously-known families of representations in the literature from our homological construction. We then recall and slightly extend a key general result for Borel-Moore homology of configuration spaces in §6. In §7 we introduce the notions of strong, very strong and weak polynomial functors in the present framework. Finally, we prove Theorem 1.3 and explain the applications of these polynomial properties in §8.

2 The general construction

2.1 Overview

The general construction of *homological functors* (also called *homological representations* when the source is a group) that we explore in this paper can be summarised as follows:

$$\begin{tikzcd} \mathcal{C}^{\mathrm{t}} \arrow[r, "F"] \arrow[d] & \mathrm{Cov}_Q \arrow[r, "\mathrm{Lift}"] & \mathrm{Top}_{k[Q]} \arrow[r, "\otimes M"] & \mathrm{Top}_R \arrow[r, "H_i"] & R\text{-Mod}, \\ \mathcal{C} = \pi_0(\mathcal{C}^{\mathrm{t}}) \arrow[urrrr, dashed] & & & & \end{tikzcd} \tag{2.1}$$

where $\mathcal{C}$ is the category on which we are interested in defining a representation and $\mathcal{C}^{\mathrm{t}}$ is a topological category recovering $\mathcal{C}$ on π_0. (See (2.26) in §2.5 for the full diagram.)

Categories: For a group Q, the objects of the category Cov_Q are based, path-connected spaces X (admitting a universal cover) equipped with a surjection $\pi_1(X) \twoheadrightarrow Q$, and morphisms are based maps commuting with these quotients. For a ring R, Top_R denotes the category of topological spaces equipped with bundles of R-modules, and bundle maps. In each case, these are *topological* categories, using the compact-open topology for the mapping spaces.

Functors: The continuous functor Lift takes an object $(X, \varphi\colon \pi_1(X) \twoheadrightarrow Q)$ to the regular covering $X^{\varphi} \to X$ to which it corresponds, viewed as a bundle of Q-sets, and then takes free k-modules fibrewise, producing a bundle of $k[Q]$-modules over X. The functor $\otimes M$ is given by fibrewise tensor product with a $(k[Q], R)$-bimodule M and the right-hand functor is twisted homology in degree i.

The idea is that, in each case of interest, we may construct interesting representations of $\mathcal{C}$ by:

(a) choosing a topological category $\mathcal{C}^{\mathrm{t}}$ with $\pi_0(\mathcal{C}^{\mathrm{t}}) = \mathcal{C}$ and a continuous functor $F\colon \mathcal{C}^{\mathrm{t}} \to \mathrm{Cov}_Q$, via some geometric construction,
(b) choosing "coefficients" (a ground ring k and a $(k[Q], R)$-bimodule M) and the degree i in which to take twisted homology.

This concentrates the interesting geometrical part of the construction, and automates the construction of representations of $\mathcal{C}$ based on this, for any choice of coefficients and homological degree.

Twisted homomorphisms: In many interesting cases, there are interesting geometrically-defined continuous functors of the form $F\colon \mathcal{C}^{\mathrm{t}} \to \widetilde{\mathrm{Cov}}_Q$, where $\widetilde{\mathrm{Cov}}_Q$ is the larger category whose objects are $(X, \varphi\colon \pi_1(X) \twoheadrightarrow Q)$ as before, but whose morphisms $(X, \varphi) \to (Y, \psi)$ are based maps $f\colon X \to Y$ satisfying the weaker condition that $f_*(\ker(\varphi)) \subseteq \ker(\psi)$. We will also consider a more general construction, taking as input such functors, and resulting in a representation of $\mathcal{C}$ on $R\text{-Mod}^{\mathrm{tw}}$, the category of R-modules and twisted homomorphisms (also called *crossed homomorphisms*).

Outline: In this section, we define precisely the (topological) categories and (continuous) functors in diagram (2.1) and its enlargement involving $\widetilde{\mathrm{Cov}}_Q$. This defines step (b) of the general procedure outlined above. In more detail, in §2.2 we discuss categories of bundles of modules and twisted bundle maps, and the fibrewise tensor product and twisted homology functors in this setting (for a brief summary of this structure, see Remark 2.5). In §2.3 we then describe carefully the functor $\mathrm{Lift}\colon \mathrm{Cov}_Q \to \mathrm{Top}_{k[Q]}$ and its twisted version. In §3.4 and §4 we then perform the more geometric step (a) in a variety of interesting examples, where $\mathcal{C}$ is a category corresponding to an interesting family of groups, including surface braid groups, mapping class groups of surfaces and loop braid groups. In §3.4 we construct the appropriate topological categories $\mathcal{C}^{\mathrm{t}}$, and in §4 we construct, geometrically, the functors $F\colon \mathcal{C}^{\mathrm{t}} \to \mathrm{Cov}_Q$. Applying the above procedure, we therefore obtain interesting "homological representations" of the categories $\mathcal{C}$. In §5, we review the already existing homological representations at the level of individual groups which are recovered using this functorial approach. Therefore this general construction allows us to extend known representations of a family of groups to a functor of an interesting category containing all of the groups in the family. The *polynomial* properties of this functor are discussed in §8.

2.2 Twisted module homomorphisms and twisted homology

In this section, we define precisely the categories of "spaces equipped with bundles of modules over bundles of algebras" that will appear in the intermediate steps of the more general version of the construction (2.1) – see Definition 2.1. We fix a (unital, commutative) ground ring k (which is typically $\mathbb{Z}$ in our applications) and use the viewpoint that bundles of k-modules over X are functors from the fundamental groupoid of X to the category of k-modules.[1]

Categories and functors. Let us write SMCat for the category of symmetric monoidal functors, and recall that we write Cat and Top for the categories of small categories and of topological spaces respectively. Write

$$U\colon \mathrm{SMCat} \longrightarrow \mathrm{Cat}$$

for the obvious forgetful functor. For a fixed $\mathcal{C} \in \mathit{Obj}(\mathrm{SMCat})$ there is a functor

$$\mathrm{Loc}_{\mathcal{C}}\colon \mathrm{Top}^{\mathrm{op}} \longrightarrow \mathrm{SMCat}$$

given on objects by $X \mapsto \mathbf{Fct}(\Pi_1(X), \mathcal{C})$, where $\Pi_1(X)$ is the fundamental groupoid of a space X. We think of this as the *category of $\mathcal{C}$-local sysetms on X*. This is motivated by the fact, mentioned in footnote 1 on page 6, that when $\mathcal{C}$ is the category of modules over a commutative ring k (with monoidal structure given by the tensor product over k) and the path-components of X admit universal covers, then the objects of $\mathrm{Loc}_{\mathcal{C}}(X)$ are in natural bijection with bundles of k-modules over X.

Monoid and module objects. Given a symmetric monoidal category $\mathcal{C}$, objects in $\mathcal{C}$ equipped with a certain structure, for example monoid objects, may be thought of as symmetric monoidal functors into $\mathcal{C}$ from appropriate finitely-presented symmetric monoidal categories.

For example, consider the symmetric monoidal category *Monoid* defined by the following presentation: it has one generating object a and two generating morphisms $\mu\colon a \otimes a \to a$ and $u\colon 1 \to a$, where 1 denotes the empty monoidal product, with defining relations:

$$(\mathrm{id} \otimes m) \circ m = (m \otimes \mathrm{id}) \circ m \qquad m \circ (\mathrm{id} \otimes u) = \mathrm{id} \qquad m \circ (u \otimes \mathrm{id}) = \mathrm{id}. \tag{2.2}$$

There is then an endofunctor

$$\mathit{Mon}\colon \mathrm{SMCat} \longrightarrow \mathrm{SMCat} \tag{2.3}$$

defined by $\mathcal{C} \longmapsto \mathbf{Fct}^{\otimes}(\mathit{Monoid}, \mathcal{C})$, where $\mathbf{Fct}^{\otimes}$ denotes the category of strict symmetric monoidal functors, which we think of as the "category of monoid objects in $\mathcal{C}$".

[1] This can be taken as a definition of "bundles of k-modules", and, when X is locally path-connected and semi-locally simply-connected (i.e. when each path-component of X admits a univeral cover), it is equivalent to the more usual geometric definition, via the correspondence that associates to a bundle of k-modules (in the usual sense) over X the functor $\Pi_1(X) \to k\text{-Mod}$ defined using unique path-lifting (which holds since a bundle of k-modules is in particular a covering space).

Similarly, we may consider the symmetric monoidal category *Module* with the following presentation: it has two generating objects a and b and three generating morphisms $l\colon a\otimes b\to b$, $\mu\colon a\otimes a\to a$ and $u\colon 1\to a$, with defining relations (2.2) together with:

$$l\circ(\mathrm{id}_a\otimes l)=l\circ(\mu\otimes\mathrm{id}_b). \tag{2.4}$$

There is then an endofunctor

$$\mathit{Mod}\colon \mathrm{SMCat}\longrightarrow\mathrm{SMCat} \tag{2.5}$$

defined by $\mathcal{C}\longmapsto \mathbf{Fct}^{\otimes}(\mathit{Module},\mathcal{C})$, which we think of as the "category of monoid objects in $\mathcal{C}$ together with a module object".

Categories of structured bundles. Denoting by $\mathrm{Cat}/\mathcal{C}$ the slice category of Cat over $\mathcal{C}$, recall that the *Grothendieck construction* can be viewed as a functor

$$\textstyle\int\colon \mathbf{Fct}(\mathcal{C}^{\mathrm{op}},\mathrm{Cat})\longrightarrow\mathrm{Cat}/\mathcal{C},$$

which, on objects, takes a functor $F\colon\mathcal{C}^{\mathrm{op}}\to\mathrm{Cat}$ to the functor $\int F\to\mathcal{C}$, where

$$\textstyle \mathit{Obj}(\int F)=\{(c,a)\mid c\in \mathit{Obj}(\mathcal{C}), a\in\mathit{Obj}(F(c))\}$$

and a morphism in $\int F$ from (c,a) to (c',a') is a morphism $f\colon c\to c'$ in $\mathcal{C}$ and a morphism $g\colon a\to F(f)(a')$ in $F(c)$. The functor $\int F\to\mathcal{C}$ simply takes (c,a) to c on objects and (f,g) to f on morphisms.

Applying this to the functor

$$U\circ\mathrm{Loc}_{k\text{-Mod}}\colon\mathrm{Top}^{\mathrm{op}}\longrightarrow\mathrm{Cat},$$

we obtain a category $\int U\circ\mathrm{Loc}_{k\text{-Mod}}$ equipped with a forgetful functor to the category of topological spaces – this may be thought of as the category of *topological spaces equipped with a bundle of k-modules.*[2]

We may equally well apply this to the functor

$$U\circ E\circ\mathrm{Loc}_{k\text{-Mod}}\colon\mathrm{Top}^{\mathrm{op}}\longrightarrow\mathrm{Cat},$$

where E is any endofunctor of SMCat. When $E=\mathit{Mon}$ we obtain the category $\int U\circ\mathit{Mon}\circ\mathrm{Loc}_{k\text{-Mod}}$ of *topological spaces equipped with a bundle of k-algebras*, and when $E=\mathit{Mod}$ we instead obtain the category $\int U\circ\mathit{Mod}\circ\mathrm{Loc}_{k\text{-Mod}}$ of *topological spaces equipped with a bundle of k-algebras together with a bundle of modules over this bundle of k-algebras.* To be more concise, we abbreviate the latter to *topological spaces equipped with a bundle of twisted k-modules.* These are each equipped with a forgetful functor to Top that remembers just the underlying space of the bundle(s).

The constructions and discussion of this section so far may be summarised as follows.

Definition 2.1 (*Categories of structured bundles.*) Let *Strc* be a symmetric monoidal category representing a certain structure (such as the finitely-presented *Monoid* or *Module* described above). Then the Grothendieck construction

$$\textstyle\int U\circ\mathbf{Fct}^{\otimes}(\mathit{Strc},-)\circ\mathrm{Loc}_{k\text{-Mod}}$$

is the category of *topological spaces equipped with a bundle of k-modules equipped with the structure Strc*. In particular, for the structure $\mathit{Strc}=\mathit{Module}$ we define

$$\textstyle\mathrm{Top}_{(k)}\coloneqq\int U\circ\mathit{Mod}\circ\mathrm{Loc}_{k\text{-Mod}}, \tag{2.6}$$

and call this the category of *topological spaces equipped with bundles of twisted k-modules.* Expanding the definition, this may also be written as

$$\textstyle\mathrm{Top}_{(k)}=\int\mathbf{Fct}^{\otimes}(\mathit{Module},\mathbf{Fct}(\Pi_1(-),k\text{-Mod})).$$

In particular, $\mathrm{Top}_{(\mathbb{Z})}$ is the category of spaces equipped with *bundles of twisted $\mathbb{Z}$-modules*, i.e. bundles of rings together with bundles of modules over that ring-bundle.

[2] When restricting to the subcategory of topological spaces whose path-components each admit a universal cover, this is literally true. In general, we simply choose to view a bundle of k-modules over a space X as a functor $\Pi_1(X)\to k$-Mod instead of its more usual definition. This is reasonable, since, in our applications, we will only ever deal with spaces admitting universal covers.

Restricted subcategories. The intermediate categories used in the construction of homological functors will be certain subcategories of $\mathrm{Top}_{(k)}$, in which we fix either an underlying space or a trivial bundle of k-modules. In this subsection, we define these subcategories and discuss the structures on these categories that we will use, namely a functor (*twisted homology*) to a category of modules and a symmetric monoidal structure (*fibrewise tensor product*). The diagram of subcategories is summarised in (2.7).

Definition 2.2 (*Restricted categories of twisted k-modules.*) We define subcategories of $\mathrm{Top}_{(k)}$ given (a) by fixing the underlying space, and considering pairs of bundles over a fixed space X, (b) by fixing a k-algebra R and requiring the bundle of k-algebras to be the trivial bundle with fibre R and (c) fixing both the space X and the k-algebra R and allowing only the bundle of R-modules over X to vary.

(a) For any functor $F\colon \mathcal{C} \to \mathrm{Cat}$ and object c of $\mathcal{C}$, there is a canonical inclusion of categories

$$F(c) \longhookrightarrow \textstyle\int F$$

given on objects by $a \mapsto (c, a)$. In the case of $\mathrm{Top}_{(k)}$, this means that, for any space X, there is a canonical inclusion of categories

$$\mathrm{Top}_{(k);X} \coloneqq \mathit{Mod}(\mathrm{Loc}_{k\text{-}\mathrm{Mod}}(X)) \longhookrightarrow \mathrm{Top}_{(k)}.$$

Note that $\mathrm{Top}_{(k);X}$ is a symmetric monoidal category.

(b) To define the subcategory $\mathrm{Top}_{(k);R} \subset \mathrm{Top}_{(k)}$ for a given k-algebra R, we first define a subfunctor $F_R\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{SMCat}$ of the functor $\mathit{Mod} \circ \mathrm{Loc}_{k\text{-}\mathrm{Mod}}$, where by *subfunctor* we mean that $F_R(X)$ is a subcategory of $\mathit{Mod}(\mathrm{Loc}_{k\text{-}\mathrm{Mod}}(X))$ for every space X and that $F_R(f)$ is the restriction of the functor $\mathit{Mod}(\mathrm{Loc}_{k\text{-}\mathrm{Mod}}(f))$ to the subcategory $F_R(X)$ for every continuous map $f\colon X \to Y$.

To define this, we need to choose a subcategory $F_R(X)$ of

$$\mathit{Mod}(\mathrm{Loc}_{k\text{-}\mathrm{Mod}}(X)) = \mathbf{Fct}^{\otimes}(\mathit{Module}, \mathbf{Fct}(\Pi_1(X), k\text{-}\mathrm{Mod}))$$

for each space X. We specify this category to be the full subcategory of those strict symmetric monoidal functors $\mathit{Module} \to \mathbf{Fct}(\Pi_1(X), k\text{-}\mathrm{Mod})$ that take the object a of Module to the constant functor $\Pi_1(X) \to k\text{-}\mathrm{Mod}$ at the k-module R. Note that:

- An object of $F_R(X)$ is a bundle of R-modules over X.
- A morphism of $F_R(X)$ from E_1 to E_2 is an endomorphism θ of R (as a k-module) together with a morphism $E_1 \to E_2$ of bundles of k-modules, that also respects the action of R on E_1 and E_2, modulo the endomorphism θ.
- The category of bundles of R-modules over X (and morphisms of bundles of R-modules) is therefore the subcategory of $F_R(X)$ consisting of those morphisms for which $\theta = \mathrm{id}_R$.

To make the last point precise, let $\mathrm{Top}_{X,R} = \mathbf{Fct}(\Pi_1(X), R\text{-}\mathrm{Mod})$ be the category of bundles of R-modules over the space X, and form the Grothendieck construction $\mathrm{Top}_R = \int \mathbf{Fct}(\Pi_1(-), R\text{-}\mathrm{Mod})$ to define the category of bundles of R-modules over spaces. Then $\mathrm{Top}_{X,R} \subset F_R(X)$ is the subcategory described in the last point above.

We now define

$$\mathrm{Top}_{(k);R} \coloneqq \textstyle\int U \circ F_R.$$

This contains Top_R as the subcategory (on the same objects) of bundles of R-modules, with all *untwisted* morphisms.

(c) Finally, we define

$$\mathrm{Top}_{(k);X,R} \coloneqq F_R(X).$$

This is by definition a subcategory of $\mathrm{Top}_{(k);X} = \mathit{Mod}(\mathrm{Loc}_{k\text{-}\mathrm{Mod}}(X))$, and it also has a canonical embedding into $\mathrm{Top}_{(k);R} = \int U \circ F_R$, by the general properties of the Grothendieck construction mentioned above. It also contains the subcategory $\mathrm{Top}_{X,R}$, as mentioned above.

Summarising the constructions above, we have a diagram of subcategories:

$$\begin{tikzcd}
\mathrm{Top}_{(k)} & \mathrm{Top}_{(k);R} \arrow[l, hook'] & \mathrm{Top}_R \arrow[l, hook'] \\
\mathrm{Top}_{(k);X} \arrow[u, hook] & \mathrm{Top}_{(k);X,R} \arrow[l, hook'] \arrow[u, hook] & \mathrm{Top}_{X,R} \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd} \tag{2.7}$$

Remark 2.3 Intuitively, the objects of $\mathrm{Top}_{(k)}$ may be thought of as "parametrised families of modules over k-algebras". The three subcategories in the left-hand square above correspond to fixing either the k-algebra R, or the space X parametrising the family, or both. However, we still allow all morphisms of $\mathrm{Top}_{(k)}$ (they are *full* subcategories), so for example a morphism in $\mathrm{Top}_{(k);R}$ between families of R-modules may involve a non-identity k-linear endomorphism of R. If we restrict this category further to allow only those morphisms that act by the identity on R, we obtain the categories Top_R and $\mathrm{Top}_{X,R}$ on the right-hand side of the diagram above.

Remark 2.4 We also consider two variations of diagram (2.7).

- (A) Each of the categories in the top row of the diagram is the Grothendieck construction of a functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Cat}$. If we precompose each of these functors with the forgetful functor $(X, A) \mapsto X$ from the category of pairs of spaces, we obtain categories of pairs (X, A) equipped with bundles of twisted k-modules over X. We denote these categories as in (2.7), but with a superscript $(\ \ )^2$. (The bottom row is unchanged.)
- (B) We may also restrict each of the categories in (2.7) to their subcategories of spaces and *proper* maps. In this case we add a superscript $(\ \ )^{\mathrm{pr}}$ to the notation.

Remark 2.5 (*Twisted homology and fibrewise tensor product*) The structure on these categories that we use in the second and third steps of the general construction of homological functors is a *fibrewise tensor product* and *twisted homology.* Twisted homology (in degree i, say) may be thought of as a functor $\mathrm{Top}_{(k)} \to k\text{-Mod}$, which, on the subcategory $\mathrm{Top}_{(k);R}$, may be upgraded to a twisted homology functor taking values in R-Mod (related to the previous one via diagram (2.11) below). We have a symmetric monoidal structure $\otimes_k$ on the category $\mathrm{Top}_{(k);X}$, which we think of as a "fibrewise tensor product over k". Moreover, for objects of the subcategory $\mathrm{Top}_{(k);X,R}$, we may also take their fibrewise tensor product over R, and this operation in particular yields a functor

$$\mathrm{Top}_{(k);X,R} \longrightarrow \mathrm{Top}_{(k);X,S}$$

for each (R, S)-bimodule. We summarise this in more detail in the rest of this subsection, including the corresponding functors for the variations of (2.7) described in Remark 2.4 (twisted homology of pairs and twisted Borel-Moore homology respectively).

Fibrewise tensor product. By definition,

$$\mathrm{Top}_{(k);X} = \mathit{Mod}(\mathrm{Loc}_{k\text{-Mod}}(X)) = \mathbf{Fct}^{\otimes}(\mathit{Module}, \mathbf{Fct}(\Pi_1(X), k\text{-Mod}))$$

is a symmetric monoidal category, with the objectwise monoidal product induced by the tensor product of k-modules in k-Mod. We write this as $\otimes_k$ and – using the viewpoint of objects of $\mathrm{Top}_{(k);X}$ as pairs of bundles of k-modules over X – we call this the *fibrewise tensor product* of (pairs of) bundles of k-modules over X.

Remark 2.6 An alternative description of the fibrewise tensor product of two bundles of k-modules, using the viewpoint of bundles of k-modules directly instead of interpreting them as functors from the fundamental groupoid to the category of k-modules, is as follows. Let $\alpha\colon A \to X$ and $\beta\colon B \to X$ be two bundles of k-modules; in other words, α is a fibre bundle with fibre a k-module M and structure group $\mathrm{Aut}_k(M)$ and β is a fibre bundle with fibre another k-module N and structure group $\mathrm{Aut}_k(N)$. Their fibrewise tensor product should be a fibre bundle over X with fibre $M \otimes_k N$ and structure group $\mathrm{Aut}_k(M \otimes_k N)$. Let $\mathcal{U}$ be an open cover of X that trivialises both α and β. The fibre bundle α is determined by a collection of continuous maps $\sigma_{U,V}\colon U \cap V \to \mathrm{Aut}_k(M)$, as U and V vary over $\mathcal{U}$, satisfying the *cocycle condition* on triple intersections. (Note that the maps $\sigma_{U,V}$ are in fact locally constant, since their codomains are discrete.) Similarly, β is determined by a collection of continuous maps $\tau_{U,V}\colon U\cap V\to \mathrm{Aut}_k(N)$ satisfying the cocycle condition on triple intersections. To define the fibrewise tensor product $\alpha\otimes_k\beta$, it suffices to specify a collection of maps $\upsilon_{U,V}\colon U\cap V\to \mathrm{Aut}_k(M\otimes_k N)$ that satisfy the cocycle condition on triple intersections. Now, the monoidal structure on k-Mod induces a natural group homomorphism

$$\iota\colon \mathrm{Aut}_k(M)\times \mathrm{Aut}_k(N)\longrightarrow \mathrm{Aut}_k(M\otimes_k N),$$

and we may define $\upsilon_{U,V}(x)=\iota(\sigma_{U,V}(x),\tau_{U,V}(x))$.

In general, this monoidal structure on $\mathrm{Top}_{(k);X}$ does *not* restrict to a monoidal structure on the subcategory $\mathrm{Top}_{(k);X,R}$, since the fibrewise tensor product (over k) of two R-modules is naturally not an R-module, but an $R\otimes_k R$-module. On the other hand, there is of course another monoidal structure defined on $\mathrm{Top}_{X,R}=\mathbf{Fct}(\Pi_1(X),R\text{-Mod})$, which we write as $\otimes_R$, namely the objectwise monoidal product induced by the tensor product of R-modules in R-Mod.

Remark 2.7 The relation between these two tensor products is as follows. Let M be a bundle of R-modules. Then we have that

$$M\otimes_R M=(M\otimes_k M)\otimes_{R^e} R,$$

where $R^e=R\otimes_k R^{\mathrm{op}}$.

Moreover, for any (R,S)-bimodule M (for k-algebras R and S), there is a functor

$$-\otimes_R M\colon R\text{-Mod}\longrightarrow S\text{-Mod}, \tag{2.8}$$

which we may apply fibrewise to bundles of R-modules, and more generally to objects of $\mathrm{Top}_{(k);X,R}$. Summarising and applying this, we have:

Proposition 2.8 *For each k-module R, the monoidal structure $\otimes_R$ on $\mathrm{Top}_{X,R}$ extends to one on $\mathrm{Top}_{(k);X,R}$. The monoidal structure $\otimes_k$ on $\mathrm{Top}_{(k);X}$ and these monoidal structures $\otimes_R$ on each subcategory $\mathrm{Top}_{(k);X,R}$ are related as described in Remark 2.7 above. For k-algebras R and S and any (R,S)-bimodule M there is a functor*

$$-\otimes_R M\colon \mathrm{Top}_{(k);X,R}\longrightarrow \mathrm{Top}_{(k);X,S}, \tag{2.9}$$

for each X, and these assemble to give a functor

$$-\otimes_R M\colon \mathrm{Top}_{(k);R}\longrightarrow \mathrm{Top}_{(k);S}. \tag{2.10}$$

Homology. Recall that homology with local coefficients (in a fixed degree i, over a ring R) is a functor taking as input a space equipped with a bundle of R-modules and producing an R-module as output. More precisely, it is a continuous functor $H_i\colon \mathrm{Top}_R\to R\text{-Mod}$. (This is recalled in more detail in [Pal18, §5.1], following [MS93].) We note here that it extends also to categories of bundles of R-modules over bundles of k-algebras in a compatible way, as follows.

Proposition 2.9 *For any k-module R, the continuous functor $H_i\colon \mathrm{Top}_R\to R\text{-Mod}$ extends to a commutative diagram of continuous functors*

$$\begin{tikzcd}
\mathrm{Top}_{(k)} \arrow[r, "H_i"] & k\text{-Mod} \\
\mathrm{Top}_{(k);R} \arrow[u, hook] \arrow[r, "H_i"] & R\text{-Mod}^{\mathrm{tw}} \arrow[u] \\
\mathrm{Top}_R \arrow[u, hook] \arrow[r, "H_i"] & R\text{-Mod}, \arrow[u]
\end{tikzcd} \tag{2.11}$$

*where $R\text{-Mod}^{\mathrm{tw}}$ is the category of R-modules and **twisted**, or **crossed**, homomorphisms.*

Proof sketch. One first checks that the usual twisted homology functor $\mathrm{Top}_k \to k\text{-Mod}$ extends to $\mathrm{Top}_{(k)}$ (*cf.* . [Pal18, §5.1]). Unwinding the definitions, we then observe that, restricted to $\mathrm{Top}_{(k);R}$, the homology retains its R-module structure (which is not used in the definition of the ith homology k-module, but it is preserved at each step in the construction) – which gives us the middle horizontal functor and the commutativity of the diagram. $\square$

Remark 2.10 (*Relative homology*) Exactly the same statement holds for the categories of pairs of spaces equipped with twisted bundles of k-modules (variation (A) of Remark 2.4): there are twisted *relative* homology functors fitting into a diagram like (2.11), with the superscript $(\ )^2$ added to each category in the left-hand column.

Borel-Moore homology. Recall that the ith Borel-Moore homology group of a locally compact space Y with local coefficients $\mathcal{L}$, thought of as a bundle of R-modules may be defined by the following inverse limit of relative (ordinary) homology groups

$$H_i^{BM}(Y, \mathcal{L}) = \varprojlim_{A \in \mathrm{Cpt}(Y)} H_i(Y, Y \setminus A; \mathcal{L}), \tag{2.12}$$

where $\mathrm{Cpt}(Y)$ is the poset of all compact subsets of Y. Alternatively, it is the same as the *homology of locally finite chains* on Y (with coefficients in $\mathcal{L}$). Also, if Y is a non-compact Hausdorff space, Borel-Moore homology groups can be defined via the relative homology of the *one-point compactification*, in other words $H_*(Y^+, *; \mathcal{L})$ where Y^+ denotes the one-point compactification and $*$ the complement point of X. In particular if Y homeomorphic to the complement of a closed subcomplex S in a finite CW-complex X, the Borel-Moore homology group $H_*^{BM}(Y, \mathcal{L})$ is isomorphic to the relative homology $H_*^{BM}(X, S; \mathcal{L})$. We refer the reader to [Bre97, Chapter V] for a detailed introduction to Borel-Moore homology. It forms a continuous functor

$$H_i^{BM} \colon \mathrm{Top}_R^{\mathrm{pr}} \longrightarrow R\text{-Mod}$$

defined on the subcategory $\mathrm{Top}_R^{\mathrm{pr}} \subset \mathrm{Top}_R$ of spaces equipped with bundles of R-modules and morphisms of such whose underlying map of spaces is *proper*, i.e. where preimages of compact sets are compact. Applying the construction (2.12) to the constructions above and using Remark 2.10, we obtain the following, where we recall that the superscript $(-)^{\mathrm{pr}}$ denotes in each case the subcategory on the same objects, restricting to those morphisms whose underlying map of spaces is proper.

Proposition 2.11 *For any k-module R, the continuous functor $H_i^{BM} \colon \mathrm{Top}_R^{\mathrm{pr}} \to R\text{-Mod}$ extends to a commutative diagram of continuous functors*

$$\begin{tikzcd}
\mathrm{Top}_{(k)}^{\mathrm{pr}} \arrow[r, "H_i^{BM}"] & k\text{-Mod} \\
\mathrm{Top}_{(k);R}^{\mathrm{pr}} \arrow[u, hook] \arrow[r, "H_i^{BM}"] & R\text{-Mod}^{\mathrm{tw}} \arrow[u] \\
\mathrm{Top}_R^{\mathrm{pr}} \arrow[u, hook] \arrow[r, "H_i^{BM}"] & R\text{-Mod}. \arrow[u]
\end{tikzcd} \tag{2.13}$$

2.3 Lifting actions to covering spaces

This section is an interlude on the general question of when a (continuous) group action on a space X lifts to a given covering $\widetilde{X}$ of X, and, if so, when the lifted action commutes with the action of the deck transformation group. This will be encoded in §2.4 into a “lifting functor”.

The considered topological spaces: Let X be a path-connected, locally path-connected and semi-locally simply connected topological space and let $x_0 \in X$ be a basepoint.

Let G be a locally path-connected topological group. We recall that this means that G is a topological space and an abstract group such that the multiplication operation $\cdot : G \times G \to G$ and the inverse operation $(-)^{-1} : G \to G$ are continuous with respect to the topology of G. We denote by ToGr the category of topological groups and ToGr_{lpc} its subcategory of locally path-connected topological groups, both with continuous homomorphisms as its morphisms. The set of path-components of a topological group inherits it a group structure and thus induces a functor:

$$\pi_0 : \mathrm{ToGr} \to \mathrm{Gr}.$$

Recall that any object of Gr can be considered as a topological group using the discrete topology. Hence, sending a point of a (locally path-connected) topological group to the path component that it lies in induces a functor $\pi_- : \mathrm{ToGr}_{lpc} \to \mathrm{Gr}$, where $\pi_H : H \to \pi_0(H)$ is a continuous group homomorphism. Note that it is necessary to restrict to ToGr_{lpc}, otherwise π_H is not always continuous and a fortiori π_- is not well-defined on morphisms.

Finally, we recall the following fact:

Lemma 2.12 *Let G' be a discrete group $\varphi \in Hom_{\mathrm{ToGr}}(G, G')$. Then, $\varphi = \pi_0(\varphi) \circ \pi_G : G \to \pi_0(G) \to \pi_0(G') = G'$.*

Proof. This follows from the definition of the functor π_- and the fact that, as G' is discrete, $\pi_{G'} : G' \to \pi_0(G') = G'$ is the identity. □

Two particular group homomorphisms: First, let $\phi : \pi_1(X, x_0) \to Q$ be a surjective group homomorphism. By [Hat02, Propositions 1.38 and 1.39], $\ker(\phi)$ corresponds to a regular path-connected covering space $\xi : X^\phi \to X$ so that $Im(\xi_*) = \ker(\phi)$ (where ξ_* denotes the map induced by ξ for the fundamental group) and with deck transformations group $D(\xi) \cong Q$. For k a natural number, this gives the homology group $H_k\left(X^\phi, \mathbb{Z}\right)$ a $\mathbb{Z}[Q]$-module structure. Recall that if $\ker(\phi) = 0$, then X^ϕ is isomorphic to the universal covering $\widetilde{X}$ of X.

Secondly, let $\theta : G \to \mathrm{Homeo}_{x_0}(X)$ be a continuous group homomorphism, where $\mathrm{Homeo}_{x_0}(X)$ is given the subspace topology induced from the compact-open topology on $\mathrm{Map}(X, X)$. This induces a group homomorphism $\theta_\pi : G \to Aut(\pi_1(X, x_0))$, defined by

$$\theta_\pi(g)([\gamma]) = [\theta(g)(\gamma)]$$

with $g \in G$ and $[\gamma] \in \pi_1(X, x_0)$. Also, we make the following assumption:

Assumption 2.13 The group homomorphism θ_π preserves the subgroup $\ker(\phi)$.

Therefore, there are well-defined actions $\theta_\pi^r : G \to Aut(\ker(\phi))$ and $\bar{\theta}_\pi : G \to Aut(Q)$, such that:

$$\bar{\theta}_\pi(g) \circ \phi = \phi \circ \theta_\pi(g)$$

for all $g \in G$. Also, we deduce that:

Proposition 2.14 *Under Assumption 2.13, we have the following properties:*

1. *Let $x_0^\phi \in X^\phi$ such that $p\left(x_0^\phi\right) = x_0$. There is a unique group homomorphism $\theta^\phi : G \to \mathrm{Homeo}_{x_0^\phi}\left(X^\phi\right)$ such that $\xi \circ \theta^\phi(g) = \theta(g) \circ \xi$ for all $g \in G$.*
2. *For any deck transformation $\psi \in D(\xi) \cong Q$ we have $\bar{\theta}_\pi(g)(\psi) \circ \theta^\phi(g) = \theta^\phi(g) \circ \psi$ for all $g \in G$.*

Proof. As θ_π preserves the subgroup $Im(\xi_*)$, the existence of θ^ϕ is a consequence of [Hat02, Propositions 1.33] and its unicity of [Hat02, Propositions 1.34].

Note that

$$\begin{aligned}
\xi\left(\bar{\theta}_{\pi}(g)(\psi)\left(\theta^{\phi}(g)(y)\right)\right) &= \theta(g)\circ p(y) \\
&= \xi\left(\theta^{\phi}(g)(\psi(y))\right),
\end{aligned}$$

for all $y \in X^{\phi}$. As X^{ϕ} is path-connected, it is enough to prove that the second equality is true for one point of X^{ϕ} (see [Hat02, Proposition 1.34]). Let be a loop in X based at x_0 such that $\phi([\gamma]) = \psi$ and we denote by γ^{ϕ} for a path based at x_0^{ϕ} lifting γ in X^{ϕ}. On the one hand, $\phi([\theta(g)(\gamma)])\left(x_0^{\phi}\right)$ is the unique lift of x_0^{ϕ} which is the endpoint of the lift in X^{ϕ} of the path $\theta(g)(\gamma)$. On the other hand, $\theta^{\phi}(g)\left(\phi([\gamma])\left(x_0^{\phi}\right)\right)$ is the unique lift of x_0^{ϕ} given by the endpoint of the path obtained applying $\theta^{\phi}(g)$ to γ^{ϕ}: as $\xi \circ \theta^{\phi}(g) = \theta(g)\circ\xi$, this is the same point as the endpoint of the lift in X^{ϕ} of the path $\theta(g)(\gamma)$. We deduce that:

$$\begin{aligned}
\bar{\theta}_{\pi}(g)(\psi)\left(\theta^{\phi}(g)\left(x_0^{\phi}\right)\right) &= \phi([\theta(g)(\gamma)])\left(x_0^{\phi}\right) \\
&= \theta^{\phi}(g)\left(\phi([\gamma])\left(x_0^{\phi}\right)\right) \\
&= \theta^{\phi}(g)\left(\psi\left(x_0^{\phi}\right)\right).
\end{aligned}$$

□

The construction: Finally, we use *ordinary* homology and *Borel-Moore* homology to define linear representations of the group G. Moreover, if the topological space X has boundary components, we also use homology *relative to the boundary* and *reduced* homology (in other words, homology relative to a point on the boundary) to construct linear representations of the group G.

Hence, using the induced action on homology, we obtain from the first point of Proposition 2.14 a well-defined action of G on the homology groups of X^{ϕ}:

Definition 2.15 The morphisms $\theta : G \to \mathrm{Homeo}_{x_0}(X)$ and $\phi : \pi_1(X, x_0) \to Q$ induce representations

$$L_k(F_{\theta,\phi}) : G \to Aut\left(H_k\left(X^{\phi}, \mathbb{Z}\right)\right) \qquad \text{and} \qquad L_k(F_{\theta,\phi})^{\mathrm{BM}} : G \to Aut\left(H_k^{BM}\left(X^{\phi}, \mathbb{Z}\right)\right).$$

If X has boundary ∂X or a basepoint, the morphisms θ and ϕ also induce representations

$$L_k(F_{\theta,\phi})^{\partial} : G \to Aut\left(H_k\left(X^{\phi}, \partial X^{\phi}; \mathbb{Z}\right)\right) \qquad \text{and} \qquad L_k(F_{\theta,\phi})^{\mathrm{red}} : G \to Aut\left(H_k^{\mathrm{red}}\left(X^{\phi}, \mathbb{Z}\right)\right).$$

In addition, let us now make the following assumption:

Assumption 2.16 The induced action $\bar{\theta}_{\pi} : G \to Aut(Q)$ of G on Q is trivial.

Hence, by the second property of Proposition 2.14, $\theta^{\phi}(g)$ commutes with all deck transformations for all $g \in G$. A fortiori, the induced representations $L_k(F_{\theta,\phi})$ and $L_k(F_{\theta,\phi})^{BM}$ commute with the $\mathbb{Z}[Q]$-module structure of the homology groups $H_k\left(X^{\phi}, \mathbb{Z}\right)$, $H_k^{BM}\left(X^{\phi}, \mathbb{Z}\right)$, $H_k\left(X^{\phi}, \partial X^{\phi}; \mathbb{Z}\right)$ and $H_k^{\mathrm{red}}\left(X^{\phi}, \mathbb{Z}\right)$

Furthermore, since G is locally path-connected, and since $\pi_1(X, x_0)$ and the automorphism groups of the various homology groups are discrete, it follows from Lemma 2.12 that the induced actions θ_{π} and θ^{ϕ} factor through $\pi_0(G)$. Note that it is a fortiori enough to check that Assumption 2.16 is satisfied just for one point in each path-component of G. We deduce that:

Definition 2.17 The representations $L_k(F_{\theta,\phi})$ etc. of G of Definition 2.15 induce representations

$$\pi_0(L_k(F_{\theta,\phi})) \colon \pi_0(G) \longrightarrow Aut_{\mathbb{Z}[Q]}\left(H_k\left(X^{\phi}, \mathbb{Z}\right)\right),$$

and similarly for the other 3 versions, called respectively the ordinary, Borel-Moore, relative to the boundary and reduced homological representations of $\pi_0(G)$ induced by $\theta : G \to \mathrm{Homeo}_{x_0}(X)$ and $\phi : \pi_1(X, x_0) \to Q$.

2.4 The lifting functor

In this section we define a continuous functor

$$\mathrm{Lift}\colon \mathrm{Cov}_Q \longrightarrow \mathrm{Top}_{k[Q]}, \tag{2.14}$$

which encodes the key construction of lifting group actions to covering spaces, while commuting with the action of the deck transformation group of the covering. In fact, we will extend this to a functor

$$\mathrm{Lift}\colon \widetilde{\mathrm{Cov}}_Q \longrightarrow \mathrm{Top}_{(k);k[Q]}, \tag{2.15}$$

which encodes the construction of lifting (more general) group actions, while commuting with the action of the deck transformation group only up to an induced action on the deck transformation group itself.

Definition 2.18 If, in §2.2, we replace the symmetric monoidal category k-Mod with the symmetric monoidal category of sets and functions, under disjoint union, the constructions go through identically, and yield a diagram

$$\begin{tikzcd} \mathrm{Top}_{(\cdot)} & \mathrm{Top}_{(\cdot);M} \arrow[l, hook'] & \mathrm{Top}_M \arrow[l, hook'] \\ \mathrm{Top}_{(\cdot);X} \arrow[u, hook] & \mathrm{Top}_{(\cdot);X,M} \arrow[l, hook'] \arrow[u, hook] & \mathrm{Top}_{X,M} \arrow[l, hook'] \arrow[u, hook] \end{tikzcd} \tag{2.16}$$

analogous to (2.7), where the subscript $(\quad)_{(\cdot)}$ is notation indicating that this is the *non-linearised* version of the construction, and M is a monoid (playing a role analogous to that of the k-algebra R in (2.7)). There is a morphism of diagrams of categories from (2.16) to (2.7) (with $R = k[M]$, the monoid-k-algebra of M) given by (fibrewise) applying the free strict symmetric monoidal functor $\mathrm{Set} \to k\text{-Mod}$.

We therefore just need to define a functor

$$\mathrm{Lift}\colon \widetilde{\mathrm{Cov}}_Q \longrightarrow \mathrm{Top}_{(\cdot);Q}, \tag{2.17}$$

for any group Q, which we will then compose with the functor (2.16) $\to$ (2.7) of Definition 2.18 in the top-middle position of the diagram (with $M = Q$).

Definition 2.19 (*The lifting functor*) The *lifting functor* (2.17) is defined on objects as follows. Let $(X, x_0, \varphi\colon \pi_1(X, x_0) \to Q)$ be an object of $\widetilde{\mathrm{Cov}}_Q$. The kernel of φ is a normal subgroup of $\pi_1(X, x_0)$, and therefore corresponds to a regular covering of X with deck transformation group Q, which is in particular a bundle of Q-sets over X (whose fibres happen to all be isomorphic to Q itself considered as a Q-set). To be slightly more careful (in order to specify a bundle of Q-sets, and not just an isomorphism class of such), we take the universal cover $\widetilde{X}$ of X (specifically, the canonical model for $\widetilde{X}$ consisting of endpoint-preserving homotopy classes of paths in X starting at x_0), which is equipped with an action of $\pi_1(X, x_0)$, and then take its quotient by the action of the subgroup $\ker(\varphi)$. Denote this covering by $\xi_\varphi\colon X^\varphi \to X$. This defines (2.17) on objects:

$$\mathrm{Lift}(X, x_0, \varphi) \;=\; (X, \xi_\varphi).$$

In order to define (2.17) on morphisms, we first note that, although we did not need it to define the functor on objects, the bundle of Q-sets associated to (X, x_0, φ) comes equipped with a particular choice of basepoint, covering the basepoint x_0 of X. This is because the standard construction of the universal cover $\widetilde{X}$ has a canonical basepoint (the constant path at x_0), and therefore so does its quotient X^φ by the action of Q. Let us denote this basepoint by $\widetilde{x}_0 \in X^\varphi$.

Suppose we are given a morphism $(X, x_0, \varphi) \to (Y, y_0, \psi)$ of $\widetilde{\mathrm{Cov}}_Q$, that is, a continuous map

$$f\colon X \longrightarrow Y$$

such that $f(x_0) = y_0$ and $f_*(\ker(\varphi)) \subseteq \ker(\psi)$. By the basic theory of covering spaces, this implies that, for each $\widetilde{y} \in \xi_\psi^{-1}(y_0)$, there is a unique continuous map $X^\varphi \to Y^\psi$ that lifts the composition $f \circ \xi_\varphi \colon X^\varphi \to Y$ and that takes $\widetilde{x}_0$ to $\widetilde{y}$. We therefore obtain a uniquely-determined lift

$$\widetilde{f} \colon X^\varphi \longrightarrow Y^\psi$$

by requiring $\widetilde{f}(\widetilde{x}_0) = \widetilde{y}_0$. Now, the fact that $f_*(\ker(\varphi)) \subseteq \ker(\psi)$ implies also that there is a unique endomorphism f_Q of the group Q such that

$$f_Q \circ \varphi = \psi \circ f_*. \tag{2.18}$$

One may now check (cf. the discussion of the previous subsection) that, for any point $\widetilde{x} \in X^\varphi$ and any element $q \in Q$, we have that

$$(f_Q(q))_\sharp(\widetilde{f}(\widetilde{x})) \;=\; \widetilde{f}(q_\sharp(\widetilde{x})),$$

where $(\quad)_\sharp$ denotes the action of Q by deck transformations on X^φ and on Y^ψ.

The triple $(f, \widetilde{f}, f_Q)$ is therefore a morphism in $\mathrm{Top}_{(\cdot);Q}$ from (X, ξ_φ) to (Y, ξ_ψ), i.e. a morphism

$$\mathrm{Lift}(X, x_0, \varphi) \longrightarrow \mathrm{Lift}(Y, y_0, \psi),$$

namely a map of covering spaces (the pair $(f, \widetilde{f})$) together with an endomorphism f_Q of Q, such that the map $\widetilde{f}$ commutes with bundle-of-Q-sets structure on X^φ and Y^ψ up to this endomorphism. In other words, the triple $(f, \widetilde{f}, f_Q)$ is a twisted (or *crossed*) morphism of bundles of Q-sets. This completes the definition of (2.17) on morphisms:

$$\mathrm{Lift}(f) \;=\; (f, \widetilde{f}, f_Q).$$

Finally, we compose (2.17) with the functor (2.16) $\to$ (2.7) of Definition 2.18 in the top-middle position of the diagram (with $M = Q$), to obtain a functor

$$\mathrm{Lift} \colon \widetilde{\mathrm{Cov}}_Q \longrightarrow \mathrm{Top}_{(k);k[Q]}, \tag{2.19}$$

which is the promised functor (2.15).

Remark 2.20 In the above definition, if the morphism f has the stronger property that $\psi \circ f_* = \varphi$, where f_* denotes the induced homomorphism on π_1 — in other words, if it lies in the subcategory $\mathrm{Cov}_Q \subseteq \widetilde{\mathrm{Cov}}_Q$ — then the unique endomorphism f_Q satisfying (2.18) must necessarily be the identity. The morphism $(f, \widetilde{f}, f_Q) = (f, \widetilde{f}, \mathrm{id}_Q)$ therefore lies in the subcategory $\mathrm{Top}_Q \subseteq \mathrm{Top}_{(\cdot);Q}$. Hence the functor (2.17) restricts to a functor

$$\mathrm{Lift} \colon \mathrm{Cov}_Q \longrightarrow \mathrm{Top}_Q. \tag{2.20}$$

Composing this with the functor (2.16) $\to$ (2.7) of Definition 2.18 in the top-right position of the diagram (with $M = Q$), we therefore obtain a functor

$$\mathrm{Lift} \colon \mathrm{Cov}_Q \longrightarrow \mathrm{Top}_{k[Q]}, \tag{2.21}$$

which is the promised functor (2.14).

2.5 Summary

Definition 2.21 (*The homological representation construction, untwisted*) Fix a topological category $\mathcal{C}^{\mathrm{t}}$ with $\pi_0(\mathcal{C}^{\mathrm{t}}) = \mathcal{C}$ and a ground ring k. Assume that we have as input:

- A continuous functor $F \colon \mathcal{C}^{\mathrm{t}} \to \mathrm{Cov}_Q$, where Q is a group.
- A $(k[Q], R)$-bimodule M, where R is a k-algebra.
- A non-negative integer i.

The corresponding *homological representation* of $\mathcal{C}$ is obtained as follows. We compose the continuous functors F, (2.21), (2.10) and (2.11) to obtain a functor $\mathcal{C}^{\mathrm{t}} \to R\text{-}\mathrm{Mod}$. Since R-Mod is a discrete category, this factors (uniquely) through a functor

$$L_i(F;M)\colon \mathcal{C} \longrightarrow R\text{-}\mathrm{Mod}. \tag{2.22}$$

This is the *homological representation* of $\mathcal{C}$ associated to the continuous functor $F\colon \mathcal{C}^{\mathrm{t}} \to \mathrm{Cov}_Q$, in degree i and with coefficients M. In the case when we take $R = k[Q]$ as a ring and $M = k[Q]$ as a bimodule over itself (i.e., when we do not twist the coefficients), we denote this simply as

$$L_i(F)\colon \mathcal{C} \longrightarrow R\text{-}\mathrm{Mod}. \tag{2.23}$$

The definition of the *twisted* version of the homological representation construction follows Definition 2.21 almost verbatim:

Definition 2.22 (*The homological representation construction, twisted*) Fix a topological category $\mathcal{C}^{\mathrm{t}}$ with $\pi_0(\mathcal{C}^{\mathrm{t}}) = \mathcal{C}$ and a ground ring k. Assume that we have as input:

- A continuous functor $F\colon \mathcal{C}^{\mathrm{t}} \to \widetilde{\mathrm{Cov}}_Q$, where Q is a group.
- A $(k[Q], R)$-bimodule M, where R is a k-algebra.
- A non-negative integer i.

The corresponding *twisted homological representation* of $\mathcal{C}$ is obtained as follows. We compose the continuous functors F, (2.19), (2.10) and (2.11) to obtain a functor $\mathcal{C}^{\mathrm{t}} \to R\text{-}\mathrm{Mod}^{\mathrm{tw}}$. Since $R\text{-}\mathrm{Mod}^{\mathrm{tw}}$ is a discrete category, this factors (uniquely) through a functor

$$\widetilde{L}_i(F;M)\colon \mathcal{C} \longrightarrow R\text{-}\mathrm{Mod}^{\mathrm{tw}}. \tag{2.24}$$

This is the *twisted homological representation* of $\mathcal{C}$ associated to the continuous functor $F\colon \mathcal{C}^{\mathrm{t}} \to \widetilde{\mathrm{Cov}}_Q$, in degree i and with coefficients M. Again, in the case when we take $R = k[Q]$ as a ring and $M = k[Q]$ as a bimodule over itself (i.e., when we do not twist the coefficients), we denote this simply as

$$\widetilde{L}_i(F)\colon \mathcal{C} \longrightarrow R\text{-}\mathrm{Mod}^{\mathrm{tw}}. \tag{2.25}$$

The two constructions above may be summarised in the following extension of diagram (2.1).

$$\begin{tikzcd}
 & \widetilde{\mathrm{Cov}}_Q \arrow[r] & \mathrm{Top}_{(k);k[Q]} \arrow[r] & \mathrm{Top}_{(k);R} \arrow[r] & R\text{-}\mathrm{Mod}^{\mathrm{tw}} \\
\mathcal{C}^{\mathrm{t}} \arrow[ur] \arrow[dr] \arrow[dd, two heads] \arrow[r, phantom, "F"] & {} & & & \\
 & \mathrm{Cov}_Q \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\mathrm{Lift}"] & \mathrm{Top}_{k[Q]} \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "-\otimes M"] & \mathrm{Top}_R \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "H_i"] & R\text{-}\mathrm{Mod} \arrow[uu, hook] \\
\mathcal{C} = \pi_0(\mathcal{C}^{\mathrm{t}}) \arrow[urrrr, bend right, "L_i(F;M)"'] \arrow[uuurrrr, bend left=60, "\widetilde{L}_i(F;M)"] & & & &
\end{tikzcd} \tag{2.26}$$

Remark 2.23 This construction may easily be adapted to deal with *pairs of spaces* and with *Borel-Moore homology* (in the latter case we must restrict to categories of spaces and *proper* continuous maps everywhere). The resulting homological representations are then denoted similarly, adding a superscript $(\ \ )^2$ or $(\ \ )^{\mathrm{BM}}$ to the notation.

3 Categorical framework for families of groups

The aim of this section is to introduce the categorical framework that is central to this paper to handle families of groups. We first recall notions and properties of Quillen's bracket construction introduced in [Gra76, p.219] and pre-braided monoidal categories. Then we describe each categorical setting associated with the families of groups we deal with to apply the homological representation constructions introduced in §2.

3.1 Background on Quillen's bracket construction

Throughout this section, we fix a (small) strict monoidal groupoid $(\mathcal{G}, \natural, 0)$ and a (small) left-module $(\mathcal{M}, \natural)$ over $(\mathcal{G}, \natural, 0)$. The following definition is a particular case of a more general construction of [Gra76].

Definition 3.1 Quillen's bracket construction $\langle \mathcal{G}, \mathcal{M} \rangle$ on the left-module $(\mathcal{M}, \natural)$ over the groupoid $(\mathcal{G}, \natural, 0)$ is the category with the same objects as $\mathcal{M}$ and the morphisms are given by:

$$\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(X, Y) = \underset{\mathcal{G}}{colim}\left[\mathrm{Hom}_{\mathcal{M}}(-\natural X, Y)\right].$$

Thus, a morphism from X to Y in $\langle \mathcal{G}, \mathcal{M} \rangle$ is denoted by $[A, \varphi] : X \to Y$: it is an equivalence class of pairs (A, φ) where A is an object of $\mathcal{G}$ and $\varphi : A\natural X \to Y$ is a morphism in $\mathcal{M}$. Also, for two morphisms $[A, \varphi] : X \to Y$ and $[B, \psi] : Y \to Z$ in $\langle \mathcal{G}, \mathcal{M} \rangle$, the composition is defined by $[B, \psi] \circ [A, \varphi] = [B, Y\natural A, \psi \circ (id_B \natural \varphi)]$.

Remark 3.2 Let ϕ be an element of $\mathrm{Hom}_{\mathcal{M}}(X, Y)$. Then, as an element of $\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(X, Y)$, we will abuse the notation and write ϕ for $[0, \phi]$. This comes from the (faithful) canonical functor $\mathcal{C}_{\langle \mathcal{G}, \mathcal{M} \rangle} : \mathcal{M} \hookrightarrow \langle \mathcal{G}, \mathcal{M} \rangle$ defined as the identity on objects and sending $\phi \in \mathrm{Hom}_{\mathcal{M}}(X, Y)$ to $[0, \phi]$.

Actually, for all the examples discussed in this paper, the category $\mathcal{M}$ is a groupoid (see §3.4). Then, a natural question is the relationship between the automorphisms of the groupoid $\mathcal{M}$ and those of its associated Quillen bracket construction $\langle \mathcal{G}, \mathcal{M} \rangle$. Recall that the monoidal groupoid $(\mathcal{G}, \natural, 0)$ is said to have no zero divisors if, for all objects A and B of $\mathcal{G}$, $A\natural B \cong 0$ if and only if $A \cong B \cong 0$. Then, following mutatis mutandis from the proof of [RW17, Proposition 1.7], we prove that:

Proposition 3.3 *If the strict monoidal groupoid $(\mathcal{G}, \natural, 0)$ has no zero divisors, if $\mathrm{Aut}_{\mathcal{G}}(0) = \{id_0\}$ and if $\mathcal{M}$ is a groupoid, then $\mathcal{M} = \mathfrak{Gr}(\langle \mathcal{G}, \mathcal{M} \rangle)$.*

Henceforth, we assume that $\mathcal{M}$ is a groupoid, that the strict monoidal groupoid $(\mathcal{G}, \natural, 0)$ has no zero divisors and that $\mathrm{Aut}_{\mathcal{G}}(0) = \{id_0\}$.

Remark 3.4 We say that $\mathcal{G}$ and $\mathcal{M}$ have the *cancellation property* if $A\natural X \cong B\natural X$ then $A \cong B$ and the *injection property* if the morphism $\mathrm{Aut}_{\mathcal{G}}(A) \to \mathrm{Aut}_{\mathcal{M}}(A\natural X)$ sending $f \in \mathrm{Aut}_{\mathcal{G}}(A)$ to $f\natural id_X$ is injective, for all objects A and B of $\mathcal{G}$ and X of $\mathcal{M}$. If the groupoids $\mathcal{G}$ and $\mathcal{M}$ satisfy these two properties, then following [RW17, Theorem 1.10], for all objects A of $\mathcal{G}$ and X of $\mathcal{M}$, $\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(A, X)$ is a set on which the group $\mathrm{Aut}_{\mathcal{M}}(X)$ acts by post-composition transitively and the image of the map $\mathrm{Aut}_{\mathcal{G}}(A) \to \mathrm{Aut}_{\mathcal{M}}(A\natural X)$ s sending $f \in Aut_{\mathcal{G}}(A)$ to $f\natural id_X$ is $\{\phi \in \mathrm{Aut}_{\mathcal{M}}(A\natural X) \mid \phi \circ (\iota_A \natural id_X) = \iota_A \natural id_X\}$. A fortiori, we deduce the set isomorphism

$$\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(X, A\natural X) \cong \mathrm{Aut}_{\mathcal{M}}(A\natural X) / \mathrm{Aut}_{\mathcal{G}}(A).$$

A natural question is to wonder when an object of $\mathbf{Fct}(\mathcal{M}, \mathcal{C})$ extends to an object of $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{C})$ for a category $\mathcal{C}$, which is the aim of the following lemma, which proof follows mutatis mutandis from the one of [Sou18, Lemma 1.6]. Analogous statements can be found in [RW17, Proposition 2.4].

Lemma 3.5 *Let $\mathcal{C}$ be a category and F an object of $\mathbf{Fct}(\mathcal{M}, \mathcal{C})$. Assume that for all $A \in \mathrm{Ob}(\mathcal{G})$ and $X \in \mathrm{Ob}(\mathcal{M})$, there exist assignments $F([A, id_{A\natural X}]) : F(X) \to F(A\natural X)$ such that for all $B \in \mathrm{Ob}(\mathcal{G})$:*

$$F([B, id_{B\natural A\natural X}]) \circ F([A, id_{A\natural X}]) = F([B\natural A, id_{B\natural A\natural X}]). \tag{3.1}$$

Then, the assignments $F([A, \gamma]) = F(\gamma) \circ F([A, id_{A\natural B}])$ for all $[A, \gamma] \in Hom_{\langle \mathcal{G}, \mathcal{M} \rangle}(X, A\natural X)$ define a functor $F : \langle \mathcal{G}, \mathcal{M} \rangle \to \mathcal{C}$ if and only if for all $A \in \mathrm{Ob}(\mathcal{G})$ and $X \in \mathrm{Ob}(\mathcal{M})$, for all $\gamma'' \in Aut_{\mathcal{M}}(X)$ and all $\gamma' \in Aut_{\mathcal{G}}(A)$:

$$F([A, id_{A\natural X}]) \circ F(\gamma'') = F(\gamma' \natural \gamma'') \circ F([A, id_{A\natural X}]). \tag{3.2}$$

Similarly, we can find a criterion for extending a morphism in the category $\mathbf{Fct}(\mathcal{M},\mathcal{C})$ to a morphism in the category $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{C})$, the proof being a slight mutatis mutandis adaptation of the one of [Sou18, Lemma 1.7].

Lemma 3.6 *Let $\mathcal{C}$ be a category, F and G be objects of $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{C})$ and $\eta : F \to G$ a natural transformation in $\mathbf{Fct}(\mathcal{M},\mathcal{C})$. The restriction $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{C}) \to \mathbf{Fct}(\mathcal{M},\mathcal{C})$ is obtained by precomposing by the canonical inclusion $\mathcal{C}_{\langle\mathcal{G},\mathcal{M}\rangle}$ of Remark 3.2. Then, η is a natural transformation in the category $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{C})$ if and only if for all $X,Y \in \mathrm{Ob}(\mathcal{M})$ such that $Y \cong A\natural X$ with $A \in \mathrm{Ob}(\mathcal{G})$:*

$$\eta_Y \circ F([A, id_Y]) = G([A, id_Y]) \circ \eta_X. \tag{3.3}$$

Finally, if the strict monoidal groupoid $(\mathcal{G},\natural,0)$ is braided, Quillen's bracket construction $\langle\mathcal{G},\mathcal{M}\rangle$ inherits a monoidal product. Beforehand, we present the notion of a pre-braided monoidal category, introduced in [RW17]. This is a generalisation of that of a braided monoidal category.

Definition 3.7 [RW17, Definition 1.5] Let $(\mathcal{C},\natural,0)$ be a strict monoidal category such that the unit 0 is initial. Recall that $\iota_X : 0 \to B$ denotes the unique morphism from 0 to an object X of $\mathcal{C}$. We say that the monoidal category $(\mathcal{C},\natural,0)$ is pre-braided if its maximal subgroupoid $\mathfrak{Gr}(\mathcal{C},\natural,0)$ is braided monoidal (the monoidal structure being induced by that of $(\mathcal{C},\natural,0)$) and if the braiding $b^{\mathcal{C}}_{A,B} : A\natural B \to B\natural A$ satisfies $b^{\mathcal{C}}_{A,B} \circ (id_A \natural \iota_B) = \iota_B \natural id_A : A \to B\natural A$ for all $A,B \in \mathrm{Obj}(\mathcal{C})$.

The following key property describes the application of Quillen's bracket construction on a left-module $(\mathcal{M},\natural)$ over strict braided monoidal groupoid $\left(\mathcal{G},\natural,0,b^{\mathcal{G}}_{-,-}\right)$. It is a mutatis mutandis generalisation of [RW17, Proposition 1.8], the proof being therefore omitted.

Proposition 3.8 *If the groupoid $(\mathcal{G},\natural,0)$ is braided, then the definition of the monoidal product $\natural$ extends to $\langle\mathcal{G},\mathcal{M}\rangle$ by letting for $[X,\varphi] \in Hom_{\langle\mathcal{G},\mathcal{M}\rangle}(A,B)$ and $[Y,\psi] \in Hom_{\langle\mathcal{G},\mathcal{M}\rangle}(C,D)$:*

$$[X,\varphi]\natural[Y,\psi] = \left[X\natural Y, (\varphi\natural\psi) \circ \left(id_X \natural \left(b^{\mathcal{G}}_{A,Y}\right)^{-1} \natural id_C\right)\right].$$

Moreover, if we consider $\mathcal{M} = \mathcal{G}$, then the category $(\langle\mathcal{G},\mathcal{G}\rangle,\natural,0)$ is pre-braided monoidal and the unit 0 of the monoidal structure is an initial object in the category $\langle\mathcal{G},\mathcal{G}\rangle$. If, in addition, $\left(\mathcal{G},\natural,0,b^{\mathcal{G}}_{-,-}\right)$ is symmetric monoidal, then the category $\left(\langle\mathcal{G},\mathcal{G}\rangle,\natural,0,b^{\mathcal{G}}_{-,-}\right)$ is symmetric monoidal.

Remark 3.9 If the category $\mathcal{M}$ is not a groupoid, then slightly more general analogous results as those of Lemmas 3.5 and 3.6 and the first part of Proposition 3.8 could be stated. However, this is not the kind of situation we deal with in this paper.

Induced framework for automorphism subgroups. For all $A \in \mathrm{Obj}(\mathcal{G})$, let H_A be a subgroup of the automorphism group $\mathrm{Aut}_{\mathcal{G}}(A)$. We denote by $\mathcal{G}'$ the subcategory of $\mathcal{G}$ with the same objects and with morphisms

$$\mathrm{Hom}_{\mathcal{G}'}(A,B) = \begin{cases} H_A & \text{if } A \cong B \text{ in } \mathcal{G}; \\ \emptyset & \text{otherwise.} \end{cases}$$

A natural question is to wonder when the monoidal structure of $\mathcal{G}$ restricts to $\mathcal{G}'$: this is useful for some situations of §3.4 to set a categorical framework for families of subgroups of automorphism groups of some monoidal groupoid.

Lemma 3.10 *The (strict) braided monoidal structure $\left(\mathcal{G},\natural,0,b^{\mathcal{G}}_{-,-}\right)$ restricts to $\mathcal{G}'$ if and only if, for all $A,A' \in Obj(\mathcal{G})$, $\varphi'\natural\varphi \in H_{A'\natural A}$ for all $\varphi' \in H_{A'}$ and $\varphi \in H_A$.*

Proof. Note that the coherence conditions and the braiding of the induced braided monoidal structure for $\mathcal{G}'$ automatically follow from those for $\mathcal{G}$. Hence the monoidal product $\natural$ defines a (strict) braided monoidal structure $\left(\mathcal{G}',\natural,0,b^{\mathcal{G}}_{-,-}\right)$ if and only if it restricts to a bifunctor $\natural : \mathcal{G}' \times \mathcal{G}' \to \mathcal{G}'$: this is equivalent to the fact that the monoidal structure $\natural$ defines group morphisms $\natural : H_{A'} \times H_A \to H_{A'\natural A}$ for all $A',A \in \mathrm{Obj}(\mathcal{G})$. $\square$

3.2 A topological enrichment

Suppose now that $\mathcal{G}$ is a *topological* monoidal groupoid and $\mathcal{M}$ is a *topological* category with a continuous left-action of $\mathcal{G}$. (Recall that, by *topological category*, we mean a category enriched over the symmetric monoidal category of topological spaces.) Definition 3.1 may be extended directly to this setting, as follows.

Definition 3.11 The category $\langle \mathcal{G}, \mathcal{M} \rangle$ is defined to have the same objects as $\mathcal{M}$, and for objects X, Y of $\mathcal{M}$, we define $\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(X, Y)$ to be the quotient space

$$\left[\bigsqcup_{A \in \mathrm{Ob}(\mathcal{G})} \mathrm{Hom}_{\mathcal{M}}(A \natural X, Y) \right] / {\sim},$$

where $\sim$ is the equivalence relation given by $(A, \varphi) \sim (A', \varphi')$ if and only if $\varphi = \varphi' \circ (\sigma \natural \mathrm{id}_X)$ for some $\sigma \in \mathrm{Hom}_{\mathcal{G}}(A, A')$. Note that this may also be written as a colimit, as in Definition 3.1.

Remark 3.12 A topological version of Quillen's bracket construction is mentioned briefly in Remark 2.10 of [Kra17], although there the categories are *topological* in the sense of being categories internal to the category of topological spaces, rather than topologically-enriched categories. Lemma 3.13 below is stated for topologically-enriched categories, but it is likely that it has an analogue for categories internal to the category of topological spaces, in which case Lemma 2.11 of [Kra17] would be a particular case of this analogue.

Lemma 3.13 *Let $\mathcal{G}$ be a topological monoidal groupoid and $\mathcal{M}$ a topological category with a continuous left-action of $\mathcal{G}$. Assume that, for each object A of $\mathcal{G}$ and each pair of objects X, Y of $\mathcal{M}$, the quotient map*

$$\mathrm{Hom}_{\mathcal{M}}(A \natural X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{M}}(A \natural X, Y)/\mathrm{Aut}_{\mathcal{G}}(A) \tag{3.4}$$

is a Serre fibration. Then there is a canonical isomorphism of categories

$$\pi_0(\langle \mathcal{G}, \mathcal{M} \rangle) \cong \langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M}) \rangle. \tag{3.5}$$

Proof. First note that $\pi_0(\langle \mathcal{G}, \mathcal{M} \rangle)$ and $\langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M}) \rangle$ have the same object set, by the definition of the discrete and topologically-enriched Quillen bracket constructions, and the functor π_0. Specifically, their common object set is $\mathrm{ob}(\mathcal{M})$. It therefore remains to show that, for objects X and Y of $\mathcal{M}$, there is a natural bijection between $\pi_0(\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(X, Y))$ and $\mathrm{Hom}_{\langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M}) \rangle}(X, Y)$. Set

$$\Phi = \bigsqcup_{A \in \mathrm{ob}(\mathcal{G})} \mathrm{Hom}_{\mathcal{M}}(A \natural X, Y).$$

Unravelling the definitions, what we need to prove is that there is a natural bijection

$$\pi_0(\Phi/{\sim_t}) \cong \pi_0(\Phi)/{\sim_h},$$

where $\sim_t$ is the equivalence relation given by $(A, \varphi) \sim_t (A', \varphi')$ if and only if there is a morphism $\sigma \in \mathrm{Hom}_{\mathcal{G}}(A, A')$ such that $\varphi = \varphi' \circ (\sigma \natural \mathrm{id})$, and $\sim_h$ is the equivalence relation given by $(A, [\varphi]) \sim_h (A', [\varphi'])$ if and only if there is a morphism $\sigma \in \mathrm{Hom}_{\mathcal{G}}(A, A')$ such that $\varphi \simeq \varphi' \circ (\sigma \natural \mathrm{id})$. Note that the only difference between these definitions is that the equality is replaced by a homotopy in the definition of $\sim_h$.

As sets, these are both quotients of (the underlying set of) Φ, so we just need to show that, given two elements (A, φ) and (A', φ') of Φ, they have the same image in $\pi_0(\Phi/{\sim_t})$ if and only if they have the same image in $\pi_0(\Phi)/{\sim_h}$.

(a) Suppose first that (A, φ) and (A', φ') have the same image in $\pi_0(\Phi)/{\sim_h}$. This means that there is a morphism $\sigma \in \mathrm{Hom}_{\mathcal{G}}(A, A')$ and a path

$$\gamma \colon [0,1] \longrightarrow \mathrm{Hom}_{\mathcal{M}}(A \natural X, Y) \subseteq \Phi$$

with $\gamma(0) = (A, \varphi)$ and $\gamma(1) = (A, \varphi' \circ (\sigma \natural \mathrm{id}))$. Composing with the projection $\Phi \to \Phi/{\sim_t}$ and writing $[-]_t$ for the equivalence classes with respect to $\sim_t$, we obtain a path in $\Phi/{\sim_t}$ from $[(A, \varphi)]_t$ to $[(A, \varphi' \circ (\sigma \natural \mathrm{id}))]_t = [(A', \varphi')]_t$. Hence (A, φ) and (A', φ') have the same image in $\pi_0(\Phi/{\sim_t})$.

(b) To prove the converse, we first make an assumption, which we will justify later. Namely, we assume that that quotient map

$$q\colon \Phi \longrightarrow \Phi/{\sim_t}$$

is a Serre fibration. Now assume that (A,φ) and (A',φ') have the same image in $\pi_0(\Phi/{\sim_t})$, so there is a path $\delta\colon [0,1] \to \Phi/{\sim_t}$ with $\delta(0) = [(A,\varphi)]_t$ and $\delta(1) = [(A',\varphi')]_t$. By our assumption that q is a Serre fibration, we may lift this to a path $\varepsilon\colon [0,1] \to \Phi$ with $\varepsilon(0) = (A,\varphi)$ and $\varepsilon(1) \sim_t (A',\varphi')$. Its image $\varepsilon([0,1])$ is path-connected, so it must lie in $\mathrm{Hom}_{\mathcal{M}}(A\natural X, Y) \subseteq \Phi$. Hence we have a path

$$\varepsilon\colon [0,1] \longrightarrow \mathrm{Hom}_{\mathcal{M}}(A\natural X, Y)$$

with $\varepsilon(0) = (A,\varphi)$ and $\varepsilon(1) = (A,\varphi'') \sim_t (A',\varphi')$, for some $\varphi'' \in \mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)$. The relation $(A,\varphi'') \sim_t (A',\varphi')$ means that there is a morphism $\sigma \in \mathrm{Hom}_{\mathcal{G}}(A,A')$ such that $\varphi'' = \varphi' \circ (\sigma\natural\mathrm{id})$. Hence ε is a homotopy witnessing that $\varphi \simeq \varphi' \circ (\sigma\natural\mathrm{id})$, so we have shown that $(A,[\varphi]) \sim_h (A',[\varphi'])$, in other words, (A,φ) and (A',φ') have the same image in $\pi_0(\Phi)/{\sim_h}$.

(c) It now just remains to prove our earlier assumption that q is a Serre fibration. Directly from the definition, one may easily verify the following two facts:

- $\bigsqcup_i f_i\colon \bigsqcup_i E_i \to B$ is a Serre fibration if and only if each $f_i\colon E_i \to B$ is a Serre fibration.
- $f\colon E \to B$ is a Serre fibration if and only if (i) $f(E)$ is a union of path-components of B and (ii) $f\colon E \to f(E)$ is a Serre fibration.

It therefore suffices to prove that

(i) $q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y))$ is a union of path-components of $\Phi/{\sim_t}$ for each $A \in \mathrm{ob}(\mathcal{G})$,
(ii) $\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y) \to q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y))$ is a Serre fibration for each $A \in \mathrm{ob}(\mathcal{G})$.

Let us partition $\mathrm{ob}(\mathcal{G})$ into equivalence classes $\mathcal{O}_\alpha$, under the equivalence relation where two objects A, A' of $\mathcal{G}$ are equivalent if and only if there is a morphism $A \to A'$ in $\mathcal{G}$. (This is an equivalence relation since $\mathcal{G}$ is a groupoid.) we may then write $\Phi = \bigsqcup_\alpha \Phi_\alpha$, where

$$\Phi_\alpha \;=\; \bigsqcup_{A\in\mathcal{O}_\alpha} \mathrm{Hom}_{\mathcal{M}}(A\natural X,Y).$$

The equivalence relation $\sim_t$ on Φ clearly preserves the topological disjoint union $\bigsqcup_\alpha \Phi_\alpha$, so we have

$$\Phi/{\sim_t} \;=\; \bigsqcup_\alpha (\Phi_\alpha/{\sim_t}).$$

Also note that, for any two objects $A, A' \in \mathcal{O}_\alpha$ (for fixed α), we have

$$q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)) \;=\; q(\mathrm{Hom}_{\mathcal{M}}(A'\natural X,Y)).$$

So, if we make a choice of object $A_\alpha \in \mathcal{O}_\alpha$ for each α, we have a decomposition of $\Phi/{\sim_t}$ as a topological disjoint union:

$$\Phi/{\sim_t} \;=\; \bigsqcup_\alpha q(\mathrm{Hom}_{\mathcal{M}}(A_\alpha\natural X,Y)).$$

This immediately implies point (i) above.

For point (ii), note that two elements $\varphi,\varphi' \in \mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)$ have the same image under q if and only if they are $\sim_t$-equivalent, which is equivalent to saying that they lie in the same orbit of the $\mathrm{Aut}_{\mathcal{G}}(A)$-action on $\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)$. Hence the map

$$q_A\colon \mathrm{Hom}_{\mathcal{M}}(A\natural X,Y) \longrightarrow q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)) \tag{3.6}$$

is isomorphic to (3.4), at least on underlying sets. If we can show that they are isomorphic also as continuous maps of spaces, then we will be done, since we know by hypothesis that (3.4) is a Serre fibration. Since (3.4) and (3.6) are surjective continuous maps with the same domain and the same point-fibres, and we know moreover that (3.4) is a quotient map, it will suffice to prove that (3.6) is also a quotient map.

Let $U \subseteq q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y))$ be a subset such that $q_A^{-1}(U)$ is open in $\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y)$. We need to show that U is open in $q(\mathrm{Hom}_{\mathcal{M}}(A\natural X,Y))$. To see this, let $A \in \mathcal{O}_\alpha$ and note that, by the fact discussed above that the equivalence relation $\sim_t$ preserves the decomposition of Φ into a topological disjoint union, the restriction

$$q_\alpha = q|_{\Phi_\alpha} \colon \Phi_\alpha \longrightarrow q(\Phi_\alpha) = q(\mathrm{Hom}_{\mathcal{M}}(A\natural X, Y))$$

is a quotient map. So it will suffice to show that $q_\alpha^{-1}(U)$ is open in Φ_α. Now, from the definitions, we observe the following description of the subset

$$q_\alpha^{-1}(U) \;\subseteq\; \Phi_\alpha \;=\; \bigsqcup_{A' \in \mathcal{O}_\alpha} \mathrm{Hom}_{\mathcal{M}}(A'\natural X, Y).$$

For each object $A' \in \mathcal{O}_\alpha$, choose an isomorphism $\sigma_{A'} \colon A' \to A$ in $\mathcal{G}$. This induces a homeomorphism

$$\Upsilon_{A'} \;=\; - \circ (\sigma_{A'} \natural \mathrm{id}) \colon \mathrm{Hom}_{\mathcal{M}}(A\natural X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{M}}(A'\natural X, Y).$$

Then we have

$$q_\alpha^{-1}(U) \;=\; \bigsqcup_{A' \in \mathcal{O}_\alpha} \Upsilon_{A'}(q_A^{-1}(U)).$$

Since $q_A^{-1}(U)$ is open in $\mathrm{Hom}_{\mathcal{M}}(A\natural X, Y)$, it follows that $\Upsilon_{A'}(q_A^{-1}(U))$ is open in $\mathrm{Hom}_{\mathcal{M}}(A'\natural X, Y)$ for each $A' \in \mathcal{O}_\alpha$. Thus $q_\alpha^{-1}(U)$ is open in Φ_α, as required. □

Verifying the condition. In order to verify the condition (3.4) in each of our examples in §3.4 below, we will use the following proposition. Let L and M be smooth, connected d-manifolds, each equipped with a distinguished boundary-component, denoted $\partial_0 L$ resp. $\partial_0 M$, and let $L \natural M$ be their boundary connected sum, which then also has an obvious distinguished boundary-component $\partial_0(L\natural M)$. Let $A \subset L$ be a (possibly empty) closed submanifold of the interior of L and let $B \subset M$ be a (possibly empty) closed submanifold of the interior of M.

Definition 3.14 Define

$$\mathrm{Diff}_{\partial_0}(L\natural M; A \sqcup B)$$

to be the group of diffeomorphisms of $L\natural M$ that fix $A \sqcup B$ as a subset and that restrict to the identity on a neighbourhood of $\partial_0(L\natural M)$. We define $\mathrm{Diff}_{\partial_0}(L; A)$ and $\mathrm{Diff}_{\partial_0}(M; B)$ similarly. However, for simplicity of notation, we will henceforth drop the A and B and write simply $\mathrm{Diff}_{\partial_0}(L\natural M)$, etc., unless there is any ambiguity about what the submanifolds A and B could be.

These groups are topologised as follows (we describe this explicitly for $\mathrm{Diff}_{\partial_0}(L)$, and for the other two cases it is exactly analogous). Let C be a closed neighbourhood of $\partial_0 L$ in L and let $\mathrm{Diff}_C(L)$ be the subgroup of $\mathrm{Diff}_{\partial_0}(L)$ consisting of diffeomorphisms that restrict to the identity on C. We then give $\mathrm{Diff}(L)$ the Whitney topology, each $\mathrm{Diff}_C(L)$ the subspace topology inherited from the Whitney topology on $\mathrm{Diff}(L)$, and we give $\mathrm{Diff}_{\partial_0}(L)$ the *final* topology with respect to the collection of subsets $\mathrm{Diff}_C(L)$ as C varies. Namely, a subset U of $\mathrm{Diff}_{\partial_0}(L)$ is open if and only if its intersection with $\mathrm{Diff}_C(L)$ is open in $\mathrm{Diff}_C(L)$ for all C. In other words, we are viewing $\mathrm{Diff}_{\partial_0}(L)$ as the colimit

$$\mathrm{Diff}_{\partial_0}(L) = \mathrm{colim}_C(\mathrm{Diff}_C(L)).$$

Note that this may differ from the subspace topology that $\mathrm{Diff}_{\partial_0}(L)$ inherits directly from the Whitney topology on $\mathrm{Diff}(L)$ (the colimit topology may be finer). However, these two topologies on $\mathrm{Diff}_{\partial_0}(L)$ are weakly equivalent. In particular, they have the same π_0.

We have a quotient map

$$\Psi \colon \mathrm{Diff}_{\partial_0}(L\natural M) \longrightarrow \mathrm{Diff}_{\partial_0}(L\natural M)/\mathrm{Diff}_{\partial_0}(L). \tag{3.7}$$

since $\mathrm{Diff}_{\partial_0}(L)$ acts on $\mathrm{Diff}_{\partial_0}(L\natural M)$ on the right by $\varphi \cdot \psi = \varphi \circ (\psi \natural \mathrm{id}_M)$.

Proposition 3.15 *The quotient map (3.7) is a Serre fibration.*

Remark 3.16 This is related to results of Cerf [Cer61, Corollaire 2, §II.2.2.2, page 294], Palais [Pal60, Theorem B] and Lima [Lim63], but we were not able to find an instance of their results that covers the setting that we require here. We therefore give a complete proof of Proposition 3.15, using key results of Cerf [Cer61, Lemme II.2.1.2, page 291] and of Palais [Pal60, Theorem A] as an input.

Proof of Proposition 3.15. First of all, we smoothly construct the boundary connected sum $L\natural M$. To do this, choose an embedding $e\colon D^{d-1}\times[-1,0]\hookrightarrow L$ such that

- $e^{-1}(\partial L)=(\partial D^{d-1}\times[-1,0])\cup(D^{d-1}\times\{0\})$,
- this intersection with ∂L is contained in the distinguished boundary-component of L,
- the image of e is disjoint from the submanifold $A\subset\mathrm{int}(L)$,
- e is a *smooth* embedding away from $\partial D^{d-1}\times\{0\}$.[3]

Similarly, choose an embedding $f\colon D^{d-1}\times[0,1]\hookrightarrow M$ satisfying similar conditions, in particular

- $f^{-1}(\partial M)=(\partial D^{d-1}\times[0,1])\cup(D^{d-1}\times\{0\})$.

We then define:

$$L\natural M\coloneqq L\cup_e(D^{d-1}\times[-1,1])\cup_f M,$$

which has an obvious induced smooth structure. For $-\frac{1}{2}\leqslant t<0$, define $M_t=(D^{d-1}\times[t,1])\cup_f M$, a submanifold-with-corners of $L\natural M$. Choose an embedding

$$c\colon\partial_0(L\natural M)\times[0,\tfrac{1}{2}]\longrightarrow L\natural M$$

such that

- $c(x,0)=x$ for all $x\in\partial_0(L\natural M)$,
- $c(x,t)=((1-t)y,s)$ for all $x=(y,s)\in\partial D^{d-1}\times[-1,1]$ and $t\in[0,\frac{1}{2}]$,
- the image of c is disjoint from the submanifold $A\sqcup B\subset\mathrm{int}(L\natural M)$.

This is all illustrated in Figure 3.1.

For each $0<\epsilon\leqslant\frac{1}{2}$, the image $C_\epsilon=c(\partial_0(L\natural M)\times[0,\epsilon])$ is a closed neighbourhood of $\partial_0(L\natural M)$ in $L\natural M$. Recall from Definition 3.14 above that $\mathrm{Diff}_{C_\epsilon}(L\natural M)$ is the group of diffeomorphisms φ of $L\natural M$ such that $\varphi(x)=x$ for all $x\in C_\epsilon$ and $\varphi(A\sqcup B)=A\sqcup B$, equipped with the Whitney topology. Since the collection $\{C_\epsilon\}$ is cofinal in the directed set of all closed neighbourhoods of $\partial_0(L\natural M)$, it follows from our definition of the topology on $\mathrm{Diff}_{\partial_0}(L\natural M)$ that:

$$\mathrm{Diff}_{\partial_0}(L\natural M)\cong\operatorname*{colim}_{\epsilon\to 0}(\mathrm{Diff}_{C_\epsilon}(L\natural M)).\tag{3.8}$$

We also define $\mathrm{Diff}_{C_\epsilon}(L\natural M;M_t)$ to be the group of diffeomorphisms φ of $L\natural M$ such that $\varphi(x)=x$ for all $x\in C_\epsilon\cup M_t$ and $\varphi(A\sqcup B)=A\sqcup B$, equipped with the Whitney topology.

For each $-\frac{1}{2}\leqslant t<0$ and $0<\epsilon\leqslant\frac{1}{2}$ we have a quotient map

$$\Psi_{\epsilon,t}\colon\mathrm{Diff}_{C_\epsilon}(L\natural M)\longrightarrow\mathrm{Diff}_{C_\epsilon}(L\natural M)/\mathrm{Diff}_{C_\epsilon}(L\natural M;M_t).$$

For any $-\frac{1}{2}\leqslant t\leqslant t'<0$ and $0<\epsilon'\leqslant\epsilon\leqslant\frac{1}{2}$ there are natural maps

$$\mathrm{Diff}_{C_\epsilon}(L\natural M)/\mathrm{Diff}_{C_\epsilon}(L\natural M;M_t)\longrightarrow\mathrm{Diff}_{C_{\epsilon'}}(L\natural M)/\mathrm{Diff}_{C_{\epsilon'}}(L\natural M;M_{t'}),$$

so we may take the directed colimit of the maps $\Psi_{\epsilon,t}$ to obtain

$$\operatorname*{colim}_{\epsilon,t\to 0}(\Psi_{\epsilon,t})\colon\mathrm{Diff}_{\partial_0}(L\natural M)\longrightarrow\operatorname*{colim}_{\epsilon,t\to 0}(\mathrm{Diff}_{C_\epsilon}(L\natural M)/\mathrm{Diff}_{C_\epsilon}(L\natural M;M_t)),$$

where we have used the identification (3.8) for the domain. Since each $\Psi_{\epsilon,t}$ is a quotient map, it follows from general facts about colimits in the category of topological spaces that $\operatorname*{colim}_{\epsilon,t\to 0}(\Psi_{\epsilon,t})$ is also a quotient map. The map

$$\Psi\colon\mathrm{Diff}_{\partial_0}(L\natural M)\longrightarrow\mathrm{Diff}_{\partial_0}(L\natural M)/\mathrm{Diff}_{\partial_0}(L),$$

[3] On $\partial D^{d-1}\times\{0\}$, it cannot be smooth, since this is a codimension-2 face of the cylinder $D^{d-1}\times[-1,0]$, whereas L has only codimension-0 faces (its interior) and codimension-1 faces (its boundary).

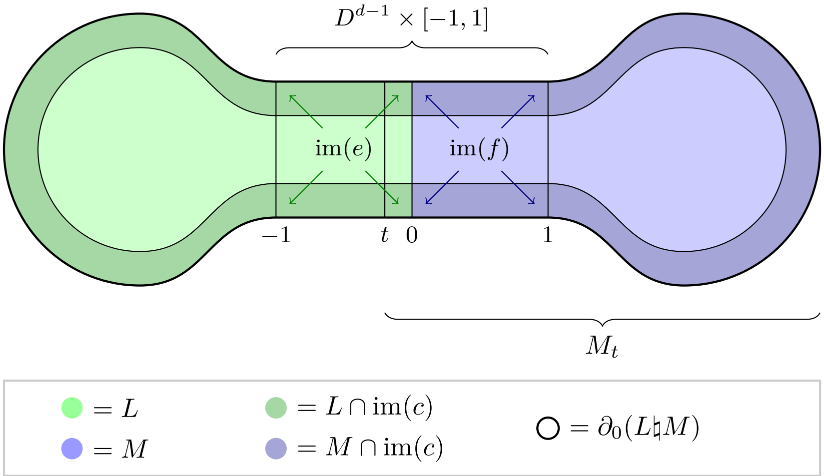


Figure 3.1 The construction of $L\natural M$, the submanifold M_t and the collar neighbourhood c of $\partial_0(L\natural M)$.

i.e., the map (3.7) that we would like to show is a Serre fibration, is also a quotient map, with the same domain. Observe that $\{(C_\epsilon \cup M_t) \cap L\}$ is cofinal in the directed set of all closed neighbourhoods of $\partial_0 L$ in L. This means that two diffeomorphisms of $\mathrm{Diff}_{\partial_0}(L\natural M)$ have the same image under Ψ if and only if they have the same image under $\underset{\epsilon,t\to 0}{\mathrm{colim}}(\Psi_{\epsilon,t})$. As they are quotient maps of the same space, it follows that $\Psi \cong \underset{\epsilon,t\to 0}{\mathrm{colim}}(\Psi_{\epsilon,t})$.

We will prove below that each $\Psi_{\epsilon,t}$ is a fibre bundle (and hence a Serre fibration), and then deduce that Ψ is a Serre fibration using the following general fact.

($*$) Any filtered colimit of based Serre fibrations between compactly-generated weak-Hausdorff spaces is again a Serre fibration.

For a reference for this fact, see Proposition 1.2.3.5(1) of [TV08], which states that a filtered colimit of fibrations is a fibration in any compactly generated model category. The classical model category of based compactly-generated weak-Hausdorff spaces, with its Quillen model structure in which the fibrations are the Serre fibrations, is compactly generated (see for example Proposition 6.3 of [MMSS01]).

To apply ($*$) in our situation, first note that we are taking a directed colimit, which is in particular a filtered colimit. We then need to check that the diffeomorphism groups $\mathrm{Diff}_{C_\epsilon}(L\natural M)$ and their quotients are compactly-generated weak-Hausdorff spaces. Diffeomorphism groups of manifolds, in the Whitney topology, are always first-countable and Hausdorff, and thus compactly-generated and weak-Hausdorff. Moreover, the property of being compactly-generated is preserved when taking quotients. The property of being weak Hausdorff is *not* preserved when taking quotients; however, in the process of proving that each $\Psi_{\epsilon,t}$ is a fibre bundle below, we will also show that its target space $\mathrm{Diff}_{C_\epsilon}(L\natural M)/\mathrm{Diff}_{C_\epsilon}(L\natural M; M_t)$ is Hausdorff.

It therefore remains to show that each $\Psi_{\epsilon,t}$ is a fibre bundle (and its target space is Hausdorff). Write

$$\mathrm{Emb}_{C_\epsilon}(M_t, L\natural M)$$

for the space of smooth embeddings $\varphi\colon M_t \to L\natural M$ such that $\varphi(x) = x$ for all $x \in C_\epsilon \cap M_t$ and $\varphi(B) \subseteq A \sqcup B$. There is a restriction map

$$\Phi_{\epsilon,t}\colon \mathrm{Diff}_{C_\epsilon}(L\natural M) \longrightarrow \mathrm{Emb}_{C_\epsilon}(M_t, L\natural M),$$

which is equivariant with respect to the left-action of $\mathrm{Diff}_{C_\epsilon}(L\natural M)$ by post-composition. Note that this factors through the quotient map $\Psi_{\epsilon,t}$, so we have an induced map

$$\begin{tikzcd}
\mathrm{Diff}_{C_\epsilon}(L\natural M) \arrow[r, "\Phi_{\epsilon,t}"] \arrow[d, "\Psi_{\epsilon,t}"'] & \mathrm{Emb}_{C_\epsilon}(M_t, L\natural M). \\
\mathrm{Diff}_{C_\epsilon}(L\natural M)/\mathrm{Diff}_{C_\epsilon}(L\natural M; M_t) \arrow[ur, "\widehat{\Phi}_{\epsilon,t}"'] &
\end{tikzcd}$$

Moreover, if two diffeomorphisms of $\mathrm{Diff}_{C_\epsilon}(L\natural M)$ have the same image under $\Phi_{\epsilon,t}$, their difference lies in $\mathrm{Diff}_{C_\epsilon}(L\natural M; M_t)$, so the induced map $\widehat{\Phi}_{\epsilon,t}$ is injective. We will prove in the next paragraphs that, after restricting its codomain $\mathrm{Emb}_{C_\epsilon}(M_t, L\natural M)$ to its image, the map $\Phi_{\epsilon,t}$ is a fibre bundle. Hence, restricting $\mathrm{Emb}_{C_\epsilon}(M_t, L\natural M)$ to $\mathrm{im}(\Phi_{\epsilon,t})$ in the above diagram, we obtain a diagram

$$\begin{tikzcd}
\mathrm{Diff}_{C_\epsilon}(L\natural M) \arrow[r, two heads, "\Phi_{\epsilon,t}"] \arrow[d, two heads, "\Psi_{\epsilon,t}"'] & \mathrm{im}(\Phi_{\epsilon,t}) \\
\mathrm{Diff}_{C_\epsilon}(L\natural M)/\mathrm{Diff}_{C_\epsilon}(L\natural M; M_t) \arrow[ur, "\widehat{\Phi}_{\epsilon,t}"'] &
\end{tikzcd}$$

in which the vertical and horizontal maps are quotient maps (since surjective fibre bundles are always quotient maps) and the diagonal map $\widehat{\Phi}_{\epsilon,t}$ is a bijection. This implies that $\widehat{\Phi}_{\epsilon,t}$ is in fact a homeomorphism, and hence $\Psi_{\epsilon,t} = \widehat{\Phi}_{\epsilon,t}^{-1} \circ \Phi_{\epsilon,t}$ is a fibre bundle, as required. Moreover, the target space of $\Psi_{\epsilon,t}$ is homeomorphic to a subspace of $\mathrm{Emb}_{C_\epsilon}(M_t, L\natural M)$, which is Hausdorff, so we have also incidentally shown that the target space of $\Psi_{\epsilon,t}$ is Hausdorff.

It finally remains to prove that $\Phi_{\epsilon,t} \colon \mathrm{Diff}_{C_\epsilon}(L\natural M) \to \mathrm{im}(\Phi_{\epsilon,t}) \subseteq \mathrm{Emb}_{C_\epsilon}(M_t, L\natural M)$ is a fibre bundle. Since it is equivariant with respect to the left-action of $\mathrm{Diff}_{C_\epsilon}(L\natural M)$, it will suffice to prove that the action of $\mathrm{Diff}_{C_\epsilon}(L\natural M)$ on $\mathrm{im}(\Phi_{\epsilon,t})$ is *locally retractile* ($\equiv$ *admits local cross-sections*). This is because, by [Pal60, Theorem A], any G-equivariant map into a G-locally retractile space is a fibre bundle.

Thus, we have to prove the following statement: given an embedding $e \in \mathrm{im}(\Phi_{\epsilon,t}) \subseteq \mathrm{Emb}_{C_\epsilon}(M_t, L\natural M)$, we may find an open neighbourhood $\mathcal{U}$ of e and a continuous map $\gamma \colon \mathcal{U} \to \mathrm{Diff}_{C_\epsilon}(L\natural M)$ such that $\gamma(e) = \mathrm{id}$ and $\gamma(f) \circ e = f$ for any $f \in \mathcal{U}$. Since $\mathrm{Diff}_{C_\epsilon}(L\natural M)$ acts transitively on $\mathrm{im}(\Phi_{\epsilon,t})$, it will suffice to prove this for just one such e, which we take to be the inclusion $M_t \hookrightarrow L\natural M$.

To prove this, we apply a result of Cerf [Cer61, Lemme II.2.1.2, page 291], which we first recall. Let X be a manifold-with-corners. This means in particular that X has a stratification into *faces* (for example, if X is a connected manifold with boundary, but no higher-codimension corners, then its set of faces is $\pi_0(\partial X) \sqcup \{X\}$). Each point $x \in X$ may lie in many faces, but it has a unique *smallest* face (according to inclusion) in which it lies, which we denote by $\mathsf{F}_X(x)$. Now if Y is any submanifold-with-corners of X, we define

$$C^\infty_{\mathrm{face}}(Y, X) = \{\text{smooth maps } \varphi \colon Y \to X \text{ such that } \mathsf{F}_X(\varphi(x)) = \mathsf{F}_X(x) \text{ for each } x \in Y\},$$

equipped with the Whitney topology. The *Extension Lemma* II.2.1.2 of [Cer61] says that, if Y is closed in X and V is any neighbourhood of Y in X, then the restriction map

$$C^\infty_{\mathrm{face}}(X, X) \longrightarrow C^\infty_{\mathrm{face}}(Y, X)$$

admits a section s defined on an open neighbourhood $\mathcal{V}$ of the inclusion in $C^\infty_{\mathrm{face}}(Y, X)$, such that $s(\mathrm{incl}) = \mathrm{id}$ and $s(f)(x) = x$ for all $f \in \mathcal{V}$ and $x \in X \smallsetminus V$.

Step 1. Note that, since each embedding $f \in \mathrm{im}(\Phi_{\epsilon,t})$ extends to a diffeomorphism of $L\natural M$, it restricts to an embedding of $\partial M \smallsetminus \partial_0 M$ into $(\partial M \smallsetminus \partial_0 M) \sqcup (\partial L \smallsetminus \partial_0 L)$, and hence it induces an injection $f_\partial \colon \pi_0(\partial M \smallsetminus \partial_0 M) \to \pi_0(\partial M \smallsetminus \partial_0 M) \sqcup \pi_0(\partial L \smallsetminus \partial_0 L)$. By definition of the embedding space $\mathrm{Emb}_{C_\epsilon}(M_t, L\natural M)$, f also sends B into $A \sqcup B$, so it also induces an injection $f_\sharp \colon \pi_0(B) \to \pi_0(B) \sqcup \pi_0(A)$. The function $f \mapsto (f_\partial, f_\sharp)$ is locally constant, so its fibres are open. Let $\mathcal{U}'$ be the open subset of $\mathrm{im}(\Phi_{\epsilon,t})$ consisting of all f such that f_∂ is the inclusion and $f_\sharp(\pi_0(B)) = \pi_0(B)$.

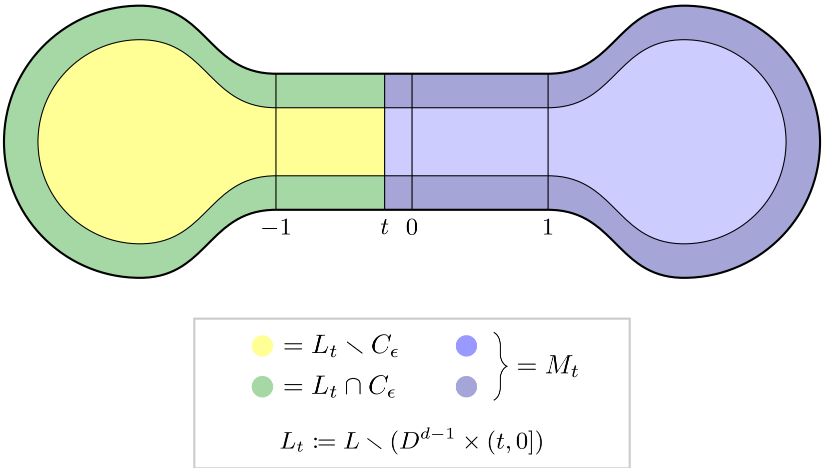


Figure 3.2 Extending an embedding from M_t to $M_t \cup C_\epsilon$ and then to all of $L \natural M$.

Note that the second condition implies that $f(B) = B$, since f is an embedding and B is a closed manifold.

Step 2. Each embedding $f \in \mathcal{U}'$ restricts to the identity on $M_t \cap C_\epsilon$, so we may extend it to a smooth map from the manifold-with-corners $M_t \cup C_\epsilon$ into $L \natural M$ by defining it to be the identity also on the rest of C_ϵ. (See Figure 3.2 for a schematic picture.) This extension is continuous in the input f, meaning that we have defined a continuous map $\gamma' : \mathcal{U}' \to C^\infty(M_t \cup C_\epsilon, L \natural M)$. Moreover, for each $f \in \mathcal{U}'$, the extension $\gamma'(f)$ lies in the subspace $C^\infty_{\mathrm{face}}(M_t \cup C_\epsilon, L \natural M)$, since it sends points of $\mathrm{int}(L \natural M)$ into $\mathrm{int}(L \natural M)$ and, for any boundary-component P of $L \natural M$ lying in $M_t \cup C_\epsilon$, it sends P into itself (this is because $f_\partial = \mathrm{incl}$). Thus, we have a continuous map

$$\gamma' : \mathcal{U}' \longrightarrow C^\infty_{\mathrm{face}}(M_t \cup C_\epsilon, L \natural M)$$

such that $\gamma'(\mathrm{incl}) = \mathrm{incl}$ and $\gamma'(f)|_{M_t} = f$ for all $f \in \mathcal{U}'$.

Step 3. Now set $X = L \natural M$ and $Y = M_t \cup C_\epsilon$ in the Extension Lemma of Cerf above, and choose V to be any open neighbourhood of $M_t \cup C_\epsilon$ in $L \natural M$ that is disjoint from the submanifold $A \subset \mathrm{int}(L)$. Composing the local section s obtained from the Extension Lemma with γ', we have a continuous map

$$\gamma'' = s \circ \gamma' : \mathcal{U}'' = (\gamma')^{-1}(\mathcal{V}) \longrightarrow C^\infty_{\mathrm{face}}(L \natural M, L \natural M)$$

such that $\gamma''(\mathrm{incl}) = \mathrm{id}$ and for any $f \in \mathcal{U}''$ we have $\gamma''(f)|_{M_t} = f$ and $\gamma''(f)(A) = A$. Moreover, by construction, we also know that $\gamma''(f)(x) = x$ for all $x \in C_\epsilon$ and $\gamma''(f)(B) = B$.

Step 4. Finally, note that $\mathrm{Diff}(L \natural M)$ is open in $C^\infty(L \natural M, L \natural M)$, so

$$\mathcal{U} = (\gamma'')^{-1}(C^\infty_{\mathrm{face}}(L \natural M, L \natural M) \cap \mathrm{Diff}(L \natural M))$$

is an open neighbourhood of the inclusion in $\mathrm{im}(\Phi_{\epsilon,t})$. For each $f \in \mathcal{U}$, the diffeomorphism $\gamma''(f)$ of $L \natural M$ fixes each point of C_ϵ and sends $A \sqcup B$ onto itself, so it is an element of $\mathrm{Diff}_{C_\epsilon}(L \natural M)$. So we have a continuous map

$$\gamma = \gamma''|_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathrm{Diff}_{C_\epsilon}(L \natural M)$$

such that $\gamma(\mathrm{incl}) = \mathrm{id}$ and, for all $f \in \mathcal{U}$, we have $\gamma(f) \circ \mathrm{incl} = f$. This completes the proof. □

For future convenience, we record a useful corollary of the proof of Proposition 3.15.

Definition 3.17 Let N be a smooth, connected d-manifold with non-empty boundary and let T be a compact, connected, proper[4] $(d-1)$-submanifold of N with non-empty boundary ∂T contained in a single boundary-component $\partial_0 N$ of N. Assume that $N \smallsetminus T$ has two components and denote their closures by N_1 and N_2, which are manifolds with corners.

[4] In this context, *proper* means that the interior of T lies in the interior of N, the boundary of T lies in the boundary of N and, moreover, near the boundary, $T \subset N$ is modelled on $\mathbb{R}^{d-2} \times \{0\} \times [0,\infty) \subset \mathbb{R}^{d-1} \times [0,\infty)$.

Note that $N = N_1 \cup N_2$ and $T = N_1 \cap N_2$. The prototypical example of this setting is $N = L \natural M$ and $T = D^{d-1} \times \{0\}$ (see Figure 3.1), with $N_1 = L$ and $N_2 = M$. Define

$$\mathrm{Emb}^{\mathrm{Diff}}_{\partial_0}(N_1, N)$$

to be the set of smooth embeddings $\varphi \colon N_1 \to N$, equipped with a germ of an extension $\bar{\varphi}$ to some neighbourhood of N_1 in N, such that

- there exists some neighbourhood U of $\partial_0 N$ such that $\bar{\varphi}(x) = x$ for all $x \in U \cap \mathrm{domain}(\bar{\varphi})$,
- there exists an extension of $\bar{\varphi}$ to a diffeomorphism of N that acts by the identity on U.

This is topologised as follows. Let U be a neighbourhood of $\partial_0 N$ in N and let V be a neighbourhood of N_1 in N. Let $\mathrm{Emb}^{\mathrm{Diff}}_U(V, N)$ be the set of smooth embeddings $\varphi \colon V \to N$ such that $\varphi(x) = x$ for all $x \in U \cap V$ and there exists an extension of φ to a diffeomorphism of N that acts by the identity on U. This is given the subspace topology induced by the Whitney topology on the space of smooth maps $C^\infty(V, N)$. Note that, if $U' \subseteq U$ and $V' \subseteq V$ are neighbourhoods as above, there are continuous restriction maps

$$\mathrm{Emb}^{\mathrm{Diff}}_U(V, N) \longrightarrow \mathrm{Emb}^{\mathrm{Diff}}_{U'}(V', N).$$

The set $\mathrm{Emb}^{\mathrm{Diff}}_{\partial_0}(N_1, N)$ is the colimit of the underlying sets of this diagram of spaces, so we may topologise it by defining

$$\mathrm{Emb}^{\mathrm{Diff}}_{\partial_0}(N_1, N) \;=\; \operatorname*{colim}_{U,V}(\mathrm{Emb}^{\mathrm{Diff}}_U(V, N)).$$

Lemma 3.18 *Under the conditions of Proposition 3.15, there is a natural homeomorphism*

$$\mathrm{Diff}_{\partial_0}(L \natural M)/\mathrm{Diff}_{\partial_0}(L) \;\cong\; \mathrm{Emb}^{\mathrm{Diff}}_{\partial_0}(M, L \natural M).$$

Proof. This follows from the sequence of homeomorphisms:

$$\begin{aligned}
\mathrm{Diff}_{\partial_0}(L \natural M)/\mathrm{Diff}_{\partial_0}(L) &\cong \operatorname*{colim}_{\epsilon, t \to 0}(\mathrm{Diff}_{C_\epsilon}(L \natural M)/\mathrm{Diff}_{C_\epsilon}(L \natural M; M_t)) \\
&\cong \operatorname*{colim}_{\epsilon, t \to 0}(\mathrm{Emb}^{\mathrm{Diff}}_{C_\epsilon}(M_t, L \natural M)) \\
&\cong \mathrm{Emb}^{\mathrm{Diff}}_{\partial_0}(M, L \natural M).
\end{aligned}$$

The first homeomorphism comes from the fact that we showed, during the proof of Proposition 3.15, that the maps Ψ and $\operatorname*{colim}_{\epsilon,t\to 0}(\Psi_{\epsilon,t})$ are homeomorphic, so in particular their target spaces are homeomorphic.

The second homeomorphism is the homeomorphism $\widehat{\Phi}_{\epsilon,t}$ from the proof of Proposition 3.15. The third homeomorphism is by definition of the topology on $\mathrm{Emb}^{\mathrm{Diff}}_{\partial_0}(M, L \natural M)$ and the facts that $\{C_\epsilon\}$ is a cofinal family of neighbourhoods playing the role of U in the definition and $\{M_t\}$ is a cofinal family of neighbourhoods playing the role of V. $\square$

A variant for decorated manifolds. In practice, we will mainly use a slight variant of this setup, where diffeomorphisms are only assumed to be the identity on a neighbourhood of two disjoint discs in the boundary.

Definition 3.19 (*Decorated manifolds and their boundary connected sum*) First, let us say that a *boundary-cylinder* for a smooth d-manifold M is a map

$$e \colon D^{d-1} \times [0,1] \longrightarrow M$$

such that $e^{-1}(\partial M) = (\partial D^{d-1} \times [0,1]) \cup (D^{d-1} \times \{0\})$ and e is a smooth embedding away from the sphere $\partial D^{d-1} \times \{0\}$. A *decorated manifold* is then a smooth d-manifold M, equipped with a closed submanifold $A \subset \mathrm{int}(M)$ and an ordered pair (e_1, e_2) of disjoint boundary-cylinders for $M \smallsetminus A$. Given two decorated manifolds, one may form their boundary connected sum, as in the beginning of the proof of Proposition 3.15, by gluing $\mathrm{image}(e_2)$ of the first manifold to $\mathrm{image}(e_1)$ of the second manifold.

Definition 3.20 (*Diffeomorphisms and embeddings of decorated manifolds*) For any decorated manifold (M, A, e_1, e_2), we define

$$\mathrm{Diff}_{\mathrm{dec}}(M)$$

to be the group of diffeomorphisms of M that send A onto itself and restrict to the identity on some neighbourhood of $e_1(D^{d-1} \times \{0\}) \sqcup e_2(D^{d-1} \times \{0\})$. This is topologised as in Definition 3.14, as a colimit of Whitney topologies. Note that, for a pair of decorated manifold L and M, the topological group $\mathrm{Diff}_{\mathrm{dec}}(L)$ may be viewed as a subgroup of $\mathrm{Diff}_{\mathrm{dec}}(L \natural M)$, by extending diffeomorphisms by the identity on M.

Similarly, if (L, A, e_1, e_2) and (M, B, f_1, f_2) are two decorated manifolds, we define

$$\mathrm{Emb}^{\mathrm{Diff}}_{\mathrm{dec}}(M, L \natural M)$$

to be the set of smooth embeddings $\varphi \colon M \to L \natural M$, equipped with a germ $\bar{\varphi}$ of an extension of φ to a neighbourhood of M in $L \natural M$, such that φ restricts to the identity on a neighbourhood of $f_2(D^{d-1} \times \{0\})$ and $\bar{\varphi}$ extends to a diffeomorphism of $L \natural M$ that restricts to the identity on a neighbourhood of $e_1(D^{d-1} \times \{0\})$. This is also topologised as a colimit of Whitney topologies, as in Definition 3.17.

The proofs of Proposition 3.15 and Lemma 3.18 may easily be adapted to prove the following analogue.

Proposition 3.21 *For any two decorated manifolds L and M, the quotient map*

$$\mathrm{Diff}_{\mathrm{dec}}(L \natural M) \longrightarrow \mathrm{Diff}_{\mathrm{dec}}(L \natural M)/\mathrm{Diff}_{\mathrm{dec}}(L) \tag{3.9}$$

is a Serre fibration and its target space is homeomorphic to $\mathrm{Emb}^{\mathrm{Diff}}_{\mathrm{dec}}(M, L \natural M)$.

Remark 3.22 We note that all of the above may be adapted to the setting where the closed submanifolds $A \subset \mathrm{int}(L)$ and $B \subset \mathrm{int}(M)$ are equipped with orientations, and all diffeomorphisms and embeddings in Definitions 3.14 and 3.17 are required to preserve these orientations. Proposition 3.15, Lemma 3.18 and Proposition 3.21 generalise immediately to this setting.

3.3 Semi-monoidal categories and semicategories

All of the examples of categories $\mathcal{C}$ for which we would like topologically to construct representations will be of the form $\langle \mathcal{G}, \mathcal{M} \rangle$, where $\mathcal{G}$ is a braided monoidal category and $\mathcal{M}$ is a left-module of $\mathcal{G}$. We therefore need to find a topological monoidal groupoid $\mathcal{G}^{\mathrm{t}}$ and a topological category $\mathcal{M}^{\mathrm{t}}$ with a left-action of $\mathcal{G}^{\mathrm{t}}$, satisfying condition (3.4) of Lemma 3.13 and such that $\pi_0(\mathcal{G}^{\mathrm{t}}) \cong \mathcal{G}$ and $\pi_0(\mathcal{M}^{\mathrm{t}}) \cong \mathcal{M}$. Given this, a continuous functor $\langle \mathcal{G}, \mathcal{M} \rangle \to \widetilde{\mathrm{Cov}}_Q$ will induce a functor

$$\mathcal{C} = \langle \mathcal{G}, \mathcal{M} \rangle \cong \pi_0(\langle \mathcal{G}^{\mathrm{t}}, \mathcal{M}^{\mathrm{t}} \rangle) = \pi_0(\mathcal{C}^{\mathrm{t}}) \longrightarrow R\text{-}\mathrm{Mod}^{\mathrm{tw}},$$

via the construction summarised in the diagram (2.26). Note that there is no need for $\mathcal{G}^{\mathrm{t}}$ to be braided, since this structure is not needed in order to form the topological Quillen bracket construction, or for Lemma 3.13. In fact, it will be convenient for our examples to drop even more structure from $\mathcal{G}$, and assume only that it is a *semi-monoidal category*. (Recall that this is defined analogously to a monoidal category, but without any of the structure or conditions involving left or right units.) This is because it will typically be easy to lift the monoidal structure of $\mathcal{G}$ to an associative binary operation on $\mathcal{G}^{\mathrm{t}}$,[5] but it is often not possible to make this lifted operation *unital* in a natural way.

Remark 3.23 If $\mathcal{G}$ is a topological semi-monoidal groupoid and $\mathcal{M}$ is a topological category with a continuous left-action of $\mathcal{G}$, then Definition 3.11 generalises directly to this setting, and produces a semicategory $\langle \mathcal{G}, \mathcal{M} \rangle$. (The associator of $\mathcal{G}$ is used to define composition in $\langle \mathcal{G}, \mathcal{M} \rangle$ and the pentagon condition for the associator implies associativity of this composition.) Moreover, Lemma 3.13 also generalises to this setting, and implies, under the same hypotheses, that there is an isomorphism of semicategories $\pi_0(\langle \mathcal{G}, \mathcal{M} \rangle) \cong \langle \pi_0(\mathcal{G}), \pi_0(\mathcal{M}) \rangle$.

[5] More precisely, a binary operation admitting an associator that satisfies the pentagon condition.

Using this remark, it will suffice, in our examples, to find a topological semi-monoidal groupoid $\mathcal{G}^{\mathrm{t}}$, such that $\pi_0(\mathcal{G}^{\mathrm{t}}) \cong \mathcal{G}$ as semi-monoidal groupoids and which satisfies condition (3.4) of Lemma 3.13. Then $\langle \mathcal{G}^{\mathrm{t}}, \mathcal{M}^{\mathrm{t}} \rangle$ is a topological semicategory, and the input for the topological construction will be a continuous semifunctor $\langle \mathcal{G}^{\mathrm{t}}, \mathcal{M}^{\mathrm{t}} \rangle \to \widetilde{\mathrm{Cov}}_Q$. Via Lemma 3.13, Remark 3.23 and diagram (2.26) we then obtain a semifunctor $\langle \mathcal{G}, \mathcal{M} \rangle \to R\text{-}\mathrm{Mod}^{\mathrm{tw}}$. However, the source and target of this semifunctor are both categories (since $\mathcal{G}$ is a monoidal category, not just a semi-monoidal category), and so we may ask whether this semifunctor is in fact a functor. The final (small) step of the topological construction will then be to verify that it does in fact preserve identities, and is therefore a functor.

Remark 3.24 In practice, in our examples that we construct in §§4.4 and 4.5 using this framework, we will ignore this issue of a lack of identities in $\langle \mathcal{G}^{\mathrm{t}}, \mathcal{M}^{\mathrm{t}} \rangle$, and proceed as if it were a topological category (with identities), to avoid unnecessary extra complications. However, formally, one should modify that procedure as described in the paragraph above.

3.4 The input categories

The families of groups for which it is natural to define the first homological functors are the braid groups of surfaces, mapping class groups of surfaces and loop-braid groups. Before doing this (in §4), we first introduce the suitable categorical framework to deal with the application of the construction of §2 to these families of groups.

In this section, we first recollect the various definitions and properties of these families of groups. Then we present an appropriate groupoid encoding the considered family of groups in each situation. These groupoids will systematically be braided monoidal or modules over a braided monoidal category: this allows one to apply Quillen's bracket construction (see §3.1) in each case, the resulting category being "richer" in the sense that it has more morphisms. This is done in §§3.4.1–3.4.3 for (surface) braid groups and mapping class groups of surfaces, and in §3.4.4 for the loop-braid groups.

In §3.4.5, we construct topological versions of all of these groupoids, recovering each of the discrete groupoids — and hence their Quillen bracket constructions (using the results of §3.2) — after taking π_0.

3.4.1 Classical braid groups

We recall that the *(classical) braid group* on $n \geqslant 2$ strings denoted by $\mathbf{B}_n$ is the group generated by $\sigma_1, ..., \sigma_{n-1}$ satisfying the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $i \in \{1, \ldots, n-2\}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $i, j \in \{1, \ldots, n-1\}$ such that $\mid i-j \mid \geqslant 2$. $\mathbf{B}_0$ and $\mathbf{B}_1$ both are the trivial group. The family of braid groups is associated with the braid groupoid $\boldsymbol{\beta}$, with objects the natural numbers $n \in \mathbb{N}$ and morphisms (for $n, m \in \mathbb{N}$):

$$\mathrm{Hom}_{\boldsymbol{\beta}}(n, m) = \begin{cases} \mathbf{B}_n & \text{if } n = m \\ \emptyset & \text{if } n \neq m. \end{cases}$$

The composition of morphisms $\circ$ in the groupoid $\boldsymbol{\beta}$ corresponds to the group operation of the braid groups. So we identify the composition in $\sigma \circ \sigma'$ with the group multiplication $\sigma\sigma'$ in $\mathbf{B}_n$ (with the convention that we read from the right to the left for the group operation).

We recall from [Mac98, Chapter XI, Section 4] that a monoidal product $\natural : \boldsymbol{\beta} \times \boldsymbol{\beta} \to \boldsymbol{\beta}$ is defined by the usual addition for the objects and laying two braids side by side for the morphisms. The object 0 is the unit of this monoidal product. The strict monoidal groupoid $(\boldsymbol{\beta}, \natural, 0)$ is braided: the braiding is defined for all natural numbers n and m such that $n + m \geqslant 2$ by

$$b^{\boldsymbol{\beta}}_{n,m} = (\sigma_m \circ \cdots \circ \sigma_2 \circ \sigma_1) \circ \cdots \circ (\sigma_{n+m-2} \circ \cdots \circ \sigma_n \circ \sigma_{n-1}) \circ (\sigma_{n+m-1} \circ \cdots \circ \sigma_{n+1} \circ \sigma_n)$$

where $\{\sigma_i\}_{i \in \{1,\ldots,n+m-1\}}$ denote the Artin generators of the braid group $\mathbf{B}_{n+m}$.

3.4.2 Mapping class groups of surfaces

The following suitable category to consider the mapping class groups of surfaces for our work is introduced in [RW17, Section 5.6]. The decorated surfaces groupoid $\mathcal{M}_2$ is defined by:

- Objects: decorated surfaces (S, I), where S is a smooth, connected, compact surface with at least one boundary component, together with a parametrised interval $I\colon [-1,1] \hookrightarrow \partial_0 S$ in the boundary. Hence there is a distinguished boundary component *decorated* by I and denoted by $\partial_0 S$. When there is no ambiguity, we omit the parametrised interval I from the notation.

- Morphisms: isotopy classes of diffeomorphisms of surfaces which restrict to the identity on a neighbourhood of their parametrised intervals I. Note that the non-distinguished boundary components may be freely moved by the mapping classes. The automorphism group of S forms the *mapping class group* of S which is denoted by $\pi_0\mathrm{Diffeo}_I(S)$.

Recall that a diffeomorphism of a surface which fixes an interval in a boundary component is isotopic to a diffeomorphism which fixes pointwise the boundary component $\partial_0 S$ of the surface. When the surface S is orientable, the orientation on S is induced by the orientation of I. The isotopy classes of diffeomorphisms then automatically preserve that orientation as they restrict to the identity on a neighbourhood of I.

Remark 3.25 Instead of boundary components, we could have equivalently considered surfaces with *punctures*. Namely, for each object S of $\mathcal{M}_2$ we associate $\bar{S}$ the surface obtained by gluing a disc with one puncture on all the boundary components but $\partial_0 S$. We denote by $\mathcal{P}$ the corresponding finite set of punctures. Let $\mathrm{Diff}_I(S, \mathcal{P})$ the group of diffeomorphisms of $\bar{S}$ including points filling in the punctures (called *marked points*), which restrict to the identity on a neighbourhood of the parametrised interval I and which send the set $\mathcal{P}$ of marked points to itself (i.e. permuting the marked points). When $\mathcal{P}$ is the empty set, we omit it from the notation. We denote by $\bar{\mathcal{M}}_2$ the category associated with this alternative. Since $\pi_0(\mathrm{Diff}_I(S)) \cong \pi_0\mathrm{Diff}_I(\bar{S}, \mathcal{P})$, the categories $\bar{\mathcal{M}}_2$ and $\mathcal{M}_2$ are equivalent. Also the mapping class group of S identifies with the group of isotopy classes of homeomorphisms of S (see for example [FM12, Section 1.4.2]). Hence we represent mapping classes using diffeomorphisms of surfaces with boundaries, but sometimes we consider homeomorphisms instead of diffeomorphisms and punctures instead of boundaries when they are more convenient to consider.

We denote by $\mathbb{D}^2$ the unit 2-disc. Let $\Sigma_{0,1}^1$ denote the cylinder $\mathbb{S}^1 \times [0,1]$ (which can be thought of as the disc $\mathbb{D}^2$ with a smaller disc is the interior which is removed), $\Sigma_{1,1}$ denote the torus with one boundary component $\left(\mathbb{S}^1 \times \mathbb{S}^1 \setminus \mathrm{Int}\left(\mathbb{D}^2\right)\right)$ and $\mathcal{N}_{1,1}$ denote a Möbius band. For S an object of the groupoid $\mathcal{M}_2$, by the classification of surfaces, there exist $g, s, c \in \mathbb{N}$ such that there is a diffeomorphism:

$$S \simeq \left(\underset{s}{\natural}\Sigma_{0,1}^1\right) \natural \left(\underset{g}{\natural}\Sigma_{1,1}\right) \natural \left(\underset{c}{\natural}\mathcal{N}_{1,1}\right).$$

Notation 3.26 If $c = 0$, then g and s are unique, we denote by $\Sigma_{g,1}^s$ the boundary connected sum $\left(\Sigma_{0,1}^1\right)^{\natural s} \natural \Sigma_{1,1}^{\natural g}$ and by $\mathbf{\Gamma}_{g,1}^s$ the mapping class group $\pi_0\mathrm{Diff}_I\left(\Sigma_{g,1}^s\right)$. If $g = 0$, we denote by $\mathcal{N}_{c,1}^s$ the boundary connected sum $\left(\Sigma_{0,1}^1\right)^{\natural s} \natural \mathcal{N}_{1,1}^{\natural c}$ and by $\boldsymbol{\mathcal{N}}_{c,1}^s$ the mapping class group $\pi_0\mathrm{Diff}_I\left(\mathcal{N}_{c,1}^s\right)$. In both cases, when $s = 0$, we omit it most of the time from the notation.

If $c = g = 0$, then the boundary connected sum $\left(\Sigma_{0,1}^1\right)^{\natural s}$ is diffeomorphic to the unit 2-disc where n interior discs are removed $\mathbb{D}_s = \mathbb{D}^2 \setminus \{p_1, \ldots, p_s\}$: we will abuse the notation and denote these two objects in the same way.

The groupoid $\mathcal{M}_2$ has a monoidal structure induced by gluing; for completeness, the definition is outlined below (see [RW17, Section 5.6.1] for technical details). For two decorated surfaces (S_1, I_1) and (S_2, I_2), the boundary connected sum $(S_1, I_1) \natural (S_2, I_2) = (S_1 \natural S_2, I_1 \natural I_2)$ is defined with $S_1 \natural S_2$ the surface obtained from gluing S_1 and S_2 along the half-interval I_1^+ and the half-interval I_2^-, and

$I_1 \natural I_2 = I_1^- \bigcup I_2^+$. The diffeomorphisms being the identity on a neighbourhood of the parametrised intervals I_1 and I_2, we canonically extend the diffeomorphisms of S_1 and S_2 to $S_1 \natural S_2$. The braiding of the monoidal structure $b^{\mathcal{M}_2}_{(S_1,I_1),(S_2,I_2)} : (S_1, I_1) \natural (S_2, I_2) \to (S_2, I_2) \natural (S_1, I_1)$ is given by doing half a Dehn twist in a pair of pants neighbourhood of ∂S_1 and ∂S_2. By [RW17, Proposition 5.18], the boundary connected sum $\natural$ induces a strict braided monoidal structure $\left(\mathcal{M}_2, \natural, \left(\mathbb{D}^2, I\right), b^{\mathcal{M}_2}_{-,-}\right)$. There are no zero divisors in the category $\mathcal{M}_2$ and $\mathrm{Aut}_{\mathcal{M}_2}\left(\mathbb{D}^2\right) = \{id_{\mathbb{D}^2}\}$ by Alexander's trick.

Let $\mathcal{M}_2^+$ (respectively $\mathcal{M}_2^-$) be the full subgroupoids of $\mathcal{M}_2$ with objects the orientable surfaces (respectively the non-orientable or genus 0 surfaces). The monoidal structure $(\mathcal{M}_2, \natural, 0)$ restricts to a braided monoidal structure both on the subgroupoids $\mathcal{M}_2^+$ and $\mathcal{M}_2^-$, denoted in the same way $\left(\mathcal{M}_2^+, \natural, 0\right)$ and $\left(\mathcal{M}_2^-, \natural, 0\right)$.

Torelli groups: Let $\mathcal{M}_2^{+,\mathrm{gen}}$ be the full subgroupoid of $\mathcal{M}_2^+$ with objects the orientable surfaces such that $\partial S = \partial_0 S$. The monoidal product $\natural$ restricts to a braided monoidal structure on $\mathcal{M}_2^{+,\mathrm{gen}}$. We denote by ab the category of finitely generated abelian groups. The direct sum $\oplus$ induces a strict symmetric monoidal structure $(\mathrm{ab}, \oplus, 0_{\mathrm{Gr}})$, the symmetry being given by the canonical permutation of the free product.

Recall that the isotopy classes of the diffeomorphisms of an object $\Sigma_{g,1}$ of $\mathcal{M}_2^{+,\mathrm{gen}}$ act naturally on its first homology group $H_1(\Sigma_{g,1}, \mathbb{Z})$: the first homology groups $H_1(-, \mathbb{Z})$ thus define a functor from the category $\mathcal{M}_2$ to the category ab. Using Van Kampen's theorem, this functor is strong monoidal with respect to the structures $\left(\mathcal{M}_2^{+,\mathrm{gen}}, \natural, 0\right)$ and $(\mathrm{ab}, \oplus, 0_{\mathrm{Gr}})$. We recall that the *Torelli group* $\mathcal{I}_{g,1}$ of the surface $\Sigma_{g,1}$ is the kernel of the action $\mathbf{\Gamma}_{g,1} \to \mathrm{Aut}\left(H_1\left(\Sigma_{g,1}; \mathbb{Z}\right)\right)$. Let $\mathcal{M}_2^{\mathcal{I}}$ be the subcategory of $\mathcal{M}_2^{+,\mathrm{gen}}$ with the same objects and restricting to the Torelli groups for the automorphism groups.

For g and g' two natural numbers, let φ be an element of $\mathcal{I}_{g,1}$ and φ' be an element of $\mathcal{I}_{g',1}$. Since the first homology functor $H_1(-, \mathbb{Z}) : \mathcal{M}_2^{\mathcal{I}} \to \mathrm{ab}$ is strong monoidal, it follows from the universal property of the kernel that $\varphi \natural \varphi'$ belongs to $\mathcal{I}_{g'+g,1}$. By Lemma 3.10, the boundary connected sum induces a strict braided monoidal structure $\left(\mathcal{M}_2^{\mathcal{I}}, \natural, \mathbb{D}^2\right)$.

3.4.3 Surface braid groups

There are several ways to introduce the surface braid groups: first, they can be defined as the fundamental groups of some configuration spaces on surfaces; secondly, there are also explicit presentations of these groups by generators and relations; finally, they can be seen as normal subgroups of the mapping class groups of surfaces with punctures. For completeness, these definitions are outlined below and we refer the reader to [Bel04] and [GJ15] for a detailed and complete introduction to these groups.

Surface braid groups as configuration spaces. Let S be an object of the decorated surfaces groupoid $\mathcal{M}_2$. Let $F_n(S)$ be the configuration spaces of n ordered points in S:

$$F_n(S) := \left\{(x_1, \ldots, x_n) \in S^{\times n} \mid x_i \neq x_j \text{ if } i \neq j\right\}.$$

Let $C_n(S)$ be the configuration spaces of n unordered points in S, induced by the natural action by permutation of coordinates of the symmetric group $\mathfrak{S}_n$ on $F_n(S)$:

$$C_n(S) := \left\{(x_1, \ldots, x_n) \in S^{\times n} \mid x_i \neq x_j \text{ if } i \neq j\right\} / \mathfrak{S}_n.$$

The *braid group on the surface* S on n strings is the fundamental group of this unordered configuration space $\mathbf{B}_n(S) = \pi_1(C_n(S), c_0)$, where $c_0 = (z_1, \ldots, z_n)$ with z_i pairwise-distinct points on the boundary component $\partial_0 S$ for each $i \in \{1, \ldots, n\}$. We recall that the braid groups on the 2-disc $\mathbb{D}^2$ are the *classical braid groups* of §3.4.1 and we therefore omit $\mathbb{D}^2$ from the notations in this situation.

Moreover, for another natural number m, the preimage of the product $\mathfrak{S}_m \times \mathfrak{S}_n$ under the canonical projection $\mathbf{B}_{m+n}(S) \twoheadrightarrow \mathfrak{S}_{m+n}$ is called the *intertwining* (m, n)*-braid group* $\mathbf{B}_{m,n}(S)$. Namely, it is the fundamental group of the configuration space $F_{m+n}(S)/(\mathfrak{S}_m \times \mathfrak{S}_n)$. In addition, the map $F_{m+n}(S)/(\mathfrak{S}_m \times \mathfrak{S}_n) \to F_n(S)/(\mathfrak{S}_n)$ defined by forgetting the first m coordinates is a locally trivial fibration with fibre $F_m\left(S^{(n)}\right)/(\mathfrak{S}_m)$, where $S^{(n)}$ denotes the surface $\left(\Sigma_{0,1}^1\right)^{\natural n} \natural S$. The long exact sequence in homotopy of this fibration gives the following *split* short exact sequence:

$$1 \longrightarrow \mathbf{B}_m\left(S^{(n)}\right) \longrightarrow \mathbf{B}_{m,n}(S) \xrightarrow{\Lambda^S_{m,n}} \mathbf{B}_n(S) \longrightarrow 1\ . \tag{3.10}$$

The splitting $\mathbf{B}_n(S) \hookrightarrow \mathbf{B}_{m,n}(S)$ is the map induced by the inclusion $F_n(S) \hookrightarrow F_{m+n}\left(\mathbb{D}^2 \natural S\right)$ defined by arbitrary fixing m points $p_1, \ldots, p_m$ in the interior of $\mathbb{D}^2$ and sending $(x_1, \ldots, x_n)$ to $(p_1, \ldots, p_m, x_1, \ldots, x_n)$. Therefore the intertwining braid group $\mathbf{B}_{m,n}(S)$ is isomorphic to the semidirect product $\mathbf{B}_m\left(S^{(n)}\right) \rtimes \mathbf{B}_n(S)$: in particular, the natural action of $\mathbf{B}_n(S)$ on $\mathbf{B}_m\left(S^{(n)}\right)$ is thus equivalent to the conjugate action in $\mathbf{B}_{m,n}(S)$ if we regard these two groups as subgroups of $\mathbf{B}_{m,n}(S)$. We refer the reader to [GJ15, Section 3.1] for further details.

Presentations of surface braid groups. [Bel04] gives a presentation by generators and relations of surface braid groups. As of now, we fix three natural numbers $s \geqslant 0$, $g \geqslant 1$ and $c \geqslant 2$, and consider the surfaces $\Sigma^s_{g,1}$ and $\mathcal{N}^s_{c,1}$ described in §3.4.2. Throughout this work, we use the following presentation:

Proposition 3.27 ([Bel04, Theorems 1.1 and A.2]) *The braid group on n points on the orientable surface $\Sigma^s_{g,1}$, denoted by $\mathbf{B}_n\left(\Sigma^s_{g,1}\right)$ with $g \geqslant 1$, admits the following presentation:*

- *Generators:* $S = \{\sigma_i\}_{i\in\{1,\ldots,n-1\}}$, $A = \{a_i\}_{i\in\{1,\ldots,g\}}$, $B = \{b_i\}_{i\in\{1,\ldots,g\}}$ *and* $X = \{\xi_i\}_{i\in\{1,\ldots,s\}}$;
- *Relations:*
  - *Braid relations:* $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *and* $\sigma_k\sigma_j = \sigma_j\sigma_k$ *for all* $i, k, j \in \{1, \ldots, n-1\}$ *so that* $|k-j| \geqslant 2$;
  - *Mixed relations:*
    (R1) $c\sigma_i = \sigma_i c$ *and* $[c\sigma_{n-1}c, \sigma_{n-1}] = 1$ *for all* $c \in A \cup B \cup X$ *and* $i \in \{1, \ldots, n-2\}$;
    (R2) $a_j\sigma_{n-1}b_j = \sigma_{n-1}b_j\sigma_{n-1}a_j\sigma_{n-1}$, $\left[\sigma_{n-1}^{-1}a_k\sigma_{n-1}, a_l\right] = 1$, $\left[\sigma_{n-1}^{-1}b_k\sigma_{n-1}, b_l\right] = 1$, $\left[\sigma_{n-1}^{-1}a_k\sigma_{n-1}, b_l\right] = 1$, $\left[\sigma_{n-1}^{-1}b_k\sigma_{n-1}, a_l\right] = 1$ *for all* $j, k, l \in \{1, \ldots, g\}$ *so that* $k < l$;
    (R3) $\left[\sigma_{n-1}^{-1}\xi_j\sigma_{n-1}, c\right] = 1$ *and* $\left[\sigma_{n-1}^{-1}\xi_k\sigma_{n-1}, \xi_l\right] = 1$ *for all* $c \in A \cup B$ *and* $j, k, l \in \{1, \ldots, s\}$ *so that* $k < l$.

The braid group on n points on the non-orientable surface $\mathcal{N}^s_{c,1}$, denoted by $\mathbf{B}_n\left(\mathcal{N}^s_{c,1}\right)$ with $c \geqslant 2$, admits the following presentation:

- *Generators:* $S = \{\sigma_i\}_{i\in\{1,\ldots,n-1\}}$, $C = \{c_i\}_{i\in\{1,\ldots,c\}}$ *and* $X = \{\xi_i\}_{i\in\{1,\ldots,s\}}$;
- *Relations:*
  - *Braid relations:* $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *and* $\sigma_k\sigma_j = \sigma_j\sigma_k$ *for all* $i, k, j \in \{1, \ldots, n-1\}$ *so that* $|k-j| \geqslant 2$;
  - *Mixed relations:*
    (R1) $e\sigma_i = \sigma_i e$ *for all* $e \in C \cup X$ *and* $i \in \{1, \ldots, n-2\}$;
    (R2) $\sigma_{n-1}c_j\sigma_{n-1}c_j\sigma_{n-1} = c_j\sigma_{n-1}c_j$, $\left[\sigma_{n-1}^{-1}c_k\sigma_{n-1}, c_l\right] = 1$ *for all* $j, k, l \in \{1, \ldots, c\}$ *so that* $k < l$;
    (R3) $\left[\sigma_{n-1}^{-1}\xi_j\sigma_{n-1}, e\right] = 1$, $\left[\sigma_{n-1}^{-1}x\sigma_{n-1}^{-1}, x\right] = 1$ *and* $\left[\sigma_{n-1}^{-1}\xi_k\sigma_{n-1}, \xi_l\right] = 1$ *for all* $e \in C$, $x \in X$ *and* $j, k, l \in \{1, \ldots, s\}$ *so that* $k < l$.

Remark 3.28 In each case, the generators of the sets A, B, C and X are actually given by the generators of the fundamental group of the considered surface. Also, we use an opposite convention to the one of [Bel04]: namely our numbering is the converse of the one chosen there so that the respective roles σ_1 and σ_{n-1} in the mixed relations in the above presentation are switched compared to those of [Bel04].

Surface braid groups as mapping classes. Finally the surface braid groups $\mathbf{B}_n(\Sigma_{g,1})$ and $\mathbf{B}_n(\mathcal{N}_{c,1})$ are respectively isomorphic to the kernels of the homomorphisms $f^{\mathbf{\Gamma}}_{g,n}:\mathbf{\Gamma}^n_{g,1}\to\mathbf{\Gamma}_{g,1}$ and $f^{\boldsymbol{\mathcal{N}}}_{c,n}:\boldsymbol{\mathcal{N}}^n_{c,1}\to\boldsymbol{\mathcal{N}}_{c,1}$, induced by the map which glues a disc on all the boundary components which are freely moved. Namely, we have the following short exact sequences

$$1 \longrightarrow \mathbf{B}_n(\Sigma_{g,1}) \longrightarrow \mathbf{\Gamma}^n_{g,1} \xrightarrow{f^{\mathbf{\Gamma}}_{g,n}} \mathbf{\Gamma}_{g,1} \longrightarrow 1 \tag{3.11}$$

$$1 \longrightarrow \mathbf{B}_n(\mathcal{N}_{c,1}) \longrightarrow \boldsymbol{\mathcal{N}}^n_{c,1} \xrightarrow{f^{\boldsymbol{\mathcal{N}}}_{c,n}} \boldsymbol{\mathcal{N}}_{c,1} \longrightarrow 1. \tag{3.12}$$

They are constructed as follows: the parametrized isotopy extension theorem of [Cer61, II, 2.2.2 Corollaire 2] provides the locally-trivial fibre bundles $\mathrm{Diff}_I\left(\Sigma^n_{g,1}\right)\to\mathrm{Diff}_I(\Sigma_{g,1})\to C_n(\Sigma_{g,1})$ and $\mathrm{Diff}_I\left(\mathcal{N}^n_{c,1}\right)\to\mathrm{Diff}_I(\mathcal{N}_{c,1})\to C_n(\mathcal{N}_{c,1})$, where the left hand maps are defined by gluing a disc on all the boundary components of S but $\partial_0 S$; then we consider the associated long exact sequence of homotopy groups and use contractibility results of [Gra73, Théorème 1] for the component of the diffeomorphism groups. We refer the reader to [Bir74] or [GJ15, Section 2.4] for more details. This last point of view is the most convenient for the categorical framework we intend to set up.

Let $\mathcal{B}_2$ be the subgroupoid of $\mathcal{M}_2$ with the same objects and with morphisms those of $\mathcal{M}_2$ that become trivial after capping the non-parametrised boundary components. The monoidal structure $(\mathcal{M}_2,\natural,0)$ restricts to a braided monoidal structure on the subgroupoid $\mathcal{B}_2$, denoted in the same way $(\mathcal{B}_2,\natural,0)$. We refer the reader to [RW17, Section 5.6.1] for further technical details.

Let $\mathcal{B}^{g,+}_2$ and $\mathcal{B}^{c,-}_2$ be the full subcategories of $\mathcal{B}_2$ on the objects $\left\{\Sigma^n_{g,1}\right\}_{n\in\mathbb{N}}$ and $\left\{\mathcal{N}^n_{c,1}\right\}_{n\in\mathbb{N}}$ respectively. Note that $\mathcal{B}^{0,+}_2$ and $\mathcal{B}^{0,-}_2$ both are equivalent to the braid groupoid $\boldsymbol{\beta}$ introduced in §3.4.1. Moreover, in this case, the braided monoidal structure $\natural$ of $\mathcal{B}_2$ restricts to the braided monoidal structure of the braid groupoid $(\boldsymbol{\beta},\natural,0)$ described in §3.4.1.

Then, the monoidal structure of $\mathcal{B}_2$ induces left $\boldsymbol{\beta}$-module structures on the groupoids $\mathcal{B}^{g,+}_2$ and $\mathcal{B}^{c,-}_2$. More precisely, the associative unital functors $\natural:\boldsymbol{\beta}\times\mathcal{B}^{g,+}_2\to\mathcal{B}^{g,+}_2$ and $\natural:\boldsymbol{\beta}\times\mathcal{B}^{c,-}_2\to\mathcal{B}^{c,-}_2$ are defined by the restriction of the monoidal product $\natural:\mathcal{B}_2\times\mathcal{B}_2\to\mathcal{B}_2$ to the subcategories $\boldsymbol{\beta}\times\mathcal{B}^{g,+}_2$ and $\boldsymbol{\beta}\times\mathcal{B}^{c,-}_2$. Hence we may apply Quillen's bracket construction and define $\left\langle\boldsymbol{\beta},\mathcal{B}^{g,+}_2\right\rangle$ and $\left\langle\boldsymbol{\beta},\mathcal{B}^{c,-}_2\right\rangle$.

3.4.4 Loop braid groups

We now focus on (extended and non-extended) loop braid groups. We review in this section various ways to define these groups and refer to [Dam17] for a complete and unified presentation of the various definitions of these groups.

Loop braid groups as mapping class groups. Loop braid groups may be defined in terms of *motion groups* of circles in a 3-disc. This is the setting that we shall use to construct suitable *topological* categories for the loop braid groups. We denote by $\mathbb{D}^3$ the unit 3-disc. Let $C_{(n)}:=C_1\amalg\cdots\amalg C_n$ be a collection of n disjoint, unknotted, oriented circles, that form a trivial link of n components in the interior of $\mathbb{D}^3$. Let $\mathrm{Diff}_\partial\left(\mathbb{D}^3,C_{(n)}\right)$ be the group of self-diffeomorphisms of $\mathbb{D}^3$ that that fix $\partial\mathbb{D}^3$ pointwise and fix $C_{(n)}$ as a subset. We denote by $\mathrm{Diff}_\partial\left(\mathbb{D}^3,C^+_{(n)}\right)$ the subgroup of $\mathrm{Diff}_\partial\left(\mathbb{D}^3,C_{(n)}\right)$ of elements that also preserve the orientation of $C_{(n)}$.

The *extended loop braid group* $\mathbf{LB}^{\mathrm{ext}}_n$ is the group of isotopy classes of $\mathrm{Diff}_\partial\left(\mathbb{D}^3,C_{(n)}\right)$. The *(non-extended) loop braid group* $\mathbf{LB}_n$ is the group of isotopy classes of $\mathrm{Diff}_\partial\left(\mathbb{D}^3,C^+_{(n)}\right)$.

Remark 3.29 The usual definition of loop braid groups as isotopy classes is in terms of self-homeomorphisms instead of self-diffeomorphisms. However, as pointed out in [Dam17, Remark 3.7], it follows from [Wat72, Lemma 1.4 and Lemma 2.4] that the two definitions coincide.

Loop braid groups via configuration spaces. Let $C_{n\mathbb{S}^1}\left(\mathbb{D}^3\right)$ be the space of configurations of n unordered, disjoint, unlinked circles $\mathbb{S}^1$ in $\mathbb{D}^3$. The extended loop braid group on n circles

$\mathbf{LB}_n^{\mathrm{ext}}$ is the fundamental group of $C_{n\mathbb{S}^1}\left(\mathbb{D}^3\right)$. In other words, denoting by $\mathrm{Emb}(n\mathbb{S}^1, \mathbb{D}^3)$ the space of embeddings of n disjoint circles $\mathbb{S}^1$ into the 3-disc $\mathbb{D}^3$, $\mathbf{LB}_n^{\mathrm{ext}}$ is the fundamental group of the path-component of the quotient $\mathrm{Emb}(n\mathbb{S}^1, \mathbb{D}^3)/\mathrm{Diff}(n\mathbb{S}^1)$ consisting of unlinks.

Denote by $\mathrm{Diff}^+(n\mathbb{S}^1)$ the subgroup of $\mathrm{Diff}(n\mathbb{S}^1)$ of those diffeomorphisms that preserve the orientation of each circle. Analogously, the (non-extended) loop braid group $\mathbf{LB}_n$ is the fundamental group of the path-component of the quotient $\mathrm{Emb}(n\mathbb{S}^1, \mathbb{D}^3)/\mathrm{Diff}^+(n\mathbb{S}^1)$ consisting of oriented unlinks.

Alternative definitions. There are several other ways to define loop braid groups: we briefly review two equivalent definitions of these groups that will be useful for our work. First, we have explicit presentations of extended and non-extended loop braid groups by generators and relations. Namely, the loop braid group $\mathbf{LB}_n$ admits a presentation given by generators

$$\{\sigma_i, \tau_i \mid i \in \{1, \ldots, n-1\}\};$$

the generators $\{\sigma_i \mid i \in \{1, \ldots, n-1\}\}$ satisfy the relations of the classical braid group $\mathbf{B}_n$ (see §3.4.1), the generators $\{\tau_i \mid i \in \{1, \ldots, n-1\}\}$ satisfy the relations of the symmetric group $\mathfrak{S}_n$ and we have three additional mixed relations (see [Dam17, Proposition 3.14]). In other words,

$$\mathbf{LB}_n \cong (\mathbf{B}_n \times \mathfrak{S}_n) \,/\, (\text{Mixed relations}).$$

The extended loop braid group $\mathbf{LB}_n^{\mathrm{ext}}$ admits a presentation given by generators

$$\{\sigma_i, \tau_i \mid i \in \{1, \ldots, n-1\}\} \sqcup \{\rho_i \mid i \in \{1, \ldots, n\}\};$$

the generators $\{\sigma_i \mid i \in \{1, \ldots, n-1\}\}$ satisfy the relations of the classical braid group $\mathbf{B}_n$ (see §3.4.1), the generators $\{\tau_i \mid i \in \{1, \ldots, n-1\}\}$ satisfy the relations of the symmetric group $\mathfrak{S}_n$, the generators $\{\rho_i \mid i \in \{1, \ldots, n\}\}$ satisfy the relations of the abelian group $(\mathbb{Z}/2\mathbb{Z})^n$ and we have eight additional mixed relations (see [Dam17, Proposition 3.16]). In summary, we have:

$$\mathbf{LB}_n^{\mathrm{ext}} \cong (\mathbf{B}_n \times \mathfrak{S}_n \times (\mathbb{Z}/2\mathbb{Z})^n) \,/\, (\text{Mixed relations}).$$

The original reference for these presentations is [BH13].

Also we can identify loop braid groups as particular subgroups of the automorphisms of free groups. We denote by $\mathbf{F}_n = \langle x_1, \ldots, x_n \rangle$ the free group on $n \in \mathbb{N}$ generators. Then the group $\mathbf{LB}_n$ identifies with the subgroup of the automorphism group $\mathrm{Aut}(\mathbf{F}_n)$ of the automorphisms which map each generator of $\mathbf{F}_n = \langle x_1, \ldots, x_n \rangle$ to a conjugate of some generator, and $\mathbf{LB}_n^{\mathrm{ext}}$ with those which can also be mapped to an inverse of some generator:

$$\mathbf{LB}_n = \left\{\varphi \in \mathrm{Aut}(\mathbf{F}_n) \mid \forall i \in \{1, \ldots, n\}, \exists s \in \mathfrak{S}_n, \exists w_i \in \mathbf{F}_n, \varphi(x_i) = w_i^{-1} x_{s(i)} w_i\right\};$$

$$\mathbf{LB}_n^{\mathrm{ext}} = \left\{\varphi \in \mathrm{Aut}(\mathbf{F}_n) \mid \forall i \in \{1, \ldots, n\}, \exists s \in \mathfrak{S}_n, \exists w_i \in \mathbf{F}_n, \varphi(x_i) = w_i^{-1} x_{s(i)}^{\pm 1} w_i\right\}.$$

3.4.5 Topological groupoids of diffeomorphisms

We now construct topological lifts of all of the monoidal groupoids constructed in this section. Via Lemma 3.13, Proposition 3.15 and Proposition 3.21, we will also obtain topological lifts of each of the various categories that we constructed using the Quillen bracket construction, and we will describe their morphism spaces in terms of spaces of embeddings.

Fix an integer $d \geqslant 2$. We will first construct a topological semi-monoidal groupoid $\mathcal{D}_d$ of diffeomorphisms of d-manifolds. Then we will show that $\pi_0(\mathcal{D}_2)$ contains $\mathcal{M}_2$ as a sub-(semi-monoidal groupoid), and hence also all of its subgroupoids described above. Then we will use $\mathcal{D}_3$ and an oriented version $\mathcal{D}_3^+$ in an analogous way for the (extended) loop braid groups.

Definition 3.30 Let $\mathcal{D}_d$ be the topological groupoid defined as follows. Its objects are all *decorated manifolds* (M,A,e_1,e_2) of dimension d, as in Definition 3.19. A morphism in $\mathcal{D}_d$ from (M,A,e_1,e_2) to (M',A',e_1',e_2') is a diffeomorphism $\varphi\colon M\to M'$ such that $\varphi(A)=A'$ and

$$\varphi(e_i(x,t))=e_i'(x,t)$$

for all $(x,t)\in D^{d-1}\times[0,\epsilon]$ and $i\in\{1,2\}$, for some $\epsilon>0$. These sets of morphisms are topologised as colimits of Whitney topologies, as in Definition 3.14; composition is continuous with this topology, since composition of smooth, proper functions is continuous in the Whitney topology (and diffeomorphisms are proper). Hence $\mathcal{D}_d$ is a topological groupoid.

Definition 3.31 (*Boundary connected sum*) The boundary connected sum of two decorated d-manifolds was described already in Definition 3.19; we now recall this and make all of the details explicit. Let (L,A,e_1,e_2) and (M,B,f_1,f_2) be two decorated d-manifolds and define

$$L\natural M=(L\sqcup M)/{\sim},$$

where $\sim$ is the equivalence relation generated by $e_2(x,0)\sim f_1(x,0)$ for all $x\in D^{d-1}$. We give this a smooth structure as follows. There are obvious topological embeddings

$$L\hookrightarrow L\natural M \qquad \text{and} \qquad M\hookrightarrow L\natural M,$$

and another topological embedding

$$D^{d-1}\times[-1,1]\hookrightarrow L\natural M$$

given by $(x,t)\mapsto e_2(x,-t)$ for $t\leqslant 0$ and $(x,t)\mapsto f_1(x,t)$ for $t\geqslant 0$. We define a smooth structure on $L\natural M$ by declaring that these are all *smooth* embeddings. Finally, we define

$$(L,A,e_1,e_2)\,\natural\,(M,B,f_1,f_2)=(L\natural M,A\sqcup B,e_1,f_2).$$

Definition 3.32 (*Semi-monoidal structure*) We define the functor

$$\natural\colon\mathcal{D}_d\times\mathcal{D}_d\longrightarrow\mathcal{D}_d$$

on objects via the boundary connected sum of Definition 3.31. Now suppose we have morphisms $\varphi\colon(L,A,e_1,e_2)\to(L',A',e_1',e_2')$ and $\psi\colon(M,B,f_1,f_2)\to(M',B',f_1',f_2')$ in $\mathcal{D}_d$. Recall that these are just diffeomorphisms $\varphi\colon L\to L'$ and $\psi\colon M\to M'$ satisfying a certain property. This property implies that φ and ψ glue to a well-defined diffeomorphism $L\natural M\to L'\natural M'$, which is moreover a morphism

$$(L,A,e_1,e_2)\,\natural\,(M,B,f_1,f_2)\longrightarrow(L',A',e_1',e_2')\,\natural\,(M',B',f_1',f_2').$$

It is then easily checked that this gives $\mathcal{D}_d$ the structure of a topological semi-monoidal groupoid.

Definition 3.33 Let $\mathcal{D}_d^+$ be the topological groupoid whose objects are decorated d-manifolds (M,A,e_1,e_2) together with an orientation of $A\subset\mathrm{int}(M)$, and whose morphisms are diffeomorphisms φ as in Definition 3.30 such that the restriction $\varphi|_A\colon A\to A'$ is orientation-preserving. The boundary connected sum for such decorated d-manifolds is defined exactly as in Definition 3.31, with the orientation for $A\sqcup B$ being induced from those of A and B. This then extends, just as in Definition 3.32, to a structure of a topological semi-monoidal groupoid on $\mathcal{D}_d^+$.

Lemma 3.34 *Let $\mathcal{G}$ be any sub-(semi-monoidal groupoid) of $\mathcal{D}_d$ and let $\mathcal{M}$ be any sub-groupoid of $\mathcal{D}_d$ that is preserved under the left-action of $\mathcal{G}$. Then the Serre fibration condition (3.4) of Lemma 3.13 is satisfied for this $\mathcal{G}$ and $\mathcal{M}$. The same holds when $\mathcal{D}_d$ is replaced by $\mathcal{D}_d^+$.*

Proof. This follows directly from Proposition 3.21 for $\mathcal{D}_d$, together with Remark 3.22 for $\mathcal{D}_d^+$. □

Recovering the decorated surfaces groupoid. Let $\mathcal{M}_2^t$ be the full subgroupoid of $\mathcal{D}_2$ on those decorated surfaces (S, A, e_1, e_2) where S is compact and connected, the intervals $e_1(D^1 \times \{0\})$ and $e_2(D^1 \times \{0\})$ lie on the *same* boundary-component of S and $A = \varnothing$. This inherits a topological semi-monoidal structure from $\mathcal{D}_2$. It is not hard to check that

$$\pi_0(\mathcal{M}_2^t) \cong \mathcal{M}_2$$

as semi-monoidal groupoids, since, for diffeomorphisms of surfaces, the condition of fixing (a neighbourhood of) an interval in a boundary-component is equivalent to the condition of fixing two (neighbourhoods of) intervals in that boundary-component. By Lemmas 3.34 and 3.13 (together with Remark 3.23), we deduce that

$$\pi_0(\mathfrak{U}\mathcal{M}_2^t) \cong \mathfrak{U}\mathcal{M}_2$$

as semicategories, where $\mathfrak{U}\mathcal{G} = \langle \mathcal{G}, \mathcal{G} \rangle$ for a (topological) semi-monoidal groupoid $\mathcal{G}$.

We now describe the morphism spaces of $\mathfrak{U}\mathcal{M}_2^t$ in terms of embedding spaces. Let S and S' be two objects of $\mathcal{M}_2^t$, i.e., compact, connected, smooth surfaces, each equipped with an ordered pair of boundary-cylinders on the same boundary-component.

Lemma 3.35 *There is a homeomorphism*

$$\mathfrak{U}\mathcal{M}_2^t(S, S') \;\cong\; \begin{cases} \mathrm{Emb}_{\mathrm{dec}}^{\mathrm{Diff}}(S, S \natural T) & \textit{if there exists an object } T \textit{ of } \mathcal{M}_2^t \textit{ such that } S \natural T \cong S' \\ \varnothing & \textit{otherwise.} \end{cases}$$

Proof. The space $\mathfrak{U}\mathcal{M}_2^t(S, S')$ is the space $\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(X, Y)$ of Definition 3.11, where we write $\mathcal{G} = \mathcal{M} = \mathcal{M}_2^t$, $X = S$ and $Y = S'$. In the notation of the proof of Lemma 3.13, this is the quotient space $\Phi/{\sim_t}$. In that proof, it is shown (a) that this splits as the topological disjoint union of certain spaces denoted $q(\mathrm{Hom}_{\mathcal{M}}(A \natural X, Y))$, as A runs over representatives of isomorphism classes of objects, and (b) that this space is homeomorphic to the quotient space $\mathrm{Hom}_{\mathcal{M}}(A \natural X, Y)/\mathrm{Aut}_{\mathcal{G}}(A)$. In our case, $\mathcal{M} = \mathcal{G}$ is a groupoid, so this quotient space is either empty (if $A \natural X \ncong Y$) or homeomorphic to the quotient space $\mathrm{Aut}_{\mathcal{G}}(A \natural X)/\mathrm{Aut}_{\mathcal{G}}(A)$ (if $A \natural X \cong Y$). Since the collection of objects of $\mathcal{M}_2^t$ under $\natural$ satisfies *cancellation*, there is at most one isomorphism class of objects A such that $A \natural X \cong Y$. Putting this all together, we have shown that there is a homeomorphism

$$\mathfrak{U}\mathcal{M}_2^t(S, S') \;\cong\; \begin{cases} \mathrm{Diff}_{\mathrm{dec}}(S \natural T)/\mathrm{Diff}_{\mathrm{dec}}(T) & \text{if there exists an object } T \text{ of } \mathcal{M}_2^t \text{ such that } S \natural T \cong S' \\ \varnothing & \text{otherwise.} \end{cases}$$

Applying the second part of Proposition 3.21 completes the proof. □

Remark 3.36 (*Subgroupoids of $\mathcal{M}_2$*) This construction may be carried out just as easily for subgroupoids of $\mathcal{M}_2$. Let $\mathcal{G}$ be any sub-(monoidal groupoid) of $\mathcal{M}_2$ and let $\mathcal{M}$ be any subgroupoid of $\mathcal{M}_2$ that is preserved under the left-action of $\mathcal{G}$. Write $\mathcal{G}^t$ for the preimage of $\mathcal{G}$ under the projection

$$\mathcal{M}_2^t \longrightarrow \pi_0(\mathcal{M}_2^t) \cong \mathcal{M}_2,$$

and similarly $\mathcal{M}^t$. Applying Lemmas 3.34 and 3.13 (together with Remark 3.23), we deduce that

$$\pi_0(\langle \mathcal{G}^t, \mathcal{M}^t \rangle) \cong \langle \mathcal{G}, \mathcal{M} \rangle$$

as semicategories. One may then find similar descriptions of the morphism spaces of the topological semicategory $\langle \mathcal{G}^t, \mathcal{M}^t \rangle$ as in Lemma 3.35.

Example 3.37 (*The braid groupoid $\boldsymbol{\beta}$*) For example, $\boldsymbol{\beta}$ is the full subgroupoid of $\mathcal{M}_2$ whose objects are decorated surfaces that are diffeomorphic to a *punctured* 2*-disc*, i.e., $\mathbb{D}^2$ minus a finite collection of open subdiscs whose closures are pairwise disjoint. From Remark 3.36, we obtain a topological semi-monoidal groupoid $\boldsymbol{\beta}^t$ such that $\pi_0(\boldsymbol{\beta}^t) \cong \boldsymbol{\beta}$ and $\pi_0(\mathfrak{U}\boldsymbol{\beta}^t) \cong \mathfrak{U}\boldsymbol{\beta}$, and the morphism spaces of $\mathfrak{U}\boldsymbol{\beta}^t$ may be described as follows. Let $\mathbb{D}_m$ and $\mathbb{D}_n$ be an m-punctured disc and an n-punctured disc respectively. Then

$$\mathfrak{U}\boldsymbol{\beta}^t(\mathbb{D}_m, \mathbb{D}_n) \;\cong\; \begin{cases} \mathrm{Emb}_{\mathrm{dec}}^{\mathrm{Diff}}(\mathbb{D}_m, \mathbb{D}_n) & \text{if } m \leqslant n \\ \varnothing & \text{if } m > n, \end{cases}$$

where $\mathrm{Emb}^{\mathrm{Diff}}_{\mathrm{dec}}(\mathbb{D}_m, \mathbb{D}_n)$ is the space of embeddings $\mathbb{D}_m \to \mathbb{D}_n$ fixing a neighbourhood of two disjoint intervals in $\partial\mathbb{D}^2$ and that may be extended to a diffeomorphism of $\mathbb{D}_n$.

Remark 3.38 (*The partial and injective braid categories*) There are alternative categories to $\mathfrak{U}\boldsymbol{\beta} = \langle\boldsymbol{\beta}, \boldsymbol{\beta}\rangle$ for encoding the family of braid groups. First let $\mathcal{B}\left(\mathbb{R}^2\right)$ be the category with the non-negative integers as its objects, and a morphism $m \to n$ is a choice of $k \leqslant \min\{m, n\}$ and a path in the the (unordered) configuration space $C_n\left(\mathbb{R}^2\right)$ from a k-element subset of $\{x_1, \ldots, x_m\}$ to a k-element subset of $\{x_1, \ldots, x_n\}$ up to endpoint-preserving homotopy. Composition of two morphisms is defined by concatenating paths and deleting configuration points for which the concatenated path is defined only half-way and the identity is given by a constant path. This is called the *partial braid category*. This category is used, for example, in [Pal18] to organise representations of braid groups for twisted homological stability results.

Then the *injective braid category* $\mathcal{B}_f\left(\mathbb{R}^2\right)$ is the subcategory of $\mathcal{B}\left(\mathbb{R}^2\right)$ on the same objects but whose morphisms $m \to n$ are those where $k = m$. We have a (faithful) inclusion functor

$$\mathcal{B}_f\left(\mathbb{R}^2\right) \hookrightarrow \mathcal{B}\left(\mathbb{R}^2\right).$$

There is also a full functor

$$\langle\boldsymbol{\beta}, \boldsymbol{\beta}\rangle \longrightarrow \mathcal{B}_f\left(\mathbb{R}^2\right) \tag{3.13}$$

defined by the identity on objects and sending a morphism $[m - n, \sigma]$ to the concatenation of the trivial braid starting from m points and ending at n points (where all the additional $n - m$ points are on the same side) followed by the geometric braid corresponding to σ. Note that (3.13) is not faithful: indeed $\mathrm{Hom}_{\mathcal{B}_f(\mathbb{R}^2)}(1, 3)$ has three elements whereas $\mathrm{Hom}_{\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle}(1, 3)$ is isomorphic to $\mathbf{B}_3/\mathbf{B}_2$ as a set, and therefore has infinitely many elements given by the pure braid group $\mathbf{PB}_2$.

Example 3.39 (*The setting for surface braid groups*) As another example, let $\mathcal{B}_2^{g,+}$ and $\mathcal{B}_2^{c,-}$ be the subgroupoids of $\mathcal{M}_2$ defined in §3.4.3, whose objects are the collection of surfaces $\Sigma^n_{g,1}$ respectively $\mathcal{N}^n_{c,1}$ for all $n \in \mathbb{N}$, and whose morphisms are isotopy classes of embeddings that become isotopic to the identity if we fill in the non-parametrised boundary-components with discs. (Recall that one boundary-component is equipped a parametrised interval; the others are called *non-parametrised*.) Applying Remark 3.36, we obtain topological groupoids $(\mathcal{B}_2^{g,+})^t$ and $(\mathcal{B}_2^{c,-})^t$, each equipped with a continuous left-action of $\boldsymbol{\beta}^t$, such that $\pi_0((\mathcal{B}_2^{g,+})^t) \cong \mathcal{B}_2^{g,+}$ and $\pi_0((\mathcal{B}_2^{c,-})^t) \cong \mathcal{B}_2^{c,-}$, and moreover

$$\pi_0(\langle\boldsymbol{\beta}^t, (\mathcal{B}_2^{g,+})^t\rangle) \cong \langle\boldsymbol{\beta}, \mathcal{B}_2^{g,+}\rangle \qquad \text{and} \qquad \pi_0(\langle\boldsymbol{\beta}^t, (\mathcal{B}_2^{c,-})^t\rangle) \cong \langle\boldsymbol{\beta}, \mathcal{B}_2^{c,-}\rangle.$$

The morphism spaces of $\langle\boldsymbol{\beta}^t, (\mathcal{B}_2^{g,+})^t\rangle$ may be described as follows (there is an analogous description of the morphism spaces of $\langle\boldsymbol{\beta}^t, (\mathcal{B}_2^{c,-})^t\rangle$):

$$\langle\boldsymbol{\beta}^t, (\mathcal{B}_2^{g,+})^t\rangle(\Sigma^m_{g,1}, \Sigma^n_{g,1}) \;\cong\; \begin{cases} \mathrm{Emb}^{\mathrm{Diff,id}}_{\mathrm{dec}}(\Sigma^m_{g,1}, \Sigma^n_{g,1}) & \text{if } m \leqslant n \\ \varnothing & \text{if } m > n, \end{cases}$$

where $\mathrm{Emb}^{\mathrm{Diff,id}}_{\mathrm{dec}}(\Sigma^m_{g,1}, \Sigma^n_{g,1})$ denotes the subspace of $\mathrm{Emb}^{\mathrm{Diff}}_{\mathrm{dec}}(\Sigma^m_{g,1}, \Sigma^n_{g,1})$ of all (decorated) embeddings $\Sigma^m_{g,1} \hookrightarrow \Sigma^n_{g,1}$ that admit an extension to a diffeomorphism of $\Sigma^n_{g,1}$ that becomes isotopic to the identity after including $\Sigma^n_{g,1} \subset \Sigma_{g,1}$. One may see this exactly as in the proof of Lemma 3.35: we first see that the morphism space is empty if $m > n$, and if $m \leqslant n$ it is homeomorphic to the quotient

$$\mathrm{Aut}_{(\mathcal{B}_2^{g,+})^t}(\Sigma^n_{g,1})/\mathrm{Aut}_{\boldsymbol{\beta}}(\mathbb{D}_{n-m}) \;\cong\; \mathrm{Diff}^{\mathrm{id}}_{\mathrm{dec}}(\Sigma^n_{g,1})/\mathrm{Diff}_{\mathrm{dec}}(\mathbb{D}_{n-m}),$$

where $\mathrm{Diff}^{\mathrm{id}}_{\mathrm{dec}}(\Sigma^n_{g,1})$ is the group of all (decorated) diffeomorphisms of $\Sigma^n_{g,1}$ that become isotopic to the identity after including $\Sigma^n_{g,1} \subset \Sigma_{g,1}$. (Recall that $\boldsymbol{\beta}$ is a *full* subgroupoid of $\mathcal{M}_2$ — so $\boldsymbol{\beta}^t$ is a full subgroupoid of $\mathcal{D}_2$ —, whereas $(\mathcal{B}_2^{g,+})^t$ is not, which is why we have a smaller diffeomorphism group of $\Sigma^n_{g,1}$.) By Proposition 3.21, this is a subspace of

$$\mathrm{Diff}_{\mathrm{dec}}(\Sigma^n_{g,1})/\mathrm{Diff}_{\mathrm{dec}}(\mathbb{D}_{n-m}) \;\cong\; \mathrm{Emb}^{\mathrm{Diff}}_{\mathrm{dec}}(\Sigma^m_{g,1}, \Sigma^n_{g,1}),$$

which one may easily check to be the subspace $\mathrm{Emb}^{\mathrm{Diff,id}}_{\mathrm{dec}}(\Sigma^m_{g,1}, \Sigma^n_{g,1})$ described above.

Topological groupoids for the loop braid groups.

Definition 3.40 (*The loop braid groupoids*) We define

$$(\mathcal{L}\beta^{\mathrm{ext}})^t \subset \mathcal{D}_3$$

to be the full subgroupoid on the collection of all decorated 3-manifolds (M, A, e_1, e_2) where M is diffeomorphic to the 3-disc and A is diffeomorphic to the disjoint union of a finite collection of circles, forming an *unlink* in M. Similarly, we define

$$\mathcal{L}\beta^t \subset \mathcal{D}_3^+$$

to be the full subgroupoid on the collection of all decorated 3-manifolds (M, A, e_1, e_2) where M is diffeomorphic to the 3-disc and A is diffeomorphic to the disjoint union of a finite collection of *oriented* circles, forming an *oriented unlink* in M. These are the (extended and non-extended) *topological loop braid groupoids*. We define their discrete versions simply by:

$$\mathcal{L}\beta^{\mathrm{ext}} = \pi_0((\mathcal{L}\beta^{\mathrm{ext}})^t) \qquad \text{and} \qquad \mathcal{L}\beta = \pi_0(\mathcal{L}\beta^t).$$

The topological groupoids $(\mathcal{L}\beta^{\mathrm{ext}})^t$ and $\mathcal{L}\beta^t$ inherit semi-monoidal structures from $\mathcal{D}_3$ and $\mathcal{D}_3^+$ respectively. Hence $\mathcal{L}\beta^{\mathrm{ext}}$ and $\mathcal{L}\beta$ are (discrete) semi-monoidal groupoids.

Remark 3.41 The semi-monoidal groupoids $\mathcal{L}\beta^{\mathrm{ext}}$ and $\mathcal{L}\beta$ are in fact monoidal (a unit is given by $(\mathbb{D}^3, \varnothing, e_1, e_2)$), and moreover *symmetric*. Hence the monoidal categories $\mathfrak{U}\mathcal{L}\beta^{\mathrm{ext}}$ and $\mathfrak{U}\mathcal{L}\beta$ are also symmetric. More generally, let us write $\mathcal{D}_d^{\mathrm{sph}}$ for the full subgroupoid of $\mathcal{D}_d$ on all decorated d-manifolds M whose two boundary-cylinders lie on the same boundary-component $\partial_0 M \cong \mathbb{S}^{d-1}$. This inherits a topological semi-monoidal structure from $\mathcal{D}_d$. Then $\pi_0(\mathcal{D}_d^{\mathrm{sph}})$ is a *braided* monoidal groupoid for $d = 2$ and a *symmetric* monoidal groupoid for $d \geqslant 3$. Hence $\mathfrak{U}(\pi_0(\mathcal{D}_d^{\mathrm{sph}}))$ is pre-braided for $d = 2$ and symmetric for $d \geqslant 3$. The same statements hold for the analogous subgroupoid $\mathcal{D}_d^{\mathrm{sph},+}$ of $\mathcal{D}_d^+$.

Remark 3.42 Applying Lemmas 3.34 and 3.13 (and Remark 3.23), we see that

$$\pi_0(\mathfrak{U}(\mathcal{L}\beta^{\mathrm{ext}})^t) \cong \mathfrak{U}\mathcal{L}\beta^{\mathrm{ext}} \qquad \text{and} \qquad \pi_0(\mathfrak{U}\mathcal{L}\beta^t) \cong \mathfrak{U}\mathcal{L}\beta.$$

Similarly to Lemma 3.35, the morphism spaces of $\mathfrak{U}(\mathcal{L}\beta^{\mathrm{ext}})^t$ and $\mathfrak{U}\mathcal{L}\beta^t$ may be described as follows:

$$\begin{aligned}
\mathfrak{U}(\mathcal{L}\beta^{\mathrm{ext}})^t(\mathbb{D}^3_m, \mathbb{D}^3_n) &\cong \begin{cases} \mathrm{Emb}^{\mathrm{Diff}}_{\mathrm{dec}}(\mathbb{D}^3_m, \mathbb{D}^3_n) & \text{if } m \leqslant n \\ \varnothing & \text{if } m > n, \end{cases} \\
\mathfrak{U}\mathcal{L}\beta^t(\mathbb{D}^3_m, \mathbb{D}^3_n) &\cong \begin{cases} \mathrm{Emb}^{\mathrm{Diff},+}_{\mathrm{dec}}(\mathbb{D}^3_m, \mathbb{D}^3_n) & \text{if } m \leqslant n \\ \varnothing & \text{if } m > n, \end{cases}
\end{aligned}$$

where $\mathbb{D}^3_m$ denotes the decorated 3-manifold given by an oriented unlink with m components in the 3-disc and $\mathrm{Emb}^{\mathrm{Diff}}_{\mathrm{dec}}(\mathbb{D}^3_m, \mathbb{D}^3_n)$ is defined as in Definition 3.20 (roughly, this means: embeddings $\mathbb{D}^3 \to \mathbb{D}^3$ that fix a 2-disc in the boundary of $\mathbb{D}^3$, send the embedded m-component unlink into the embedded n-component unlink, and that may be extended to a self-diffeomorphism of $\mathbb{D}^3$ that fixes the n-component unlink as a subset). Then $\mathrm{Emb}^{\mathrm{Diff},+}_{\mathrm{dec}}(\mathbb{D}^3_m, \mathbb{D}^3_n)$ is its subspace where embeddings must also preserve the given orientations of the unlinks.

Finally, we justify the name *loop braid groupoid*.

Lemma 3.43 *There are isomorphisms*

$$\mathrm{Aut}_{\mathcal{L}\beta^{\mathrm{ext}}}(\mathbb{D}^3_n) \cong \mathbf{LB}^{\mathrm{ext}}_n \qquad \textit{and} \qquad \mathrm{Aut}_{\mathcal{L}\beta}(\mathbb{D}^3_n) \cong \mathbf{LB}_n.$$

Proof. By definition, the automorphism group of $\mathbb{D}^3_n$ in $\mathcal{L}\beta^{\mathrm{ext}}$ is

$$\pi_0(\mathrm{Diff}_{\mathrm{dec}}(\mathbb{D}^3_n)),$$

where $\mathrm{Diff}_{\mathrm{dec}}(\mathbb{D}^3_n)$ is the topological group of diffeomorphisms of $\mathbb{D}^3$ that send the embedded n-component unlink onto itself and that restrict to the identity on a neighbourhood of two disjoint 2-discs in $\partial\mathbb{D}^3$. (See Definition 3.20.) By definition (cf. Remark 3.29), $\mathbf{LB}_n^{\mathrm{ext}}$ is π_0 of the topological group of diffeomorphisms of $\mathbb{D}^3$ that send the embedded n-component unlink onto itself and that restrict to the identity on $\partial\mathbb{D}^3$. It therefore suffices to show that, for isotopy classes of diffeomorphisms of 3-manifolds M with a spherical boundary-component $\partial_0 M$, fixing two disjoint 2-discs in $\partial_0 M$ is equivalent to fixing all of $\partial_0 M$.[6] This is similar to the fact that we used for surfaces: that fixing one interval in a boundary-component is equivalent, for isotopy classes of diffeomorphisms, to fixing two disjoint intervals in that boundary-component. However, for 3-manifolds it is a less trivial fact. To see this, we argue as follows. Let $\mathrm{Diff}(M, \partial_0 M)$ be the group of diffeomorphisms of M that send $\partial_0 M$ to itself. The restriction map

$$\mathrm{Diff}(M, \partial_0 M) \longrightarrow \mathrm{Diff}(\partial_0 M) = \mathrm{Diff}(\mathbb{S}^2)$$

is a fibre bundle, by [Cer61, Corollaire 2, §II.2.2.2, page 294], and hence its restriction

$$\mathrm{Diff}_{\mathbb{D}^2 \sqcup \mathbb{D}^2}(M) \longrightarrow \mathrm{Diff}_{\mathbb{D}^2 \sqcup \mathbb{D}^2}(\mathbb{S}^2) = \mathrm{Diff}_{\partial C}(C)$$

is also a fibre bundle, where the subscript ${}_{\mathbb{D}^2 \sqcup \mathbb{D}^2}$ means that diffeomorphisms must restrict to the identity on a given pair of disjoint discs in $\partial_0 M = \mathbb{S}^2$, and C is the 2-dimensional cylinder $\mathbb{S}^1 \times [0, 1]$. The fibre is $\mathrm{Diff}_{\partial_0 M}(M)$ and we obtain an exact sequence

$$\cdots \to \pi_1(\mathrm{Diff}_{\partial C}(C)) \longrightarrow \pi_0(\mathrm{Diff}_{\partial_0 M}(M)) \xrightarrow{(*)} \pi_0(\mathrm{Diff}_{\mathbb{D}^2 \sqcup \mathbb{D}^2}(M)) \longrightarrow \pi_0(\mathrm{Diff}_{\partial C}(C)).$$

Our aim is to show that $(*)$ is a bijection, so it suffices to know that π_0 and π_1 of $\mathrm{Diff}_{\partial C}(C)$ are trivial. But in fact $\mathrm{Diff}_{\partial C}(C)$ is contractible, by a theorem of Gramain [Gra73, Théorème 1]. $\square$

Remark 3.44 A similar fact (to the one used in the proof of Lemma 3.43) holds also for a single 2-disc in a spherical boundary-component of a 3-manifold. Namely, for isotopy classes of diffeomorphisms of a 3-manifold M with a spherical boundary-component $\partial_0 M$, fixing a 2-disc in $\partial_0 M$ is equivalent to fixing the whole of $\partial_0 M$. This follows, via a similar argument as in Lemma 3.43, using the fact (due to Smale [Sma59]) that $\mathrm{Diff}_{\partial\mathbb{D}^2}(\mathbb{D}^2)$ is contractible. The same statement is in fact true for isotopy classes of diffeomorphisms of 4-manifolds with a spherical boundary-component, since $\mathrm{Diff}_{\partial\mathbb{D}^3}(\mathbb{D}^3)$ is also contractible ([Hat83]). However, in higher dimensions this does not continue to hold: $\pi_0(\mathrm{Diff}_{\partial\mathbb{D}^{d-1}}(\mathbb{D}^{d-1}))$ is isomorphic to the group of exotic d-spheres, which is very often non-trivial in higher dimensions. Also, $\pi_1(\mathrm{Diff}_{\partial\mathbb{D}^4}(\mathbb{D}^4))$ has recently been shown to be non-trivial [Wat18].

4 Topological construction of representations

This section is devoted to the application of the general construction of §2 to the families of groups introduced in §3.4. In §§4.1–4.3, we apply the lifting construction of §2.3 to obtain representations of (surface) braid groups and mapping class groups. In each case, we also extend these to functors defined on categories of the form $\langle \mathcal{G}, \mathcal{M} \rangle$ (whose automorphism groups are one of the families of groups in question), where $\langle -, - \rangle$ is the Quillen bracket construction of §3.1. We do this by explicitly writing down an extension to certain "generating" morphisms of this category, and then verifying that conditions (3.1) and (3.2) of Lemma 3.5 are satisfied.

In §4.4 we then reinterpret the constructions of §§4.1–4.3 using the *functorial* version of the lifting construction, summarised in §2.5. Using this construction, together with the topological categories constructed in §3.4, we recover each of the functors of §§4.1–4.3, each being induced a certain continuous functor defined on one of the topological categories of §3.4.

Then, in §4.5, we directly apply the functorial version of the lifting construction to construct families of representations of the (extended and non-extended) loop braid groups. These appear to be new, and are analogues of the *reduced Burau* and *Lawrence-Bigelow* representations of the classical braid groups.

[6] Note that the condition of fixing *a neighbourhood of* two discs in the boundary is clearly homotopy equivalent to fixing just the two discs.

The tools for §§4.1–4.3. We briefly review the tools, steps and spirit of the construction used for §§4.1–4.3. (The more functorial version of the construction used in §§4.4–4.5 is summarised in detail in §2.5.) For a locally path-connected topological group G, we consider a space X on which G acts through a continuous group homomorphism

$$\theta\colon G \longrightarrow \mathrm{Homeo}_{x_0}(X)$$

where $x_0 \in X$. In order to use covering space theory, we assume that X is path-connected, locally path-connected and semi-locally simply connected, so that it admits a universal cover. Moreover, we consider a surjective group homomorphism $\phi\colon \pi_1(X, x_0) \twoheadrightarrow Q$ so that the induced action θ_π of G on the fundamental group of X satisfies Assumption 2.13. These two morphisms θ and ϕ define a functor $F_{\theta,\phi}\colon G \to \widetilde{\mathrm{Cov}}_Q$. (The category $\widetilde{\mathrm{Cov}}_Q$ is defined in §2.1.) Then we define a representation $L_k(F_{\theta,\phi})$ of $\pi_0(G)$ using the k-th integral homology group of the regular path-connected covering space of X associated with ϕ, denoted X^ϕ:

$$L_k(F_{\theta,\phi}) : \pi_0(G) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}\left(H_k\left(X^\phi;\mathbb{Z}\right)\right).$$

If, in addition, the action θ_π of G on $\pi_1(X, x_0)$ satisfies Assumption 2.16, the action defined by $L_k(F_{\theta,\phi})$ commutes with the $\mathbb{Z}[Q]$-module structure of $H_k\left(X^\phi;\mathbb{Z}\right)$:

$$L_k(F_{\theta,\phi}) : \pi_0(G) \longrightarrow \mathrm{Aut}_{\mathbb{Z}[Q]}\left(H_k\left(X^\phi;\mathbb{Z}\right)\right).$$

4.1 Classical braid groups

This first application of the general construction of §2 relies on considering the total number of half-twists and the total winding number for an ordered configuration space of points. As proved in §5.1, it allows one to recover the well-known families of *Lawrence-Bigelow representations* originally introduced by Ruth Lawrence [Law90].

We fix m and n two natural numbers such that $m \geqslant 1$ and $n \geqslant 0$. We recall that $\mathbb{D}_n$ is the surface $\mathbb{D}^2 \smallsetminus \{d_1, \ldots, d_n\}$, for $\{d_i\}_{i\in\{1,\ldots,n\}}$ a collection of pairwise disjoint open discs in the interior of the closed unit 2-disc $\mathbb{D}^2$ and that $C_m(\mathbb{D}_n)$ is the configuration space of m unordered points in $\mathbb{D}_n$. We consider the topological group $\mathrm{Diff}_{\partial_0}(\mathbb{D}_n)$ of self-diffeomorphisms of $\mathbb{D}_n$ which restrict to the identity on the boundary of $\mathbb{D}^2$. We fix m distinct points $\{z_i\}_{i\in\{1,\ldots,m\}}$ in the boundary of $\mathbb{D}^2$ and take the configuration $c_0 = \{z_1, \ldots, z_m\}$ as the basepoint of $C_m(\mathbb{D}_n)$.

Let $\theta_{m,n} : \mathrm{Diff}_{\partial_0}(\mathbb{D}_n) \to \mathrm{Homeo}_{c_0}(C_m(\mathbb{D}_n))$ be the continuous group morphism giving the natural action of $\mathrm{Diff}_{\partial_0}(\mathbb{D}_n)$ on the coordinates of the configuration space $C_m(\mathbb{D}_n)$, i.e. defined by

$$\varphi \longmapsto (\{x_1, \ldots, x_m\} \longmapsto \{\varphi(x_1), \ldots, \varphi(x_m)\}).$$

In particular, $\theta_{m,n}(\varphi)$ preserves the basepoint c_0 since φ fixes pointwise the boundary of $\mathbb{D}^2$.

The choice of the quotient of the fundamental group $\pi_1(C_m(\mathbb{D}_n), c_0)$ depends on m. Beforehand, we recall from §3.4.3 that $\pi_1(C_m(\mathbb{D}_n), c_0)$ identifies with the surface braid group $\mathbf{B}_m(\mathbb{D}_n)$ and that γ_2 denotes the abelianisation map of a group G.

For $m = 1$: Let $\Sigma : \mathbb{Z}^n \to \mathbb{Z}$ be the sum map $(a_1, \ldots, a_n) \mapsto \sum_{i\in\{1,\ldots,n\}} a_i$. Then if $m = 1$ we consider the composite

$$\phi = \phi_{1,n}\colon \mathbf{B}_1(\mathbb{D}_n) \xrightarrow{\gamma_2} \mathbb{Z}^n \xrightarrow{\Sigma} \mathbb{Z}.$$

For $m \geqslant 2$: We first have to introduce two new homomorphisms. The inclusion of $\mathbb{D}_n$ in $\mathbb{D}^2$ by gluing a disc on all the interior boundary components induces an inclusion map $i : C_m(\mathbb{D}_n) \hookrightarrow C_m\left(\mathbb{D}^2\right)$: a configuration in $\mathbb{D}_n$ is in particular a configuration in $\mathbb{D}^2$. We denote by i_* the induced surjective homomorphism on π_1 and consider the composite with the abelianisation map

$$T\colon \mathbf{B}_m(\mathbb{D}_n) \xrightarrow{i_*} \mathbf{B}_m \xrightarrow{\gamma_2} \mathbb{Z}.$$

We may have the following geometrical interpretation of this morphism: for $\gamma \in \pi_1 (C_m (\mathbb{D}_n), c_0)$ and c_γ a simple closed curve in $C_m (\mathbb{D}_n)$ representative of γ, one can think of $T(\gamma)$ as counting the total number of half-twists that occur in the path c_γ of configurations of m points in $\mathbb{D}^2$. Here a *half-twist* means doing half a Dehn twist in a tubular neighbourhood along the path c_γ: namely this is something like one of the standard generators of the braid group (consisting of a pair of adjacent strands crossing each other, and no other crossings). This assumes an orientation on the curve c_γ and a fortiori half-twists with the opposite orientation count negatively.

Furthermore, let $j : C_m (\mathbb{D}_n) \hookrightarrow C_{m+n} \left(\mathbb{D}^2\right)$ be the map defined by

$$\{x_1, \ldots, x_m\} \longmapsto \{x_1, \ldots, x_m, p_1, \ldots, p_n\},$$

which glues a disc with a marked point p_i onto each interior boundary component ∂d_i of $\mathbb{D}_n$. The new points are thus considered as coordinates of the configuration space. We denote by $j_* : \mathbf{B}_m (\mathbb{D}_n) \hookrightarrow \mathbf{B}_{m+n}$ the induced injective homomorphism on π_1 and consider the composite with the abelianisation map

$$R\colon \mathbf{B}_m(\mathbb{D}_n) \xhookrightarrow{j_*} \mathbf{B}_{m+n} \xrightarrow{\gamma_2} \mathbb{Z}.$$

This morphism can geometrically be interpreted in the following way: for $\gamma \in \pi_1 (C_m (\mathbb{D}_n), c_0)$ and c_γ as above, $R(\gamma)$ counts the total number of half-twists that occur in a path c_γ, and also between configuration points and the additional marked points $\{p_1, \ldots, p_n\}$. In principle, it also counts half-twists between pairs of marked points, but of course these remain fixed, so there are zero of these. Hence $R(\gamma) - T(\gamma)$ is the total number (counted with signs) of half-twists that occur between configuration points and the marked points in the path c_γ of configurations. This is twice the total number of times that a configuration point winds around a marked point: $R(\gamma) - T(\gamma)$ is thus always even and corresponds to twice the total winding number of c_γ. Hence we may define $W\colon \mathbf{B}_m (\mathbb{D}_n) \twoheadrightarrow \mathbb{Z}$ to be the surjective morphism defined by the total winding number, i.e. $\gamma \longmapsto \frac{1}{2} (R(\gamma) - T(\gamma))$. These descriptions of the homomorphisms T, R and W come from [Bud05, Section 2].

Finally we choose the product $(T \times W) \circ \Delta\colon \mathbf{B}_m (\mathbb{D}_n) \twoheadrightarrow \mathbb{Z}^2$, defined by $\gamma \longmapsto (T(\gamma), W(\gamma))$, for the choice $\phi = \phi_{m,n}$ of the quotient of the fundamental group. In addition, we have the following property:

Lemma 4.1 *For each $m \geqslant 1$, the homomorphism $\phi_{m,n}$ is invariant under the action $(\theta_{m,n})_\pi$ of* $\mathrm{Diff}_{\partial_0} (\mathbb{D}_n)$*.*

Proof. First, note that this statement is equivalent to saying that $\phi_{m,n}$ is invariant under the action $\pi_0((\theta_{m,n})_\pi)$ of $\pi_0(\mathrm{Diff}_{\partial_0} (\mathbb{D}_n)) \cong \mathbf{B}_n$.

For $m = 1$, the morphism $\pi_0 \left((\theta_{1,n})_\pi\right)$ corresponds to the Artin representation $a_n : \mathbf{B}_n \to Aut(\mathbf{F}_n)$: for each elementary braid σ_i, the automorphism $a_n(\sigma_i)$ sends the generator g_j to g_{i+1} if $j = i$, to $g_{i+1}^{-1} g_i g_{i+1}$ if $j = i + 1$ and to g_j if $j \notin \{i, i + 1\}$. The result thus follows from the fact that $\phi_{1,n}$ is the composite of the abelianisation by the sum map.

For $m \geqslant 2$, the result is a consequence of the more general facts that i_* and R are invariant under the action of $\pi_0 \left((\theta_{m,n})_\pi\right)$. Indeed, let g' be an extension of an element g of $\mathrm{Diff}_{\partial_0} (\mathbb{D}_n)$ to $\mathrm{Diff}_{\partial_0} \left(\mathbb{D}^2\right)$. Then g' is isotopic to $id_{\mathbb{D}^2}$, since the space $\mathrm{Diff}_{\partial_0} \left(\mathbb{D}^2\right)$ is path-connected (see [Mun60, Theorem 1.3], or [Sma59, Theorem B] for the stronger fact that it is contractible) and it makes the following diagrams commutative:

$$\begin{tikzcd}
C_m(\mathbb{D}_n) \arrow[r, "i"] \arrow[d, "\theta_{m,n}(g)"'] & C_m\left(\mathbb{D}^2\right) \arrow[d, "\theta_{m,0}(g')"] \\
C_m(\mathbb{D}_n) \arrow[r, "i"] & C_m\left(\mathbb{D}^2\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
C_m(\mathbb{D}_n) \arrow[r, "j"] \arrow[d, "\theta_{m,n}(g)"'] & C_{m+n}\left(\mathbb{D}^2\right) \arrow[d, "\theta_{m+n,0}(g')"] \\
C_m(\mathbb{D}_n) \arrow[r, "j"] & C_{m+n}\left(\mathbb{D}^2\right).
\end{tikzcd}$$

Let $H_{g'}$ be an isotopy of $\mathbb{D}^2$ from g' to $id_{\mathbb{D}^2}$. Taking the product of m times $H_{g'}$ thus induces a homotopy of $C_m \left(\mathbb{D}^2\right)$. We deduce that $i_* \circ (\theta_{m,n})_\pi (g) = i_*$ and a fortiori $T \circ (\theta_{m,n})_\pi (g) = T$. On the other hand, we claim that the action of $(\theta_{m+n,0})_\pi(g')$ on $\mathbf{B}_{m+n}$ corresponds (up to isotopy) to conjugation by some element $\sigma \in \mathbf{B}_n \hookrightarrow \mathbf{B}_{m+n}$. Therefore $\gamma_2 \circ (\theta_{m+n,0})_\pi(g') = \gamma_2$ and a fortori $R \circ (\theta_{m,n})_\pi(g) = R$. The result thus follows from the definition of $\phi_{m,n}$.

It remains to verify our claim that $(\theta_{m+n,0})_\pi(g')$ acts on $\mathbf{B}_{m+n}$ by conjugation by an element of $\mathbf{B}_n \subset \mathbf{B}_{m+n}$. To see this, let b be a geometric braid in $\mathbb{D}^2 \times [1,2]$ on $m+n$ strands, where m of them begin and end at a configuration in $\partial\mathbb{D}^2$ and n of them begin and end at a configuration in the interior of $\mathbb{D}^2$. Extend this to a geometric braid in $\mathbb{D}^2 \times [0,3]$ by gluing two trivial $(m+n)$-strand braids to the top and bottom. Then $(\theta_{m+n,0})_\pi(g')([b])$ is represented by the geometric braid $G(b)$, where G is the self-homeomorphism of $\mathbb{D}^2 \times [0,3]$ given by $G(x,h) = (g'(x),h)$. Recall that we have an isotopy $H_{g'}$ of self-diffeomorphisms of $\mathbb{D}^2$ fixing $\partial\mathbb{D}^2$ pointwise, with $H_{g'}(0) = g'$ and $H_{g'}(1) = id_{\mathbb{D}^2}$. Using this, we define a homotopy K of self-homeomorphisms of $\mathbb{D}^2 \times [0,3]$ by

$$K(t)(x,h) = \begin{cases} (H_{g'}(ht)(x),h) & \text{if } h \in [0,1] \\ (H_{g'}(t)(x),h) & \text{if } h \in [1,2] \\ (H_{g'}((3-h)t)(x),h) & \text{if } h \in [2,3] \end{cases}$$

This induces a homotopy of geometric braids from $G(b) = K(0)(b)$ to $K(1)(b)$, since, at all times t, the self-homeomorphism acts on the top and bottom discs (corresponding to $h = 0,3$) by g', which fixes the endpoints of b (setwise) by construction. Now note that the geometric braid $K(1)(b)$ is equal to b in the middle section $\mathbb{D}^2 \times [1,2]$. Moreover, the top section of $K(1)(b)$ (i.e. its intersection with $\mathbb{D}^2 \times [2,3]$) is the inverse of the bottom section of $K(1)(b)$ (i.e. its intersection with $\mathbb{D}^2 \times [0,1]$). Let σ denote the braid represented by the geometric braid $K(1)(b) \cap (\mathbb{D}^2 \times [0,1])$. We have shown that

$$(\theta_{m+n,0})_\pi(g')([b]) = [G(b)] = [K(1)(b)] = \sigma \circ [b] \circ \sigma^{-1}.$$

Finally, we need to check that σ lies in the subgroup $\mathbf{B}_n \subset \mathbf{B}_{m+n}$. But this is immediate from the construction, since $K(1)$ fixes $\partial\mathbb{D}^2 \times [0,3]$ pointwise, and the trivial $(m+n)$-strand braid has m of its strands lying in the boundary of $\mathbb{D}^2$. $\square$

Therefore, the morphisms $\theta_{m,n}$ and $\phi_{m,n}$ satisfy Assumptions 2.13 and 2.16, for all natural numbers m. Following Definition 2.17, we thus define the representations

$$L_k\left(F_{\theta_{m,n},\phi_{m,n}}\right) : \mathbf{B}_n \longrightarrow \mathrm{Aut}_R\left(H_k\left((C_m(\mathbb{D}_n))^{\phi_{m,n}};\mathbb{Z}\right)\right)$$

for all integers $m \geqslant 1$ and $k,n \geqslant 0$, where $R = \mathbb{Z}[\mathbb{Z}]$ when $m = 1$ and $R = \mathbb{Z}[\mathbb{Z}^2]$ when $m \geqslant 2$.

Functoriality. We fix a natural number m. All the homological representations $L_m\left(F_{\theta_{m,n},\phi_{m,n}}\right)$ assemble to define a functor $\boldsymbol{\beta} \to \mathbb{Z}[A_m]$-Mod where $A_1 = \mathbb{Z}$ and $A_{m\geqslant 2} = \mathbb{Z}^2$. We denote this functor by $\mathfrak{LB}_m$. The reason for this notation will be explained in §5.1.

Actually, this functor extends to Quillen's bracket construction: by Lemma 3.5, it is enough to define properly $\mathfrak{LB}_m$ on the morphisms $[n'-n, id_{n'}]$ for all objects n and n' of $\boldsymbol{\beta}$ so that $n' \geqslant n$.

Recall that the boundary connected sum $\natural$ defines an embedding $\iota_{\mathbb{D}_1}\natural id_{\mathbb{D}_n} : \mathbb{D}_n \hookrightarrow \mathbb{D}_1\natural\mathbb{D}_n$ for all natural numbers n, adding an interior boundary component in $\mathbb{D}_n$. This embedding induces maps for the configuration spaces $e_{m,n} : C_m(\mathbb{D}_n) \to C_m(\mathbb{D}_{1+n})$, which itself induces an injective morphism for the fundamental groups $\pi_1(e_{m,n}) : \mathbf{B}_m(\mathbb{D}_n) \hookrightarrow \mathbf{B}_m(\mathbb{D}_{1+n})$. We denote the composite $e_{m,n'-1} \circ \cdots \circ e_{m,1+n} \circ e_{m,n}$ by $e_{m,n\to n'} : C_m(\mathbb{D}_n) \to C_m(\mathbb{D}_{n'})$. From the functoriality of the fundamental groups with respect to the category of based topological spaces, the induced morphism for the fundamental groups $\pi_1(e_{m,n\to n'}) : \mathbf{B}_m(\mathbb{D}_n) \hookrightarrow \mathbf{B}_m(\mathbb{D}_{n'})$ is equal to the composite $\pi_1(e_{m,n'-1}) \circ \cdots \circ \pi_1(e_{m,1+n}) \circ \pi_1(e_{m,n})$.

It follows from the above definitions that the restriction of the morphism $\phi_{m,n'}$ to the braid group $\mathbf{B}_m(\mathbb{D}_n)$ along $\pi_1(e_{m,n\to n'})$ is the morphism $\phi_{m,n}$: in other words, the following diagram is commutative

$$\begin{tikzcd}
\pi_1\left(C_m\left(\mathbb{D}_n\right), c_0\right) \arrow[rr, hook, "\pi_1\left(e_{m,n\to n'}\right)"] \arrow[dr, "\phi_{m,n}"'] & & \pi_1\left(C_m\left(\mathbb{D}_{n'}\right), c_0\right) \arrow[dl, "\phi_{m,n'}"] \\
& A_m . &
\end{tikzcd}$$

Hence there exists a unique based lift $e^{\phi}_{m,n\to n'} : C_m\left(\mathbb{D}_n\right)^{\phi} \to C_m\left(\mathbb{D}_{n'}\right)^{\phi}$ so that the following diagram is commutative

$$\begin{tikzcd}
C_m\left(\mathbb{D}_n\right)^{\phi_{m,n}} \arrow[r, "e^{\phi}_{m,n\to n'}"] \arrow[d, "\xi_{m,n}"'] & C_m\left(\mathbb{D}_{n'}\right)^{\phi_{m,n'}} \arrow[d, "\xi_{m,n'}"] \\
C_m\left(\mathbb{D}_n\right) \arrow[r, "e_{m,n\to n'}"] & C_m\left(\mathbb{D}_{n'}\right) .
\end{tikzcd}$$

All these lifts are the key maps to define the functor $\mathfrak{LB}_m$ on the morphisms of the category $\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle$:

Definition 4.2 For every natural numbers n and n' so that $n\geqslant n'$, let $\mathfrak{LB}_m\left([n'-n, id_{n'}]\right)$ be the homomorphism induced by the lift $e^{\phi}_{m,n\to n'}$ for the standard homology.

Proposition 4.3 *The functor $\mathfrak{LB}_m$ extends to define objects of* $\mathbf{Fct}\left(\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle, \mathbb{Z}\left[A_m\right]\text{-Mod}\right)$.

Proof. First of all it follows from the above definitions that

$$\mathfrak{LB}_m\left([n'-n, id_{n'}]\right) = \mathfrak{LB}_m\left([n', id_{n'-1}]\right)\circ\cdots\circ\mathfrak{LB}_m\left([1, id_{1+n}]\right).$$

Following Lemma 3.5, we only have to check that Relation (3.2) is satisfied. For all $\sigma\in\mathbf{B}_n$ and all $\sigma'\in\mathbf{B}_{n'}$, by the definition of $\theta_{m,n}$, the action of $\pi_0((\theta_{m,n'+n})_\pi)\left(\sigma'\natural\sigma\right)\in\mathbf{B}_{n'+n}$ on the n last points of $C_m\left(\mathbb{D}_{n'+n}\right)$ is completely induced by σ since σ' is the identity on these n last points. Hence, the following diagram is commutative:

$$\begin{tikzcd}
C_m\left(\mathbb{D}_n\right) \arrow[r, "e_{m,n\to n'}"] \arrow[d, "\pi_0(\theta_{m,n})(\sigma)"'] & C_m\left(\mathbb{D}_{n'+n}\right) \arrow[d, "\pi_0(\theta_{m,n})\left(\sigma'\natural\sigma\right)"] \\
C_m\left(\mathbb{D}_n\right) \arrow[r, "e_{m,n\to n'}"] & C_m\left(\mathbb{D}_{n'+n}\right) .
\end{tikzcd}$$

The unicity of the lifts with respect to the covering spaces $\left\{C_m\left(\mathbb{D}_i\right)^{\phi_{m,i}}\right\}_{i\in\mathbb{N}}$ induced by the morphisms $\left\{\phi_{m,i}\right\}_{i\in\mathbb{N}}$ implies that

$$e^{\phi}_{m,n\to n'}\circ\left(\pi_0\left(\theta^{\phi}_{m,n}\right)(\sigma)\right) = \left(\pi_0\left(\theta^{\phi}_{m,n}\right)\left(\sigma'\natural\sigma\right)\right)\circ e^{\phi}_{m,n\to n'}.$$

The result thus follows from the assignments for $\mathfrak{LB}_m$, given by the maps for the standard and Borel-Moore homology (see (2.12)) induced by these lifts. ☐

Alternative using the abelianisation of mixed braid groups. As pointed out in [BGG17], the morphisms $\phi_{m,n}$ can be introduced using kernels of the abelianisation of some mixed braid groups. We assume that $n\geqslant 2$. Recall that we denote by $\gamma_2\left(\mathbf{B}_{m,n}\right)$ and $\gamma_2\left(\mathbf{B}_n\right)$ the respective canonical abelianisation morphisms of $\mathbf{B}_{m,n}$ and $\mathbf{B}_n$, and that we omit the groups from the notations where there is no ambiguity.

They canonically induce a morphism $\overline{\Lambda}^{\mathbb{D}^2}_{m,n}$ so that $\overline{\Lambda}^{\mathbb{D}^2}_{m,n}\circ\gamma_2\left(\mathbf{B}_{m,n}\right) = \gamma_2\left(\mathbf{B}_n\right)\circ\Lambda^{\mathbb{D}^2}_{m,n}$, where $\Lambda^{\mathbb{D}^2}_{m,n} : \mathbf{B}_{m,n}\left(\mathbb{D}^2\right)\twoheadrightarrow\mathbf{B}_n\left(\mathbb{D}^2\right)$ is the morphism introduced in §3.4.3 induced by forgetting the first m coordinates. We consider the kernel of $\overline{\Lambda}^{\mathbb{D}^2}_{m,n}$, which depends only on m since

$$\mathbf{B}_{m,n}\left(\mathbb{D}^2\right)/\Gamma_2\left(\mathbf{B}_{m,n}\left(\mathbb{D}^2\right)\right)\cong\begin{cases}\mathbb{Z}^{\oplus 2} & \text{if } m=1;\\ \mathbb{Z}^{\oplus 3} & \text{if } m\geqslant 2.\end{cases}$$

Hence $\ker\left(\overline{\Lambda}^{\mathbb{D}^2}_{m,n}\right) \cong A_m$. Then $\phi_{m,n}$ is the unique surjective morphism (given by the universal property of the kernel) so that the following diagram is commutative (and where the two lines are exact):

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{B}_m\left(\mathbb{D}_n\right) \arrow[r] \arrow[d, "\phi_{m,n}"] & \mathbf{B}_{m,n}\left(\mathbb{D}^2\right) \arrow[r, "\Lambda^{\mathbb{D}^2}_{m,n}"] \arrow[d, "\gamma_2"] & \mathbf{B}_n\left(\mathbb{D}^2\right) \arrow[r] \arrow[d, "\gamma_2"] & 1 \\
1 \arrow[r] & A_m \arrow[r] & \mathbf{B}_{m,n}\left(\mathbb{D}^2\right)/\mathit{\Gamma}_2\left(\mathbf{B}_{m,n}\left(\mathbb{D}^2\right)\right) \arrow[r, "\overline{\Lambda}^{\mathbb{D}^2}_{m,n}"] & \mathbb{Z} \arrow[r] & 1.
\end{tikzcd}$$

An interesting application of this eqivalent definition of the morphisms $\phi_{m,n}$ is the following alternative purely algebraic proof of Lemma 4.1.

Proof of Lemma 4.1. We fix $\psi \in \mathbf{B}_n$, $x \in A_m$ and $\tilde{x} \in \mathbf{B}_m\left(\mathbb{D}_n\right)$ so that $\phi_{2,m}\left(\tilde{x}\right) = x$. Recall that the map $\Lambda^{\mathbb{D}^2}_{m,n}$ admits a left inverse and therefore the upper short exact sequence of the above diagram is split. Note that, by the multiplication rule of a semidirect product, the action of ψ on x is induced by the conjugation seen as elements of the mixed braid group: $\left(\theta_{m,n}\left(\psi\right)\left(x\right), 1_{\mathbf{B}_n}\right) = \left(1_{\mathbf{B}_m\left(\mathbb{D}_n\right)}, \psi^{-1}\right)\left(\tilde{x}, 1_{\mathbf{B}_n}\right)\left(1_{\mathbf{B}_m\left(\mathbb{D}_n\right)}, \psi\right)$ which proves that Assumption 2.13 is satisfied. We deduce that $\gamma_2\left(\psi^{-1}\tilde{x}\psi\right) = \gamma_2\left(\tilde{x}\right)$. Then it follows from the commutativity of the above diagram that $\phi_{m,n}\left(\tilde{x}\right) = x = \psi^{-1}x\psi$, which corresponds to the proof of Assumption 2.16. $\square$

Remark 4.4 A natural idea would be to apply the same principle using the further lower central quotient groups $\mathbf{B}_{m,n}\left(\mathbb{D}^2\right)/\mathit{\Gamma}_k\left(\mathbf{B}_{m,n}\left(\mathbb{D}^2\right)\right)$ for $k \geqslant 3$. However [BGG17, Corollary 3.8] proves that $\mathit{\Gamma}_2\left(\mathbf{B}_{m,n}\left(\mathbb{D}^2\right)\right) = \mathit{\Gamma}_3\left(\mathbf{B}_{m,n}\left(\mathbb{D}^2\right)\right)$ if $m, n \geqslant 3$, hence this idea is not relevant. It might potentially be relevant for $m = 1$ or $m = 2$, but in this case the lower central quotients are not yet well-understood.

4.2 Surface braid groups

Taking inspiration from the situation for classical braid groups described in §4.1, the main idea to construct homological representations for the surface braid groups consists in using the quotients defined by the lower central series of some mixed surface braid groups. As for the classical braid groups, the abelianisation gives an interesting family of representations (see §4.2.1) which, as far as the authors are aware, does not appear in the literature. However, in contrast to classical braid groups, it is relevant to consider the further lower central quotients. In particular the third lower central quotients represent an interesting case and this case is detailed in §4.2.2. For orientable surfaces, this option is actually a reinterpretation of Bellingeri, Godelle and Guaschi in [BGG17] of the work by An and Ko [AK10] to extend some homological representations from the classical braid groups to the surface braid groups (see §5.2 for further details on this point).

For the remainder of §4.2, we fix two natural numbers $g \geqslant 1$ and $c \geqslant 2$ and consider a surface S which is either the orientable surface $\Sigma_{g,1}$ or the non-orientable surface $\mathcal{N}_{c,1}$, as denoted in §3.4.2. For all natural numbers n, we denote by $S^{(n)}$ the surface $\left(\Sigma^1_{0,1}\right)^{\natural n} \natural S$ obtained using the boundary connected sum.

Let $n \geqslant 0$ and $m \geqslant 1$ be two natural numbers. Let $\mathrm{BrDiff}_I\left(S^{(n)}\right)$ be the topological group of diffeomorphisms of $S^{(n)}$ that fix a given interval $I \subset \partial S$ pointwise and that become isotopic to the identity (fixing I throughout the isotopy) after capping the non-parametrised boundary-components of $\partial S^{(n)}$ (i.e. the n boundary-components corresponding to the n copies of $\partial\Sigma^1_{0,1}$). Note that $\pi_0\left(\mathrm{BrDiff}_I\left(S^{(n)}\right)\right)$ is the automorphism group of $S^{(n)}$ in the groupoid $\mathcal{B}_2$ introduced in §3.4.3. In particular,

$$\pi_0\left(\mathrm{BrDiff}_I\left(S^{(n)}\right)\right) \cong \mathbf{B}_n\left(S\right)$$

is the n-th surface braid group of S. We now fix m distinct points $\{z_i\}_{i\in\{1,\ldots,m\}}$ in the parametrised interval I in the boundary of S and choose the configuration $c_0 = \{z_1, \ldots, z_m\}$ as the basepoint of

$C_m\left(S^{(n)}\right)$.

Let $\theta_m\left(S^{(n)}\right) : \mathrm{BrDiff}_I\left(S^{(n)}\right) \to \mathrm{Homeo}_{c_0}\left(C_m\left(S^{(n)}\right)\right)$ be the continuous group morphism giving the natural action of $\mathrm{BrDiff}_I\left(S^{(n)}\right)$ on the coordinates of the configuration space $C_m\left(S^{(n)}\right)$, i.e. defined by

$$\varphi \longmapsto \left(\{x_1, \ldots, x_m\} \longmapsto \{\varphi(x_1), \ldots, \varphi(x_m)\}\right).$$

In particular, $\theta_m\left(S^{(n)}\right)(\varphi)$ preserves the basepoint c_0 since φ fixes pointwise the interval I containing c_0.

Remark 4.5 In the general construction recalled at the beginning of this section, we are taking $G = \mathrm{BrDiff}_I\left(S^{(n)}\right)$ and $X = C_m(S^{(n)})$. Note that $\pi_1(C_m(S^{(n)}), c_0) \cong \mathbf{B}_m(S^{(n)})$. To complete the construction, we now define the relevant quotient of $\mathbf{B}_m(S^{(n)})$.

We recall from §3.4.3 that $\Lambda^S_{m,n} : \mathbf{B}_{m,n}(S) \twoheadrightarrow \mathbf{B}_n(S)$ denotes the split surjective morphism induced by forgetting the first m coordinates and that $\mathbf{B}_{m,n}(S) \cong \mathbf{B}_m\left(S^{(n)}\right) \rtimes \mathbf{B}_n(S)$. Hence the natural action of $\mathbf{B}_n(S)$ on $\mathbf{B}_m\left(S^{(n)}\right)$ induced by $\theta_m\left(S^{(n)}\right)$ is equivalent to the conjugate action in $\mathbf{B}_{m,n}(S)$, if we regard these two groups as subgroups of $\mathbf{B}_{m,n}(S)$.

Recall that we denote by $\gamma_l(\mathbf{B}_{m,n}(S))$ and $\gamma_l(\mathbf{B}_n(S))$ the respective canonical projections on the quotient by the lth term of the lower central series of $\mathbf{B}_{m,n}(S)$ and $\mathbf{B}_n(S)$, and that we omit the groups from the notations where there is no ambiguity. They canonically induce a morphism $\overline{\Lambda}^S_{l,m,n}$ so that $\overline{\Lambda}^S_{l,m,n} \circ \gamma_l(\mathbf{B}_{m,n}(S)) = \gamma_l(\mathbf{B}_n(S)) \circ \Lambda^S_{m,n}$.

We consider the kernel of $\overline{\Lambda}^S_{l,m,n}$. By the universal property of the kernel, there exists a unique surjective morphism that we denote by $\phi^S_{l,m,n}$, so that the following diagram is commutative (and where the two lines are exact):

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{B}_m\left(S^{(n)}\right) \arrow[r] \arrow[d, "\phi^S_{l,m,n}"] & \mathbf{B}_{m,n}(S) \arrow[r, "\Lambda^S_{m,n}"] \arrow[d, "\gamma_l"] & \mathbf{B}_n(S) \arrow[r] \arrow[d, "\gamma_l"] & 1 \\
1 \arrow[r] & \ker\left(\overline{\Lambda}^S_{l,m,n}\right) \arrow[r] & \mathbf{B}_{m,n}(S)/\Gamma_l(\mathbf{B}_{m,n}(S)) \arrow[r, "\overline{\Lambda}^S_{l,m,n}"] & \mathbf{B}_n(S)/\Gamma_l(\mathbf{B}_n(S)) \arrow[r] & 1.
\end{tikzcd} \tag{4.1}$$

Then, we have the following key property.

Proposition 4.6 *For all integers $m \geqslant 1$, $\ker\left(\phi^S_{l,m,n}\right)$ is preserved by the action $\pi_0\left(\left(\theta_m\left(S^{(n)}\right)\right)_\pi\right)$ of $\mathbf{B}_n(S)$ on $\mathbf{B}_m\left(S^{(n)}\right)$.*

Proof. We consider $\psi \in \mathbf{B}_n(S)$ and $x \in \ker\left(\phi^S_{l,m,n}\right)$. As elements of $\mathbf{B}_{m,n}(S)$, the action of ψ on x is defined by $\left(\theta_{m,n}(\psi)(x), 1_{\mathbf{B}_n(S)}\right) = \left(1_{\mathbf{B}_m\left(S^{(n)}\right)}, \psi^{-1}\right)\left(x, 1_{\mathbf{B}_n(S)}\right)\left(1_{\mathbf{B}_m\left(S^{(n)}\right)}, \psi\right)$. By the universal property of the kernel, there exists a unique morphism $\ker\left(\phi^S_{l,m,n}\right) \to \Gamma_l(\mathbf{B}_{m,n}(S))$ such that the following square is commutative

$$\begin{tikzcd}
\ker\left(\phi^S_{l,m,n}\right) \arrow[r, dotted] \arrow[d, hook] & \Gamma_l(\mathbf{B}_{m,n}(S)) \arrow[d, hook] \\
\mathbf{B}_m\left(S^{(n)}\right) \arrow[r, hook] & \mathbf{B}_{m,n}(S).
\end{tikzcd}$$

Therefore $\left(x, 1_{\mathbf{B}_n(S)}\right) \in \Gamma_l(\mathbf{B}_{m,n}(S))$. Since $\Gamma_l(\mathbf{B}_{m,n}(S))$ is a normal subgroup of $\mathbf{B}_{m,n}(S)$, it follows from the commutativity of the left-hand square of the diagram (4.1) that $\psi^{-1}x\psi \in \ker\left(\phi^S_{l,m,n}\right)$. $\square$

Remark 4.7 In the current situation, we are only able to deduce Assumption 2.13 and not also Assumption 2.16, as we could in the proof of Lemma 4.1 above. Indeed the conjugation action automatically becomes trivial only for the abelianisation and not for the further lower central quotients.

Hence Assumption 2.13 is satisfied and then Definition 2.15 gives a homological representation, for all index $k \geqslant 0$:

$$L_k\left(F_{\theta_m,\phi^S_{l,m,n}}\right) : \mathbf{B}_n(S) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}\left(H_k\left(\left(C_m\left(S^{(n)}\right)\right)^{\phi^S_{l,m,n}}; \mathbb{Z}\right)\right).$$

4.2.1 Case of the abelianisation

Using the abelianisation of a mixed surface braid group $\mathbf{B}_{m,n}(S)$, the constructed homological representation $L_k\left(F_{\theta_m,\phi^S_{2,m,n}}\right)$ satisfies two additional interesting properties. First, the following lemma shows that the quotient groups $\overline{\Lambda}^S_{2,m,n}$ does not depend on n if $n \geqslant 2$. Such property is inter alia crucial for stating some polynomiality results in §8.

Lemma 4.8 *For all $m \geqslant 1$ and $n \geqslant 2$, there is an isomorphism* $\ker\left(\overline{\Lambda}^S_{2,m,n}\right) \cong \ker\left(\overline{\Lambda}^S_{2,m,n+1}\right)$.

Proof. Either if $S = \Sigma_{g,1}$ or if $S = \mathcal{N}_{c,1}$, we recall from §3.4.3 that the subgroup $\mathbf{B}_{m,n}(S)$ of $\mathbf{B}_{m+n}(S)$ is isomorphic the semidirect product of $\mathbf{B}_m\left(S^{(n)}\right) \rtimes \mathbf{B}_n(S)$. We also recall that the presentations of the surface braid groups are detailed in Proposition 3.27. Since we have the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for the braid generators of $\mathbf{B}_n(S)$, we deduce that $\gamma_2(\sigma_i) = \gamma_2(\sigma_{i+1})$ for all $i \in \{1, \ldots, n\}$.

Furthermore, it is a standard observation that the conjugation action by the braid generators $\{\sigma_i\}_{i\in\{1,\ldots,n-1\}}$ of $\mathbf{B}_n(S)$ on the generators $X = \{\xi_j\}_{j\in\{1,\ldots,n\}}$ of $\mathbf{B}_m\left(S^{(n)}\right)$ is the same as the action of the braid group $\mathbf{B}_n$ on the fundamental group $\pi_1(\mathbb{D}_n, p_0)$. More precisely, using the identification

$$\xi_j = \begin{cases} \sigma_1^2 & \text{if } j = 1 \\ \sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{i-1}^{-1} \circ \sigma_i^2 \circ \sigma_{i-1} \circ \cdots \circ \sigma_2 \circ \sigma_1 & \text{if } j \in \{2, \ldots, n\}, \end{cases}$$

it follows from the presentation of $\mathbf{B}_{m+n}(S)$ that

$$\sigma_i^{-1}\xi_j\sigma_i = \begin{cases} \xi_{i+1} & \text{if } j = i; \\ \xi_{i+1}^{-1}\xi_i\xi_{i+1} & \text{if } j = i+1; \\ \xi_j & \text{if } j \notin \{i, i+1\}. \end{cases}$$

Hence $\gamma_2(\xi_i) = \gamma_2(\xi_{i+1})$ and we deduce from the presentations of $\mathbf{B}_m\left(S^{(n)}\right)$ and $\mathbf{B}_n(S)$ that the numbers of generators of $\mathbf{B}_{m,n}(S)/\Gamma_2(\mathbf{B}_{m,n}(S))$ is fixed when n varies.

Moreover, a straightforward computation from their presentations shows that

$$\mathbf{B}_n(\Sigma_{g,1})/\Gamma_2(\mathbf{B}_n(\Sigma_{g,1})) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/2\mathbb{Z} \text{ and } \mathbf{B}_n(\mathcal{N}_{c,1})/\Gamma_2(\mathbf{B}_n(\mathcal{N}_{c,1})) \cong \mathbb{Z}^g \oplus \mathbb{Z}$$

and are thus independent of n. A fortiori $\ker\left(\overline{\Lambda}^S_{2,m,n}\right)$ and $\ker\left(\overline{\Lambda}^S_{2,m,n+1}\right)$ are isomorphic. $\square$

Remark 4.9 For the orientable case, [BGG17, Proposition 3.3] shows that

$$\mathbf{B}_{m,n}(\Sigma_{g,1})/\Gamma_2(\mathbf{B}_{m,n}(\Sigma_{g,1})) \cong \mathbb{Z}^{4g} \oplus (\mathbb{Z}/2\mathbb{Z})^{d_{m,n}}$$

where $d_{m,n} = \begin{cases} 0 & \text{if } n = m = 1; \\ 1 & \text{if } m = 1 \text{ and } n \geqslant 2 \text{ or } m \geqslant 2 \text{ and } n = 1; \\ 2 & \text{if } m \geqslant 2 \text{ and } n \geqslant 2. \end{cases}$

Notation 4.10 We denote $\ker\left(\overline{\Lambda}^{S}_{2,m,n}\right)$ by $A_{2,m}(S)$ and $\phi^{S}_{2,m,n}$ by $\phi^{S}_{2,m}$ for all natural numbers n. This notation is consistent for $n \geqslant 2$ by Lemma 4.8. Although for $n = 0$ and $n = 1$ the kernels are not isomorphic to the others, we use this slight abuse of notations for simplicity.

Furthermore, Assumption 2.16 is satisfied using the morphism $\theta_m\left(S^{(n)}\right)$ and the quotient group $A_{2,m}(S)$. Indeed, we have the following key property.

Proposition 4.11 *For all natural numbers $m \geqslant 1$, the action of $\mathbf{B}_n(S)$ on the group $A_{2,m}(S)$ is trivial.*

Proof. We consider $\psi \in \mathbf{B}_n(S)$ and $x \in A_{2,m}(S)$. We fix $\tilde{x} \in \mathbf{B}_m\left(S^{(n)}\right)$ so that $\phi^{S}_{2,m}(\tilde{x}) = x$. Recall that the action of ψ on x is induced by the conjugation as elements of $\mathbf{B}_{m,n}(S)$: $\left(\theta_{m,n}(\psi)(x), 1_{\mathbf{B}_n(S)}\right) = \left(1_{\mathbf{B}_m\left(S^{(n)}\right)}, \psi^{-1}\right)\left(\tilde{x}, 1_{\mathbf{B}_n(S)}\right)\left(1_{\mathbf{B}_m\left(S^{(n)}\right)}, \psi\right)$. Therefore $\gamma_2\left(\psi^{-1}\tilde{x}\psi\right) = \gamma_2(\tilde{x})$. Then it follows from the commutativity of the diagram (4.1) that $\phi^{S}_{2,m}(\tilde{x}) = x = \psi^{-1}x\psi$. $\square$

4.2.2 Case of the third lower central quotients

Contrary to classical braid groups (see Remark 4.4), another possibility to build representations for the surface braid groups is to consider the third lower central quotients of the mixed braid groups, which are generally speaking different from the second ones in this case. As for the situation of §4.2.1, the following lemma shows that the quotient groups $\overline{\Lambda}^{S}_{3,m,n}$ do not depend on n if $n \geqslant 2$. Again such a property will be used to prove some polynomiality results in §8.

Lemma 4.12 *For all $m \geqslant 1$ and $n \geqslant 2$, there is an isomorphism $\ker\left(\overline{\Lambda}^{S}_{3,m,n}\right) \cong \ker\left(\overline{\Lambda}^{S}_{3,m,n+1}\right)$.*

Proof. Recall the group $\mathbf{B}_{m,n}(S)$ is isomorphic the semidirect product of $\mathbf{B}_m\left(S^{(n)}\right) \rtimes \mathbf{B}_n(S)$. Recall that we have the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for the braid generators of $\mathbf{B}_n(S)$ by Proposition 3.27. Since $\gamma_3([\sigma_i, [\sigma_{i+1}, \sigma_i]])$ is trivial, we deduce that $\gamma_3(\sigma_i) = \gamma_3(\sigma_{i+1})$ for all $i \in \{1, \ldots, n\}$ in the metabelian quotients $\mathbf{B}_{m,n}(S)/\Gamma_3(\mathbf{B}_{m,n}(S))$ and $\mathbf{B}_n(S)/\Gamma_3(\mathbf{B}_n(S))$.

Furthermore, it follows from the presentation of $\mathbf{B}_{m+n}(S)$ that the conjugation action by the braid generators $\{\sigma_i\}_{i\in\{1,\ldots,n-1\}}$ of $\mathbf{B}_n(S)$ on the generators $\{\xi_j\}_{j\in\{1,\ldots,n\}}$ of $\mathbf{B}_m\left(S^{(n)}\right)$ is defined by

$$\sigma_i^{-1}\xi_j\sigma_i = \begin{cases} \xi_{i+1} & \text{if } j = i; \\ \xi_{i+1}^{-1}\xi_i\xi_{i+1} & \text{if } j = i+1; \\ \xi_j & \text{if } j \notin \{i, i+1\}. \end{cases}$$

We deduce from the relation $\gamma_3([\xi_{i+1}, [\sigma_i, \xi_{i+1}]]) = 1_{\mathbf{B}_{m,n}(S)}$ that $\gamma_3(\xi_i) = \gamma_3(\xi_{i+1})$. Then, using the fact that $\gamma_3(\sigma_i) = \gamma_3(\sigma_{i+1})$, we obtain

$$\gamma_3(\xi_{i+1}) = \gamma_3\left(\sigma_i^{-1}\xi_i\sigma_i\right) = \gamma_3\left(\sigma_{i+1}^{-1}\xi_i\sigma_{i+1}\right) = \gamma_3\left(\xi_{i+1}^{-1}\xi_i\xi_{i+1}\right) = \gamma_3(\xi_i).$$

The numbers of generators of $\mathbf{B}_{m,n}(S)/\Gamma_3(\mathbf{B}_{m,n}(S))$ does not depend on n because of the presentations of $\mathbf{B}_m\left(S^{(n)}\right)$ and $\mathbf{B}_n(S)$ (see Proposition 3.27) and the semidirect product structure. Moreover, the metabelian quotient $\mathbf{B}_n(S)/\Gamma_3(\mathbf{B}_n(S))$ is also independent of n. Hence $\ker\left(\overline{\Lambda}^{S}_{m,n}\right)$ and $\ker\left(\overline{\Lambda}^{S}_{m,n+1}\right)$ are isomorphic. $\square$

Remark 4.13 For the orientable case, [BGG17, Corollary 3.9] proves that for $m, n \geqslant 3$ and $g \geqslant 1$

$$\mathbf{B}_{m,n}(\Sigma_{g,1})/\Gamma_3(\mathbf{B}_{m,n}(\Sigma_{g,1})) \cong \left(\mathbb{Z}^3 \times \mathbb{Z}^{2g}\right) \rtimes \mathbb{Z}^{2g}.$$

Notation 4.14 For simplicity, we denote $\ker\left(\overline{\Lambda}^{S}_{3,m,n}\right)$ by $A_{3,m}(S)$ for all natural numbers n.

4.3 Mapping class groups

The use of configuration spaces to construct homological representations for the surface braid groups in §4.2 can be repeated for mapping class groups of surfaces. Indeed the quotients given by the lower central series associated with some *ordered* and *unordered* configuration spaces give rise to interesting families of homological representations of these groups. We will see in §5.3 that the Magnus representations and those introduced by [Mor07] are particular cases of homological representations. For the remainder of §4.3, we fix two natural numbers $g \geqslant 1$ and $c \geqslant 2$ and consider a surface S which is either the orientable surface $\Sigma_{g,1}$ or the non-orientable surface $\mathcal{N}_{c,1}$, as denoted in §3.4.2.

4.3.1 Using ordered configuration spaces

A first idea to construct homological representations for the mapping class groups of surfaces makes use of ordered configuration spaces. We fix a natural number $m \geqslant 1$. Recall from §3.4.3 that $F_m(S)$ denotes the *ordered* configuration space of m points on the surface S. Considering the following subspace of $S^{\times m}$

$$\mathcal{D}_m(S) := \left\{(x_1, \ldots, x_m) \in S^{\times m} \mid x_i = x_j \text{ for some } i \neq j\right\},$$

then $F_m(S) = S^{\times m} \setminus \mathcal{D}_m(S)$. We fix m distinct points $\{z_i\}_{i \in \{1,\ldots,m\}}$ in the preferred boundary-component of S, and another point p_0 in the same boundary-component, distinct from all z_i. We choose the configuration $c_0 = (z_1, \ldots, z_m)$ as the basepoint of $F_m(S)$.

Each diffeomorphism φ of S that fixes the preferred boundary-component of S also fixes c_0 and the diagonal action of φ on $S^{\times m}$ preserves the subspace $\mathcal{D}_m(S)$. Hence the natural action of $\mathrm{Diff}_{\partial_0}(S)$ on the coordinates of the configuration space $F_m(S)$ defines a continuous group morphism

$$\theta_{F_m(S)} \colon \mathrm{Diff}_{\partial_0}(S) \longrightarrow \mathrm{Homeo}_{c_0}(F_m(S)).$$

We consider the canonical projection on the quotient by the lth term of the lower central series of $\pi_1(F_m(S), c_0)$, denoted by γ_l. Assumption 2.13 is automatically satisfied since $\Gamma_l(\pi_1(F_m(S), c_0))$ is a characteristic subgroup of $\pi_1(F_m(S), c_0)$. Then Definition 2.15 gives a representation, for all index $k \geqslant 0$:

$$L_k\left(F_{\theta_{F_m(S)}, \gamma_l}\right) : \pi_0 \mathrm{Diff}_{\partial_0}(S) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}\left(H_k\left((F_m(S))^{\gamma_l}; \mathbb{Z}\right)\right).$$

An interesting modification of this construction consists in removing the basepoint p_0 from the configuration space: the version with Borel-Moore homology of this alternative is then endowed with a natural free generating set (see §6) and recovers *Moriyama representations* (see §5.3.2). We denote the surface $S \setminus \{p_0\}$ by $S^\bullet$. Namely, we consider now the configuration space $F_m(S^\bullet) = S^{\times m} \setminus (\mathcal{D}_m(S) \cup \mathcal{A}_m(S, p_0))$ where $\mathcal{A}_m(S, p_0)$ is the space $\{(x_1, \ldots, x_m) \in S^{\times m} \mid x_i = p_0 \text{ for some } i\}$.

Again, any diffeomorphism φ of S that fixes the preferred boundary-component of S also fixes c_0 and the diagonal action of φ on $S^{\times m}$ preserves the subsets $\mathcal{D}_m(S)$ and $\mathcal{A}_m(S, p_0)$. Hence the natural action of $\mathrm{Diff}_{\partial_0}(S)$ on the coordinates of the configuration space $F_m(S^\bullet)$ defines a continuous group morphism $\theta_{F_m(S^\bullet)} : \mathrm{Diff}_{\partial_0}(S) \to \mathrm{Homeo}_{c_0}(F_m(S^\bullet))$. Assumption 2.13 being again satisfied since $\Gamma_l(\pi_1(F_m(S^\bullet), c_0))$ is a characteristic subgroup of $\pi_1(F_m(S^\bullet), c_0)$, Definition 2.15 gives a representation, for all index $k \geqslant 0$:

$$L_k\left(F_{\theta_{F_m(S^\bullet)}, \gamma_l}\right) : \pi_0 \mathrm{Diff}_{\partial_0}(S) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}\left(H_k\left(F_m(S^\bullet)^{\gamma_l}; \mathbb{Z}\right)\right).$$

4.3.2 Using unordered configuration spaces

Another natural idea to apply the general construction of §2 for mapping class groups of surfaces is to consider the lower central quotient groups of unordered configuration spaces.

We fix a natural number $m \geqslant 0$. Recall from §3.4.3 that $C_m(S)$ denotes the *unordered* configuration space of m points on the surface S. Using the space $\mathcal{D}_m(S)$ introduced in §4.3.1, $C_m(S)$ can be viewed as the space $\left(S^{\times m} \setminus \mathcal{D}_m(S)\right)/\mathfrak{S}_m$. We fix m distinct points $\{z_i\}_{i\in\{1,\ldots,n\}}$ in the boundary S. We choose the configuration $c_0 = \{z_1, \ldots, z_m\}$ as the basepoint of $C_m(S)$. Recall from §3.4.3 that the braid group $\mathbf{B}_m(S)$ is the fundamental group of the configuration space $\pi_1(C_m(S), c_0)$.

The action by permutation of coordinates of the symmetric group $\mathfrak{S}_n$ on $F_m(S)$ commutes with the action of any diffeomorphism φ of S on the coordinates of the elements of $F_m(S)$. This induces a canonical surjective continuous group morphism $\mathrm{Homeo}_{c_0}(F_m(S)) \twoheadrightarrow \mathrm{Homeo}_{c_0}(C_m(S))$. The composite of the continuous group morphism $\theta_{F_m(S)}$ defined in §4.3.1 by this canonical surjection gives the natural action of $\mathrm{Diff}_{\partial_0}(S)$ on the coordinates of the configuration space $C_m(S)$ defining a continuous group morphism:

$$\theta_{C_m(S)} : \mathrm{Diff}_{\partial_0}(S) \to \mathrm{Homeo}_{c_0}(C_m(S)).$$

Again, we consider the canonical projection on the quotient by the lth term of the lower central series of $\pi_1(C_m(S), c_0)$, denoted by γ_l. Assumption 2.13 is automatically satisfied since $\Gamma_l(\pi_1(C_m(S), c_0))$ is a characteristic subgroup of $\pi_1(C_m(S), c_0)$. Therefore, Definition 2.15 gives a representation, for all index $k \geqslant 0$:

$$L_k\left(F_{\theta_{C_m(S)},\gamma_l}\right) : \pi_0\mathrm{Diff}_{\partial_0}(S) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}\left(H_k\left((C_m(S))^{\gamma_l}; \mathbb{Z}\right)\right).$$

Additional properties for orientable surfaces: The lower central series and quotients for the surface braid groups for an orientable surface $S = \Sigma_{g,1}$ have already been the subject of an intensive study in the literature. This allows us to give additional properties on the homological representations in this situation.

Taking $m = 1$, the first homology group is the only one which produces a non-trivial homological representation $L_1(F_{\theta_S,\gamma_l})$ for any l. Since the lower central series of a free group on two or more generators does not stabilise, i.e. $\Gamma_l(\mathbf{F}_{2g}) \neq \Gamma_{l+1}(\mathbf{F}_{2g})$ for all $l \geqslant 1$ and $g \geqslant 1$, the study of the lower central quotient of the fundamental group of the surface $\Sigma_{g,1}$ is an active research topic. We refer the reader to [MKS04] for further details on this question. For $l = 0$, the action corresponds to the natural action on the first homology group of the surface and its kernel is the Torelli group. For convenience we denote it by $\mathfrak{a}_g : \mathbf{\Gamma}_{g,1} \to \mathrm{Aut}_{\mathbb{Z}}(H_1(\Sigma_{g,1}, \mathbb{Z}))$.

Surface braid groups on $m = 2$ strands represent a more difficult situation. For instance, for the torus with one boundary component $\Sigma_{1,1}$, [BGG08, Section 4] proves that the lower central series does not stabilise: $\Gamma_l(\mathbf{B}_2(\Sigma_{1,1})) \neq \Gamma_{l+1}(\mathbf{B}_2(\Sigma_{1,1}))$ for all $l \geqslant 1$. Actually the question of whether surface braid group $\mathbf{B}_2(\Sigma_{g,1})$ for any $g \geqslant 1$ is residually nilpotent is still open.

We now fix $m \geqslant 3$. [BGG08] and [BGG17] give a complete study of the lower central quotient groups of $\mathbf{B}_m(\Sigma_{g,1})$. In particular they show that the lower central series stabilises, namely that $\Gamma_3(\mathbf{B}_m(\Sigma_{g,1})) = \Gamma_4(\mathbf{B}_m(\Sigma_{g,1}))$. Therefore it is relevant to define the constructed representations $L_k\left(F_{\theta_{C_m(\Sigma_{g,1})},\gamma_l}\right)$ only for $l \leqslant 3$. Moreover, they prove the following key results:

Proposition 4.15 ([BGG08, Theorem 1] [BGG17, Corollary 3.12]) *The abelianisation of $\mathbf{B}_m(\Sigma_{g,1})$ is isomorphic to the product $\mathbb{Z}^{2g} \times \mathbb{Z}/2\mathbb{Z}$. The third lower central quotient $\mathbf{B}_m(\Sigma_{g,1})/\Gamma_3(\mathbf{B}_m(\Sigma_{g,1}))$ is isomorphic to the semidirect product*

$$(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g.$$

More precisely, recalling the presentation of Proposition 3.27, the first factor $\mathbb{Z}$ is central and is generated by $\sigma := \gamma_3(\sigma_i)$ for all $i \in \{1, \ldots, m-1\}$, the second factor $\mathbb{Z}^g$ is generated by $\{a_i := \gamma_3(a_i)\}_{i\in\{1,\ldots,g\}}$, and the third factor $\mathbb{Z}^g$ is generated by $\{b_i := \gamma_3(b_i)\}_{i\in\{1,\ldots,g\}}$; for all $j \in \{1, \ldots, g\}$, the generator b_j acts trivially on a_i for $i \in \{1, \ldots, g\} \setminus \{j\}$ and $a_j b_j = \sigma^2 b_j a_j$.

The result on the third lower central quotient allows us to obtain an additional property for the associated homological representation: we can find the best subgroup of the mapping class groups $\mathbf{\Gamma}_{g,1}$ which acts on $H_m\left(C_m(\Sigma_{g,1})^{\gamma_3}, \mathbb{Z}\right)$ as a $\mathbf{B}_m(\Sigma_{g,1})/\Gamma_3(\mathbf{B}_m(\Sigma_{g,1}))$-module, i.e. so that Assumption 2.16 is satisfied. Beforehand, we have to refine the result of Proposition 4.15.

According to the unpublished work [BGG11], the ideas developed below were already explored by Christian Blanchet but have not yet been published.

Let $\{\alpha_i, \beta_i\}_{i\in\{1,\ldots,g\}}$ be loops in $\Sigma_{g,1}$ based at p_0 that form a system of meridians and parallels of the surface: α_i and β_i respectively encircle the meridian and the parallel of the ith handle. They give a free generating set $\{[\alpha_i], [\beta_i]\}_{i\in\{1,\ldots,n\}}$ for the fundamental group $\pi_1(\Sigma_{g,1}, p_0)$. We also assume that these loops are such that the product of commutators is a positively oriented loop around the boundary component. The abelianisation γ_2 thus induces a symplectic basis $\{A_i, B_i\}_{i\in\{1,\ldots,n\}}$ for the first homology group of the surface $H_g := H_1(\Sigma_{g,1}; \mathbb{Z})$ with respect to the algebraic intersection form $\omega_g : H_g \times H_g \to \mathbb{Z}$. Moreover, the operation

$$(k, c) \cdot (k, c) = (k + k' + \omega_g(c, c'), c + c')$$

for all $k, k' \in \mathbb{Z}$ and $c, c' \in H_g$ defines a group structure on the set $\mathbb{Z} \times H_g$: we denote by $\mathbb{Z} \underset{\omega_g}{\times} H_g$ this central extension. Then:

Lemma 4.16 *The third lower central quotient $\mathbf{B}_m(\Sigma_{g,1})/\Gamma_3(\mathbf{B}_m(\Sigma_{g,1}))$ is isomorphic to the central extension $\mathbb{Z} \underset{\omega_g}{\times} H_g$. Moreover the action of the mapping class group $\mathbf{\Gamma}_{g,1}$ on the quotient $\mathbf{B}_m(\Sigma_{g,1})/\Gamma_3(\mathbf{B}_m(\Sigma_{g,1})) \twoheadrightarrow H_g$ is the natural action $\mathfrak{a}_g$.*

Proof. The isomorphism is given by sending σ to the generator of $\mathbb{Z}$ in the central extension, a_i to A_i and b_i to B_i for all $i \in \{1, \ldots, g\}$. The relation $a_j b_j = \sigma^2 b_j a_j$ is preserved through this morphism by the definition of the intersection form (up to the sign convention).
As recalled in Remark 3.28, the generators $\{a_i, b_i\}_{i\in\{1,\ldots,g\}}$ define a generating set for the fundamental group of $\Sigma_{g,1}$: the action of $\mathbf{\Gamma}_{g,1}$ induced by $\theta_{C_m(\Sigma_{g,1})}$ on the image of these generators in the quotient H_g is thus exactly the symplectic representation of the mapping class group. $\square$

Hence Lemma 4.16 shows that we must restrict to a subgroup of the Torelli group $\mathcal{I}_{g,1}$ to obtain a trivial action on the third lower central quotient. Since $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, there exists a map $k : \mathbf{\Gamma}_{g,1} \to \mathrm{Hom}(H_g, \mathbb{Z})$ so that the homological representation is defined by

$$L_m\left(F_{\theta_{C_m(\Sigma_{g,1})},\gamma_3}\right) = \begin{bmatrix} \pm Id_{\mathbb{Z}} & k \\ 0 & \mathfrak{a}_g \end{bmatrix}.$$

The following result shows that this map k is related to the well-known *Chillingworth homomorphism* Chill introduced in [Chi72] which describes the action of the Torelli group $\mathcal{I}_{g,1}$ on the winding numbers of the curves of $\Sigma_{g,1}$.

Lemma 4.17 *The map k is a crossed homomorphism and its kernel coincides with the kernel of Chillingworth homomorphism.*

Proof. Since $L_m\left(F_{\theta_{C_m(\Sigma_{g,1})},\gamma_3}\right)$ is a morphism, we deduce that that $k(\varphi \circ \psi) = k(\psi) + k(\varphi)\,\mathfrak{a}_g(\psi)$ for all $\varphi, \psi \in \mathbf{\Gamma}_{g,1}$: this proves that k is a crossed homomorphism. Moreover [Mor89] proves that $H^1(\mathbf{\Gamma}_{g,1}, H_g) \cong \mathbb{Z}$. Hence $k = \lambda \cdot \mathrm{Chill} + c$ where $\lambda \in \mathbb{Z}$ and c is a principal crossed homomorphism: restricting to the Torelli group, we deduce that $\ker(k) = \ker(\mathrm{Chill})$. $\square$

Hence the homological representation $L_m\left(F_{\theta_{C_m(\Sigma_{g,1})},\gamma_3}\right)$ acts on the homology $H_m\left((C_m(\Sigma_{g,1}))^{\gamma_3}, \mathbb{Z}\right)$ as a $\left(\mathbb{Z} \underset{\omega_g}{\times} H_g\right)$-module if we restrict to any subgroup of the (index-1 or index-2) subgroup of the kernel of the Chillingworth homomorphism so that the sign of the $\pm Id_Z$ is positive. For instance, [Joh83] proves that the Johnson subgroup $\mathcal{K}_{g,1}$, which is the kernel of the natural map $\mathbf{\Gamma}_{g,1} \to \mathrm{Aut}_{\mathbb{Z}}(\pi_1(\Sigma_{g,1}, p_0)/\Gamma_3(\pi_1(\Sigma_{g,1}, p_0)))$, is a subgroup of $\ker(\mathrm{Chill})$.

Remark 4.18 If we assume that $g \geqslant 3$, then it follows from [BGP14] that every proper subgroup of the mapping class group $\mathbf{\Gamma}_{g,1}$ has index at least 28: a fortiori there are no non-trivial morphisms $\mathbf{\Gamma}_{g,1} \to \mathbb{Z}/2\mathbb{Z}$ in this case. Thus the subgroup of the kernel of the Chillingworth homomorphism mentioned above is of index 1 (i.e., equal to it). So it suffices to restrict to any subgroup the kernel of the Chillingworth homomorphism in order that the homological representation $L_m\left(F_{\theta_{C_m(\Sigma_{g,1})},\gamma_3}\right)$ acts on the homology $H_m\left(\left(C_m\left(\Sigma_{g,1}\right)\right)^{\gamma_3},\mathbb{Z}\right)$ as a $\left(\mathbb{Z}\underset{\omega_g}{\times}H_g\right)$-module if $g \geqslant 3$.

4.4 Families of representations via topological categories of embeddings

In this subsection, we reinterpret some of the constructions of sections 4.1–4.3 using the general procedure summarised in §2.5. In each case, this amounts to defining a continuous functor

$$\mathcal{C}^{\mathrm{t}} \longrightarrow \mathrm{Cov}_Q \qquad \text{or} \qquad \mathcal{C}^{\mathrm{t}} \longrightarrow \widetilde{\mathrm{Cov}}_Q,$$

where $\mathcal{C}^{\mathrm{t}}$ is an appropriate topological category (obtained by applying the topological version of Quillen's bracket construction, as described in §3.4.5), and Q is a group. In each of our examples for §§4.1–4.3, objects of $\mathcal{C}^{\mathrm{t}}$ are surfaces and morphism spaces are spaces of embeddings. The general pattern of the construction is as follows: we send each object (surface) to a configuration space on that surface, equipped with a choice of quotient of its fundamental group onto Q. Embeddings between surfaces clearly induce maps of configuration spaces, and we then check that either (a) the induced homomorphisms of fundamental groups commute with the chosen quotients to Q, in which case we obtain a functor $\mathcal{C}^{\mathrm{t}} \to \mathrm{Cov}_Q$, or (b) the weaker property that the induced homomorphisms of fundamental groups at least preserve the kernels of these quotients, in which case we obtain a functor $\mathcal{C}^{\mathrm{t}} \to \widetilde{\mathrm{Cov}}_Q$.

4.4.1 Classical braid groups

Let us fix an integer $m \geqslant 1$ and set $Q = A_m = \mathbb{Z} \oplus \mathbf{B}_m^{\mathrm{ab}}$, which is $\mathbb{Z}$ for $m = 1$ and $\mathbb{Z}^2$ for $m \geqslant 2$. Let $\mathfrak{U}\boldsymbol{\beta}^t$ be the topological category defined in Example 3.37 and let $\mathfrak{U}\boldsymbol{\beta}^t_{\geqslant 1}$ be its full subcategory on all objects *except* the zero-punctured disc $\mathbb{D}_0 = \mathbb{D}^2$. We define a continuous functor

$$F\colon \mathfrak{U}\boldsymbol{\beta}^t_{\geqslant 1} \longrightarrow \mathrm{Cov}_{A_m} \tag{4.2}$$

as follows. For an object (i.e., a punctured disc) $\mathbb{D}_n$ of $\mathfrak{U}\boldsymbol{\beta}^t_{\geqslant 1}$ we define $F(\mathbb{D}_n)$ to be the unordered configuration space $C_m(\mathbb{D}_n)$, based at a fixed configuration c_0 contained in $\partial\mathbb{D}^2$, and we choose the quotient

$$\pi_1(C_m(\mathbb{D}_n), c_0) \longrightarrow\!\!\!\!\!\rightarrow A_m = \mathbb{Z} \oplus \mathbf{B}_m^{\mathrm{ab}}$$

to be the homomorphism $\phi_{m,n}$ constructed in §4.1. Morphisms of $\mathfrak{U}\boldsymbol{\beta}^t_{\geqslant 1}$ are certain embeddings $\varphi\colon \mathbb{D}_n \to \mathbb{D}_{n'}$ (see Example 3.37 for a precise description), and induce maps of configuration spaces $C_m(\varphi)\colon C_m(\mathbb{D}_n) \to C_m(\mathbb{D}_{n'})$, so we may set $F(\varphi) = C_m(\varphi)$. This clearly defines a continuous functor if one ignores the condition that $C_m(\varphi)_*$ must commute with the quotients $\phi_{m,n}$ and $\phi_{m,n'}$, so it remains to check this condition. Now, all morphisms of $\mathfrak{U}\boldsymbol{\beta}^t_{\geqslant 1}$ may be written as a composition of a "standard embedding" $\mathbb{D}_n \to \mathbb{D}_{n'}$ followed by an automorphism of $\mathbb{D}_{n'}$. This is because, as explained in Example 3.37, the morphism space $\mathfrak{U}\boldsymbol{\beta}^t(\mathbb{D}_n, \mathbb{D}_{n'})$ consists of embeddings that may be extended to a diffeomorphism of $\mathbb{D}_{n'}$. Hence it suffices to check the commutativity condition for φ either a standard embedding or an automorphism. For automorphisms, this is exactly the content of Lemma 4.1, and for standard embeddings, this is what was checked explicitly just above Definition 4.2. Thus we have a well-defined continuous functor F.

This gives us the first input of Definition 2.21, with $\mathcal{C}^{\mathrm{t}} = \mathfrak{U}\boldsymbol{\beta}^t_{\geqslant 1}$. Taking the ground ring k to be $\mathbb{Z}$ and defining $R = M = \mathbb{Z}[A_m]$ (i.e., not twisting the coefficients), we therefore obtain a functor

$$L_i(F)\colon \pi_0\left(\mathfrak{U}\boldsymbol{\beta}^t_{\geqslant 1}\right) = \mathfrak{U}\boldsymbol{\beta}_{\geqslant 1} \longrightarrow \mathbb{Z}[A_m]\text{–Mod},$$

for any $i \geqslant 0$. Since $\mathfrak{U}\boldsymbol{\beta}$ is equivalent to $\mathfrak{U}\boldsymbol{\beta}_{\geqslant 1}$ with an initial object adjoined, we may extend this to a functor

$$L_i(F)\colon \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{Z}[A_m]\text{–Mod}$$

by sending the initial object of $\mathfrak{U}\boldsymbol{\beta}$ (the zero-punctured disc) to the initial object of $\mathbb{Z}[A_m]$–Mod (the trivial $\mathbb{Z}[A_m]$-module). This is exactly the functor $\mathfrak{LB}_m$ of Proposition 4.3.

We summarise this discussion as:

Proposition 4.19 *The continuous functor* (4.2) *determines, through the construction of §2.5, the functor*

$$\mathfrak{LB}_m \colon \mathfrak{U}\boldsymbol{\beta} \longrightarrow \mathbb{Z}[A_m]\text{–Mod}$$

of Proposition 4.3, and hence in particular a family of representations of the classical braid groups.

4.4.2 Surface braid groups

Now let us fix a surface $S = \Sigma_{g,1}$ or $\mathcal{N}_{c,1}$, an integer $m \geqslant 1$ and $l \in \{2,3\}$. Recall that the group $\ker\left(\overline{\Lambda}^S_{l,m,n}\right)$ is defined in diagram (4.1); it is the subgroup

$$\ker\left(\overline{\Lambda}^S_{l,m,n}\right) = \mathbf{B}_m(S^{(n)}) \cap \varGamma_l(\mathbf{B}_{m,n}(S))$$

of the surface braid group $\mathbf{B}_m(S^{(n)})$, where $S^{(n)} = \mathbb{D}_n \natural S$. According to Lemmas 4.8 and 4.12, it is independent of n for $n \geqslant 2$, so we may define $A_{l,m}(S)$ to be this group for any $n \geqslant 2$.

We set $Q = A_{l,m}(S)$ and let $\langle \boldsymbol{\beta}^t, \mathcal{B}_2(S)^t \rangle$ be the topological category defined in Example 3.39, where $\mathcal{B}_2(S) = \mathcal{B}_2^{g,+}$ if $S = \Sigma_{g,1}$ and $\mathcal{B}_2(S) = \mathcal{B}_2^{c,-}$ if $S = \mathcal{N}_{c,1}$. Let $\langle \boldsymbol{\beta}^t, \mathcal{B}_2(S)^t \rangle_{\geqslant 2}$ denote its subcategory on all objects of the form $S^{(n)} = \mathbb{D}_n \natural S$ for $n \geqslant 2$. We define continuous functors

$$\begin{aligned} G_2 \colon \langle \boldsymbol{\beta}^t, \mathcal{B}_2(S)^t \rangle_{\geqslant 2} &\longrightarrow \mathrm{Cov}_{A_{2,m}(S)} \\ G_3 \colon \langle \boldsymbol{\beta}^t, \mathcal{B}_2(S)^t \rangle_{\geqslant 2} &\longrightarrow \widetilde{\mathrm{Cov}}_{A_{3,m}(S)} \end{aligned} \tag{4.3}$$

as follows. We send an object $S^{(n)}$ to the unordered configuration space $C_m(S^{(n)})$, based at a configuration c_0 contained in the parametrised interval $I \subset \partial S$, and we choose the quotient

$$\pi_1(C_m(S^{(n)}), c_0) = \mathbf{B}_m(S^{(n)}) \longrightarrow\!\!\!\!\!\rightarrow A_{l,m}(S)$$

to be the quotient map $\phi^S_{l,m,n}$ constructed in §4.2 (see in particular diagram (4.1)).

Morphisms of $\langle \boldsymbol{\beta}^t, \mathcal{B}_2(S)^t \rangle_{\geqslant 2}$ are certain embeddings $\varphi \colon S^{(n)} \to S^{(n')}$ for $2 \leqslant n \leqslant n'$ (see Example 3.39 for a precise description). Hence they induce maps of configuration spaces $C_m(\varphi) \colon C_m(S^{(n)}) \to C_m(S^{(n')})$, so we may set $G_l(\varphi) = C_m(\varphi)$. This clearly defines a continuous functor, if we ignore the condition that $C_m(\varphi)_*$ must commute with the quotients $\phi^S_{l,m,n}$ and $\phi^S_{l,m,n'}$ (in the case $l = 2$) or simply preserve the kernels of these quotients (in the case $l = 3$), so it remains to check these conditions. As in the case of $\mathfrak{U}\boldsymbol{\beta}^t$, all morphisms of $\langle \boldsymbol{\beta}^t, \mathcal{B}_2(S)^t \rangle$ may be written as a composition of the "standard inclusion" $S^{(n)} = \mathbb{D}_n \natural S \to \mathbb{D}_{n'} \natural S = S^{(n')}$ followed by an automorphism of $\mathbb{D}_{n'}$, due to the fact that the morphism space of $\langle \boldsymbol{\beta}^t, \mathcal{B}_2(S)^t \rangle (S^{(n)}, S^{(n')})$ consists of embeddings that may be extended to diffeomorphisms of $S^{(n')}$, as explained in Example 3.39. Hence it suffices to check the conditions on π_1 for φ either a standard inclusion or an automorphism. For $l = 2$, the fact that the induced map $C_m(\varphi)_*$ on π_1 commutes with the quotients $\phi^S_{2,m,n}$ and $\phi^S_{2,m,n'}$ follows from Lemma 4.8 when φ is a standard inclusion and from the combination of Propositions 4.6 and 4.11 when φ is an automorphism. For $l = 3$, the fact that $C_m(\varphi)_*$ sends the kernel of $\phi^S_{3,m,n}$ to the kernel of $\phi^S_{3,m,n'}$ follows from Lemma 4.12 when φ is a standard inclusion,[7] and from Proposition 4.6 when φ is an automorphism. Thus we have well-defined continuous functors G_2 and G_3.

This gives us the first input of Definition 2.21 (for $l = 2$) or Definition 2.22 (for $l = 3$), with $\mathcal{C}^{\mathrm{t}} = \langle \boldsymbol{\beta}^t, \mathcal{B}_2(S)^t \rangle_{\geqslant 2}$. Taking the ground ring k to be $\mathbb{Z}$ and defining $R = M = \mathbb{Z}[A_{l,m}(S)]$ (i.e., not twisting the coefficients), we therefore obtain functors

$$\begin{aligned} L_i(G_2) \colon \pi_0(\langle \boldsymbol{\beta}^t, \mathcal{B}_2(S)^t \rangle_{\geqslant 2}) = \langle \boldsymbol{\beta}, \mathcal{B}_2(S) \rangle_{\geqslant 2} &\longrightarrow \mathbb{Z}[A_{2,m}(S)]\text{–Mod} \\ L_i(G_3) \colon \pi_0(\langle \boldsymbol{\beta}^t, \mathcal{B}_2(S)^t \rangle_{\geqslant 2}) = \langle \boldsymbol{\beta}, \mathcal{B}_2(S) \rangle_{\geqslant 2} &\longrightarrow \mathbb{Z}[A_{3,m}(S)]\text{–Mod}^{\mathrm{tw}}, \end{aligned} \tag{4.4}$$

for any $i \geqslant 0$. This extends functorially the families of homological representations constructed in §4.2, for any $m \geqslant 1$ and for $l \in \{2,3\}$. We summarise this discussion as:

[7] In fact, when φ is a standard inclusion, Lemma 4.12 actually shows that $C_m(\varphi)_*$ commutes with the quotients $\phi^S_{3,m,n}$ and $\phi^S_{3,m,n'}$.

Proposition 4.20 *For any $m \geqslant 1$, the continuous functors* (4.3) *determine, via the construction of §2.5, functors of the form* (4.4) *that extend the families of representations defined in §4.2 for the surface braid groups $\mathbf{B}_n(S)$ for S any compact, connected surface with one boundary-component.*

Remark 4.21 We do not have analogous functorial extensions, in the setting of §2.5, of the representations constructed in §4.3 of mapping class groups of surfaces S. This is because the quotient groups Q in those constructions are typically *not* independent of the surface S. For example, in §4.3.2 we constructed representations of $\pi_0\mathrm{Diff}_{\partial_0}(S)$ using $Q = \mathbf{B}_m(S)/\Gamma_l(\mathbf{B}_m(S))$, but this depends non-trivially on S for any $m \geqslant 3$ and $l \in \{2,3\}$ by Proposition 4.15. By contrast, for example, the quotient groups Q that we used in the case of the classical braid groups $\mathbf{B}_n$ are all isomorphic to $\mathbb{Z}$ (for $m = 1$) or $\mathbb{Z}^2$ (for $m \geqslant 2$). This allowed us, for fixed $m \geqslant 1$, to construct a functor out of $\mathfrak{U}\boldsymbol{\beta}$ using the category $\mathrm{Cov}_{\mathbb{Z}}$ (for $m = 1$) or $\mathrm{Cov}_{\mathbb{Z}^2}$ (for $m \geqslant 2$).

4.5 Loop braid groups

We first describe certain spaces of unlinks, which will be used in our construction of functorial homological representations of (extended and non-extended) loop braid groups. First, recall from above that there are (at least) two interpretations of *loop braid groups* $\mathbf{LB}_n$, as follows:

- $\mathbf{LB}_n \cong \pi_0(\mathrm{Diff}^+_{2D}(\mathbb{D}^3, L))$,

where L is an n-component unlink in the interior of the unit 3-disc $\mathbb{D}^3$ that diffeomorphisms are assumed to fix as a subset, the subscript "$2D$" indicates that diffeomorphisms are assumed to fix pointwise a neighbourhood of a chosen pair of disjoint 2-discs in $\partial\mathbb{D}^3$ and the superscript "$+$" indicates that diffeomorphisms are assumed to preserve the orientation of L (see Lemma 3.43). Secondly:

- $\mathbf{LB}_n \cong \pi_1(U_n^+)$,[8]

where U_n^+ denotes the space of oriented n-component unlinks in $\mathbb{D}^3$, topologised as a subspace of $\mathrm{Emb}(n\mathbb{S}^1, \mathbb{D}^3)/\mathrm{Diff}^+(n\mathbb{S}^1)$, where $n\mathbb{S}^1$ is the disjoint union of n copies of the circle (see §3.4.4). We similarly have two interpretations of the *extended loop braid groups* $\mathbf{LB}_n^{\mathrm{ext}}$, as follows:

- $\mathbf{LB}_n^{\mathrm{ext}} \cong \pi_0(\mathrm{Diff}_{2D}(\mathbb{D}^3, L))$,
- $\mathbf{LB}_n^{\mathrm{ext}} \cong \pi_1(U_n)$,

where, in the first interpretation, we have dropped the condition that diffeomorphisms preserve the orientation of L, and the space U_n denotes the space of *unoriented* n-component unlinks in $\mathbb{D}^3$, topologised as a subspace of $\mathrm{Emb}(n\mathbb{S}^1, \mathbb{D}^3)/\mathrm{Diff}(n\mathbb{S}^1)$.

Recall (cf. §3.4.4) that $\mathbf{LB}_n$ is generated by two finite families of elements: certain elements τ_i that each correspond to a loop in U_n^+ that interchanges two unknots without either of them passing through the other, and certain elements σ_i that each correspond to a loop in U_n^+ that interchanges two unknots, while one passes through the other. The extended version $\mathbf{LB}_n^{\mathrm{ext}}$ is generated by these elements together with certain elements ρ_i that each correspond to a loop in U_n that rotates a single circle by 180 degrees. See Figure 1 of [BH13] for a picture (note that our notation differs from theirs by a cyclic permutation of the letters $\tau \mapsto \sigma \mapsto \rho \mapsto \tau$). Using the presentations of $\mathbf{LB}_n$ and $\mathbf{LB}_n^{\mathrm{ext}}$ calculated in §3 of [BH13] (Proposition 3.3 for $UR_n \cong \mathbf{LB}_n$ and Proposition 3.7 for $R_n \cong \mathbf{LB}_n^{\mathrm{ext}}$), we see that:

- The abelianisation of $\mathbf{LB}_n$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for $n \geqslant 2$ and trivial for $n = 1$, where the $\mathbb{Z}$ summand is generated by an element σ, which is the image of all generators of the form σ_i, and the $\mathbb{Z}/2\mathbb{Z}$ summand is generated by an element τ, which is the image of all generators of the form τ_i.
- The abelianisation of $\mathbf{LB}_n^{\mathrm{ext}}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ for $n \geqslant 2$ and $\mathbb{Z}/2\mathbb{Z}$ for $n = 1$, where the three summands are generated by elements σ, τ and ρ, which are the images of all generators of the form σ_i, τ_i and ρ_i respectively, and the summand generated by ρ is the one present also in the $n = 1$ case.

[8] See Remark 4.23 below for a discussion of issues related to choosing basepoints.

Definition 4.22 For an n-component unlink L in the interior of $\mathbb{D}^3$ and $m \geqslant 1$, we define

$$X_m(L) = \{\text{oriented } m\text{-component links } L' \text{ in } \mathbb{D}^3, \text{ disjoint from } L, \text{ so that } L \sqcup L' \text{ is an unlink}\},$$
$$Y_m(L) = \{\text{unoriented } m\text{-component links } L' \text{ in } \mathbb{D}^3, \text{ disjoint from } L, \text{ so that } L \sqcup L' \text{ is an unlink}\},$$

topologised as subspaces of $\mathrm{Emb}(m\mathbb{S}^1, \mathbb{D}^3)/\mathrm{Diff}^+(m\mathbb{S}^1)$ and $\mathrm{Emb}(m\mathbb{S}^1, \mathbb{D}^3)/\mathrm{Diff}(m\mathbb{S}^1)$ respectively.

We now construct certain quotients of $\pi_1(X_m(L))$ and $\pi_1(Y_m(L))$.

The alpha quotient. First, note that we have natural inclusions

$$i\colon X_m(L) \hookrightarrow U_m^+ \qquad \text{and} \qquad j\colon X_m(L) \hookrightarrow U_{m+n}^+,$$

given respectively by *forgetting* the unlink L or *adjoining* it to the given m-component unlink. Here, n denotes the number of components of L. Note that j depends a choice of *orientation* of L (but i does not, of course). Taking induced maps on π_1 and abelianising, we obtain a map

$$\alpha_m(L)\colon \pi_1(X_m(L)) \longrightarrow \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{for } m = 1, \\ (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) & \text{for } m \geqslant 2 \end{cases} \tag{4.5}$$

defined by $\alpha_m(L) = ((\mathrm{ab} \circ i_*) \times (\mathrm{ab} \circ j_*)) \circ \Delta$. This is not surjective, but it is an easy exercise to calculate its image. When $m = 1$, its image is spanned by the element $\sigma\tau$, which has infinite order. When $m \geqslant 2$, its image is spanned by the elements $(1, \sigma\tau)$, $(\sigma\tau, \sigma\tau)$ and (τ, τ), which generate a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Writing $\hat{\alpha}_m(L)$ for the map $\alpha_m(L)$ after restricting its codomain to its image, we have therefore constructed a quotient map

$$\hat{\alpha}_m(L)\colon \pi_1(X_m(L)) \twoheadrightarrow \begin{cases} \mathbb{Z} & \text{for } m = 1, \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{for } m \geqslant 2, \end{cases} \tag{4.6}$$

depending on a choice of orientation of the unlink L.

Remark 4.23 There is a subtlety related to basepoints that arises in the above construction. For each $r \geqslant 1$ let us choose an r-component unlink c_r contained in $\partial\mathbb{D}^3$. We also fix an orientation of $\mathbb{D}^3$, which therefore induces an orientation of $\partial\mathbb{D}^3$ and hence an orientation of each c_r, so we may think of c_r as an element of U_r^+, and also of $X_r(L)$ for any unlink L in the interior of $\mathbb{D}^3$. To make precise the identification of $\mathbf{LB}_n$ with $\pi_1(U_n^+)$ above, we choose an isomorphism $\mathbf{LB}_n \cong \pi_1(U_n^+, c_n)$. Now, the inclusion i sends c_m to itself, so it induces a homomorphism

$$i_*\colon \pi_1(X_m(L), c_m) \longrightarrow \pi_1(U_m^+, c_m) \cong \mathbf{LB}_m$$

using the chosen identification with $\mathbf{LB}_m$. We may then compose this with the abelianisation map of $\mathbf{LB}_m$ to obtain the first component of the map $\alpha_m(L)$. On the other hand, the inclusion j sends c_m to $c_m \sqcup L \neq c_{m+n}$. We therefore choose some path $c_m \sqcup L \rightsquigarrow c_{m+n}$ in U_{m+n}^+ and use this choice to identify the target of j_* with $\mathbf{LB}_{m+n}$:

$$j_*\colon \pi_1(X_m(L), c_m) \longrightarrow \pi_1(U_{m+n}^+, c_m \sqcup L) \cong \pi_1(U_{m+n}^+, c_{m+n}) \cong \mathbf{LB}_{m+n}.$$

Composing this with the abelianisation map $\mathbf{LB}_{m+n}$, we obtain a map

$$\mathrm{ab} \times j_*\colon \pi_1(X_m(L), c_m) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

which (a priori) depends on the choice of path $c_m \sqcup L \rightsquigarrow c_{m+n}$. Modifying this choice corresponds to inserting a conjugation automorphism of $\pi_1(U_{m+n}^+, c_{m+n})$ into the composition. However, since the target of the homomorphism is abelian, this does not actually change anything, and so the map $\mathrm{ab} \times j_*$ is in fact well-defined, independent of the choice of path $c_m \sqcup L \rightsquigarrow c_{m+n}$.

A similar issue arises in the next construction, which is resolved in the same way.

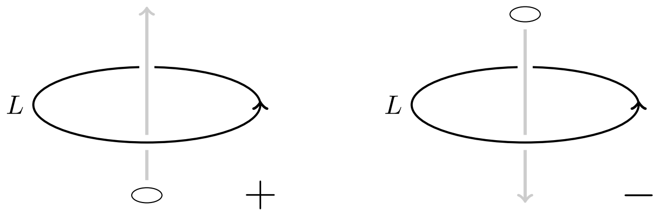


Figure 4.1 The sign convention for the geometric description of (one summand of) the quotient $\hat{\alpha}_m(L)$.

The beta quotient. In the unoriented case, we may perform a similar construction. We have natural inclusions

$$i\colon Y_m(L) \hookrightarrow U_m \qquad \text{and} \qquad j\colon Y_m(L) \hookrightarrow U_{m+n},$$

given respectively by *forgetting* the unlink L or *adjoining* it to the given m-component unlink. This time both i and j are independent of any choice of orientation of L. Taking induced maps on π_1 and abelianising, we obtain a map

$$\beta_m(L)\colon \pi_1(Y_m(L)) \longrightarrow \begin{cases} (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^3 & \text{for } m=1, \\ (\mathbb{Z}/2\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})^3 & \text{for } m \geqslant 2 \end{cases} \tag{4.7}$$

defined by $\beta_m(L) = ((\mathrm{ab} \circ i_*) \times (\mathrm{ab} \circ j_*)) \circ \Delta$. This is again not surjective, but it is an easy exercise to calculate its image. When $m=1$, its image is spanned by the two elements (ρ,ρ) and $(1,\sigma\tau)$, which generate a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. When $m \geqslant 2$, its image is spanned by these elements together with (τ,τ) and $(\sigma\tau,\sigma\tau)$, which generate a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. Writing $\hat{\beta}_m(L)$ for the map $\beta_m(L)$ after restricting its codomain to its image, we have therefore constructed a quotient map

$$\hat{\beta}_m(L)\colon \pi_1(Y_m(L)) \twoheadrightarrow \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & \text{for } m=1, \\ (\mathbb{Z}/2\mathbb{Z})^4 & \text{for } m \geqslant 2. \end{cases} \tag{4.8}$$

Note that this quotient does not depend on a choice of orientation of the unlink L.

The gamma quotient. Finally, it will be useful to construct a further quotient of $\hat{\alpha}_m(L)$ that is independent of an orientation of the unlink L. Recall that the image of $\hat{\alpha}_m(L)$ is generated by the element $\sigma\tau$ of $\mathbf{LB}^{\mathrm{ab}}_{\geqslant 2}$ when $m=1$ and by the elements $(1,\sigma\tau)$, $(\sigma\tau,\sigma\tau)$ and (τ,τ) of $\mathbf{LB}^{\mathrm{ab}}_{\geqslant 2} \oplus \mathbf{LB}^{\mathrm{ab}}_{\geqslant 2}$ when $m \geqslant 2$. Geometrically, the summand generated by $\sigma\tau$ when $m=1$ and by $(1,\sigma\tau)$ when $m \geqslant 2$ counts the number of times that a component of the m-component unlink passes through one of the components of the fixed n-component unlink L. This is counted with sign, as depicted in Figure 4.1: note that the orientation of L is important to determine this sign, whereas the orientation of the component that is passing through it does not matter. Thus, reversing the orientation of L changes the sign of this count. On the other hand, the other two summands (when $m \geqslant 2$), generated by $(\sigma\tau,\sigma\tau)$ and (τ,τ), count certain motions of the m-component unlink that do not involve L, so reversing the orientation of L does not affect these counts. If we write $-L$ for the unlink L with the opposite orientation, this discussion implies that the two quotients $\hat{\alpha}_m(L)$ and $\hat{\alpha}_m(-L)$ differ by the automorphism of their target given by $x \mapsto -x$ (in the case $m=1$) respectively $(x,y,z) \mapsto (-x,y,z)$ (in the case $m \geqslant 2$). Thus, to obtain a further quotient of $\hat{\alpha}_m(L)$ that is independent of the choice of orientation of L, we simply need to quotient one $\mathbb{Z}$ summand of its image to a $\mathbb{Z}/2\mathbb{Z}$ summand. In summary, we have constructed a quotient map

$$\gamma_m(L)\colon \pi_1(X_m(L)) \twoheadrightarrow \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } m=1, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{for } m \geqslant 2, \end{cases} \tag{4.9}$$

which is *independent* of any choice of orientation of the unlink L.

We now apply these quotients and the construction of §2.5 to produce functorial homological representations of the (extended and non-extended) loop braid groups.

The alpha representations of the non-extended loop braid groups. Fix an integer $m \geqslant 1$ and set Q to be the group $\mathbb{Z}$ if $m = 1$ and $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if $m \geqslant 2$. Let $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}^t$ be the topological category described in Remark 3.42: its objects are oriented unlinks in the interior of $\mathbb{D}^3$ and its morphisms from L to L' are embeddings $\varphi\colon \mathbb{D}^3 \to \mathbb{D}^3$ fixing pointwise a disc in $\partial\mathbb{D}^3$, sending L into L' preserving their orientations, and which admit an extension to a diffeomorphism $\mathbb{D}^3 \natural \mathbb{D}^3 \cong \mathbb{D}^3$ taking $L \sqcup L''$ onto L', where L'' is some oriented unlink in the second copy of $\mathbb{D}^3$ in the boundary connected sum. Note that this last condition implies that the preimage of L' under φ must be equal to L (and not larger). See Remark 3.42 for the precise details of these morphism spaces. Let $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}^t_{\geqslant 1}$ be the full subcategory on all objects *except* the empty unlink. We define a continuous functor

$$F^{\alpha}_m \colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta}^t_{\geqslant 1} \longrightarrow \mathrm{Cov}_Q \tag{4.10}$$

as follows. For an object (i.e., an oriented unlink) L of $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}^t_{\geqslant 1}$ we define $F^{\alpha}_m(L)$ to be the space $X_m(L)$ equipped with the quotient $\hat{\alpha}_m(L)$ of its fundamental group. Given a morphism $\varphi\colon \mathbb{D}^3 \to \mathbb{D}^3$ from L to L', we claim first of all that it induces a well-defined map $X_m(\varphi)\colon X_m(L) \to X_m(L')$, i.e., that it sends a link L'' in $\mathbb{D}^3 \smallsetminus L$ such that $L \sqcup L''$ is an unlink to a link $\varphi(L'')$ in $\mathbb{D}^3 \smallsetminus L'$ such that $L' \sqcup \varphi(L'')$ is an unlink. The fact that $\varphi(L'')$ is disjoint from L' follows from the fact that $\varphi^{-1}(L') = L$, as noted above. To see that $L' \sqcup \varphi(L'')$ is an unlink, note that $L \sqcup L''$ bounds a disjoint union of 2-discs in $\mathbb{D}^3$, since it is an unlink, so $\varphi(L) \sqcup \varphi(L'')$ also bounds a disjoint union of 2-discs, so it is an unlink, and hence so is its sub-link $L' \sqcup \varphi(L'')$. Hence setting $F^{\alpha}_m(\varphi) = X_m(\varphi)$ defines a continuous functor, if we ignore the condition that the induced map $X_m(\varphi)_*$ on π_1 must commute with the quotients $\hat{\alpha}_m(L)$ and $\hat{\alpha}_m(L')$, so it remains to check this condition.

All morphisms of $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}^t_{\geqslant 1}$ may be written as a composition of a "standard inclusion" $(\mathbb{D}^3, L) \to (\mathbb{D}^3, L')$ followed by an automorphism of $(\mathbb{D}^3, L')$, due to the fact, recalled above, that morphisms are certain embeddings that admit an extension to a diffeomorphism. Hence it suffices to check the commutativity condition for $X_m(\varphi)_*$ when φ is either a standard inclusion or an automorphism. Now, the commutativity condition says geometrically that the map

$$X_m(\varphi)_* \colon \pi_1(X_m(L)) \longrightarrow \pi_1(X_m(L'))$$

must preserve certain invariants of loops of m-component unlinks. If $m = 1$, there is just one invariant, which is the number of times (with sign determined by Figure 4.1) that a component of the m-component unlink passes through a component of the fixed link (namely L or L'). If $m \geqslant 2$, there are two further invariants, which count (a) the number of times that one component of the m-component unlink passes through another of its components (with sign), and (b) the number of times (mod 2) that two of its components are interchanged (without passing through each other). It is clear that standard inclusions preserve all of these invariants of loops. We therefore just have to check that $X_m(\varphi)_*$ preserves these invariants when φ is an automorphism of $L = L'$ in $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}^t_{\geqslant 1}$, which corresponds (on π_0) to an element of $\mathbf{LB}_n$, where n is the number of components of L. First of all, we may assume that this diffeomorphism of $\mathbb{D}^3$ is supported in a small disc neighbourhood of L, and so it clearly does not affect the invariants (a) and (b), since these involve loops of m-component unlinks that do not interact with L (they may be isotoped to be disjoint from the small disc neighbourhood of L). So we just need to show that $[\varphi] \in \mathbf{LB}_n$ preserves the invariant that counts the number of times that a component of the m-component unlink passes through a component of L. It suffices to check this for $[\varphi] = \tau_i$ and $[\varphi] = \sigma_i$ generators of $\mathbf{LB}_n$. This may easily be checked on a case-by-case basis, drawing a picture of the effects that τ_i and $\sigma_i \in \mathbf{LB}_n$ have on a simple loop of m-component unlinks in $\mathbb{D}^3 \smallsetminus L$, where $m-1$ of the components are fixed, far away from L, and the remaining component passes once through one of the components of L.

Thus we have a well-defined continuous functor F^{α}_m as above. This gives us the first input of Definition 2.21, with $\mathcal{C}^{\mathrm{t}} = \mathfrak{U}\mathcal{L}\boldsymbol{\beta}^t_{\geqslant 1}$. Taking the ground ring k to be $\mathbb{Z}$ and defining $R = M = \mathbb{Z}[Q]$ (i.e., not twisting the coefficients), we therefore obtain a functor

$$L_i(F^{\alpha}_m) \colon \pi_0(\mathfrak{U}\mathcal{L}\boldsymbol{\beta}^t_{\geqslant 1}) = \mathfrak{U}\mathcal{L}\boldsymbol{\beta}_{\geqslant 1} \longrightarrow \mathbb{Z}[Q]\text{−Mod},$$

for any $i \geqslant 0$. Since $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}$ is equivalent to $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}_{\geqslant 1}$ with an initial object adjoined, we may extend this to a functor

$$L_i(F^{\alpha}_m) \colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta} \longrightarrow \mathbb{Z}[Q]\text{−Mod}$$

by sending the initial object of $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}$ (the empty unlink) to the initial object of $\mathbb{Z}[Q]$−Mod (the trivial module). We summarise this construction as:

Theorem 4.24 *For any $m \geqslant 1$ and $i \geqslant 0$, the continuous functor (4.10) determines, through the construction of §2.5, a functor*

$$L_i(F_m^\alpha)\colon \mathfrak{U}\mathcal{L}\boldsymbol{\beta} \longrightarrow \mathbb{Z}[Q]\text{−Mod}, \tag{4.11}$$

where $Q = \mathbb{Z}$ when $m = 1$ and $Q = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ when $m \geqslant 2$. In particular, this gives coherent families of representations of the loop braid groups $\{\mathbf{LB}_n\}_{n\geqslant 1}$ defined over the Laurent polynomial rings

$$\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[x^\pm]$$

when $m = 1$ and defined over

$$\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}] = \mathbb{Z}[x^\pm, y^\pm, z^\pm]/(z^2)$$

when $m \geqslant 2$.

The beta representations of the extended loop braid groups. Fix an integer $m \geqslant 1$ and set Q to be the group $(\mathbb{Z}/2\mathbb{Z})^2$ if $m = 1$ and $(\mathbb{Z}/2\mathbb{Z})^4$ if $m \geqslant 2$. Let $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t$ be the category described in Remark 3.42: its objects are unoriented unlinks in the interior of $\mathbb{D}^3$ and its morphisms from L to L' are embeddings $\varphi\colon \mathbb{D}^3 \to \mathbb{D}^3$, defined exactly as described for $\mathfrak{U}\mathcal{L}\boldsymbol{\beta}^t$ above, but without the orientation-preserving condition. Let $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1}$ be its subcategory on all objects except the empty unlink. We define a continuous functor

$$F_m^\beta\colon \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1} \longrightarrow \mathrm{Cov}_Q \tag{4.12}$$

as follows. For an object (i.e., a non-empty unlink) L of $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1}$, we define $F_m^\beta(L)$ to be the space $Y_m(L)$ equipped with the quotient $\hat{\beta}_m(L)$ of its fundamental group. A morphism $\varphi\colon \mathbb{D}^3 \to \mathbb{D}^3$ of $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1}$ from L to L' induces a map $Y_m(\varphi)\colon Y_m(L) \to Y_m(L')$, which one may see is well-defined by exactly the same argument as above (for the functor F_m^α). Hence setting $F_m^\beta(\varphi) = Y_m(\varphi)$ defines a continuous functor, if we ignore the condition that the induced map $Y_m(\varphi)_*$ on π_1 must commute with the quotients $\hat{\beta}_m(L)$ and $\hat{\beta}_m(L')$, so it remains to check this condition, which says geometrically that the map

$$Y_m(\varphi)_*\colon \pi_1(Y_m(L)) \longrightarrow \pi_1(Y_m(L'))$$

must preserve certain invariants of loops of m-component unlinks. As before, it is enough to check this condition when φ is either a standard inclusion (in which case it is clear) or an automorphism of $L = L'$ in $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1}$, which corresponds (on π_0) to an element of $\mathbf{LB}_n^{\mathrm{ext}}$, where n is the number of components of L. As in the previous setting, all but one of these invariants count certain motions of m-component unlinks that do not interact with L (in the sense that they are supported away from a small disc neighbourhood of L), so they are clearly preserved, since we may ensure by an isotopy that φ is supported only on a small disc neighbourhood of L. The last invariant that we must check is preserved by $[\varphi] \in \mathbf{LB}_n^{\mathrm{ext}}$, is the number (mod 2) of times that a component of the m-component unlink passes through a component of L. It suffices to check this for $[\varphi] = \tau_i$, $[\varphi] = \sigma_i$ and $[\varphi] = \rho_i$ generators of $\mathbf{LB}_n^{\mathrm{ext}}$. As in the previous setting, this may easily be checked on a case-by-case basis. In fact, the generators τ_i and σ_i preserve this invariant as an integer (not just mod 2),[9] whereas the generator ρ_i preserves it only mod 2, since it reverses the orientation of one component of L.

Thus we have a well-defined continuous functor F_m^β as above. This gives us the first input of Definition 2.21, with $\mathcal{C}^{\mathrm{t}} = \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1}$. Taking the ground ring k to be $\mathbb{Z}$ and defining $R = M = \mathbb{Z}[Q]$ (i.e., not twisting the coefficients), we therefore obtain a functor

$$L_i(F_m^\beta)\colon \pi_0(\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1}) = \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})_{\geqslant 1} \longrightarrow \mathbb{Z}[Q]\text{−Mod},$$

[9] Although, of course, this integer depends on an arbitrary choice of orientation of L (since L is not equipped with one), and it is only independent of this choice mod 2.

for any $i \geqslant 0$. Since $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})$ is equivalent to $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})_{\geqslant 1}$ with an initial object adjoined, we may extend this to a functor

$$L_i(F_m^{\beta})\colon \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}}) \longrightarrow \mathbb{Z}[Q]\text{--Mod}$$

by sending the initial object of $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})$ (the empty unlink) to the initial object of $\mathbb{Z}[Q]$–Mod (the trivial module). We summarise this construction as:

Theorem 4.25 *For any $m \geqslant 1$ and $i \geqslant 0$, the continuous functor* (4.12) *determines, through the construction of §2.5, a functor*

$$L_i(F_m^{\beta})\colon \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}}) \longrightarrow \mathbb{Z}[Q]\text{--Mod}, \tag{4.13}$$

where $Q = (\mathbb{Z}/2\mathbb{Z})^2$ when $m = 1$ and $Q = (\mathbb{Z}/2\mathbb{Z})^4$ when $m \geqslant 2$. In particular, this gives coherent families of representations of the extended loop braid groups $\{\mathbf{LB}_n^{\mathrm{ext}}\}_{n\geqslant 1}$ defined over the Laurent polynomial rings

$$\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})^2] = \mathbb{Z}[x^{\pm}, y^{\pm}]/(x^2, y^2)$$

when $m = 1$ and defined over

$$\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})^4] = \mathbb{Z}[x^{\pm}, y^{\pm}, z^{\pm}, w^{\pm}]/(x^2, y^2, z^2, w^2)$$

when $m \geqslant 2$.

The gamma representations of the extended loop braid groups. Clearly the *beta representations* of the extended loop braid groups constructed in Theorem 4.25 restrict to representations of the non-extended loop braid groups, since these are subgroups of the extended loop braid groups (and indeed $\mathfrak{UL}\boldsymbol{\beta}$ is a subcategory of $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})$). Conversely, one may wonder whether the *alpha representations* of the non-extended loop braid groups constructed in Theorem 4.24 may be extended to representations of the *extended* loop braid groups. Functorially, this is the question of whether the functors $L_i(F_m^{\alpha})$ defined on $\mathfrak{UL}\boldsymbol{\beta}$ may be extended to $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})$. In fact, they may not, but our next construction shows that, after changing the ring over which we define this functor — specifically, dividing the Laurent polynomial rings of Theorem 4.24 by the ideal (x^2) — they may be so extended, using the further quotients $\gamma_m(L)$ of the quotients $\hat{\alpha}_m(L)$ constructed above.

Fix an integer $m \geqslant 1$ and set Q to be the group $\mathbb{Z}/2\mathbb{Z}$ if $m = 1$ and $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ if $m \geqslant 2$. We define a continuous functor

$$F_m^{\gamma}\colon \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1} \longrightarrow \mathrm{Cov}_Q \tag{4.14}$$

as follows. We send the object (i.e., unoriented unlink) L of $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1}$ to the space $X_m(L)$ equipped with the quotient $\gamma_m(L)$ of its fundamental group. Note that this is valid since we ensured that $\gamma_m(L)$ does *not* depend on a choice of orientation of L. A morphism $\varphi\colon \mathbb{D}^3 \to \mathbb{D}^3$ of $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1}$ from L to L' induces a well-defined map $X_m(\varphi)\colon X_m(L) \to X_m(L')$. This was shown above (in the construction of F_m^{α}) when φ satisfied the additional assumption that its restriction to a map $L \to L'$ was orientation-preserving. However, this fact was not used in that argument, so it applies equally well to this more general setting. Hence setting $F_m^{\gamma}(\varphi) = X_m(\varphi)$ defines a continuous functor, if we ignore the condition that the induced map

$$X_m(\varphi)_*\colon \pi_1(X_m(L)) \longrightarrow \pi_1(X_m(L'))$$

must commute with the quotients $\gamma_m(L)$ and $\gamma_m(L')$, so it remains to check this condition. If φ is a morphism of

$$\mathfrak{UL}\boldsymbol{\beta}^t_{\geqslant 1} \subset \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1},$$

then we have already checked, during the construction of F_m^{α}, that $X_m(\varphi)_*$ commutes with the quotients $\hat{\alpha}_m(L)$ and $\hat{\alpha}_m(L')$. Since $\gamma_m(L)$ factors through $\hat{\alpha}_m(L)$, this means that $X_m(\varphi)_*$ also commutes with the quotients $\gamma_m(L)$ and $\gamma_m(L')$. Any morphism of $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1}$ may be written as a composition of a morphism of $\mathfrak{UL}\boldsymbol{\beta}^t_{\geqslant 1}$ together with an automorphism of $L = L'$ in $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^t_{\geqslant 1}$ that corresponds, on π_0, to a generator of $\mathbf{LB}_n^{\mathrm{ext}}$ (where n is the number of components of L) of the form ρ_i, which rotates one component of L by 180 degrees. It therefore remains to check that $X_m(\varphi)_*$ commutes with the quotients $\gamma_m(L)$ and $\gamma_m(L')$ when $[\varphi] = \rho_i$. This corresponds to checking that $X_m(\rho_i)_*$ preserves (mod 2) the number (counted with sign) of times that a component of the m-component unlink passes through a component of L. Let us consider a simple loop θ of m-component unlinks in $\mathbb{D}^3 \smallsetminus L$, where $m-1$ of the components are fixed, far away from L, and the remaining component passes once through the j-th component of L. Let us say that an orientation of L has been chosen such that the number of passes of θ through L is $+1$.[10] Then, for $i \neq j$, the number of passes of $X_m(\rho_i)_*(\theta)$ through L is also $+1$, and for $i = j$, the number of passes of $X_m(\rho_i)_*(\theta)$ through L is -1. But $-1 \equiv +1 \pmod 2$.

Thus we have a well-defined continuous functor F_m^γ as above. This gives us the first input of Definition 2.21, with $\mathcal{C}^{\mathrm{t}} = \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^{t}_{\geqslant 1}$. Taking the ground ring k to be $\mathbb{Z}$ and defining $R = M = \mathbb{Z}[Q]$ (i.e., not twisting the coefficients), we therefore obtain a functor

$$L_i(F_m^\gamma)\colon \pi_0(\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})^{t}_{\geqslant 1}) = \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})_{\geqslant 1} \longrightarrow \mathbb{Z}[Q]\text{−Mod},$$

for any $i \geqslant 0$. Since $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})$ is equivalent to $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})_{\geqslant 1}$ with an initial object adjoined, we may extend this to a functor

$$L_i(F_m^\gamma)\colon \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}}) \longrightarrow \mathbb{Z}[Q]\text{−Mod}$$

by sending the initial object of $\mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}})$ (the empty unlink) to the initial object of $\mathbb{Z}[Q]$−Mod (the trivial module). We summarise this construction as:

Theorem 4.26 *For any $m \geqslant 1$ and $i \geqslant 0$, the continuous functor* (4.14) *determines, through the construction of §2.5, a functor*

$$L_i(F_m^\gamma)\colon \mathfrak{U}(\mathcal{L}\boldsymbol{\beta}^{\mathrm{ext}}) \longrightarrow \mathbb{Z}[Q]\text{−Mod}, \tag{4.15}$$

where $Q = \mathbb{Z}/2\mathbb{Z}$ when $m = 1$ and $Q = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ when $m \geqslant 2$. In particular, this gives coherent families of representations of the extended loop braid groups $\{\mathbf{LB}_n^{\mathrm{ext}}\}_{n\geqslant 1}$ defined over the Laurent polynomial rings

$$\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] = \mathbb{Z}[x^{\pm}]/(x^2)$$

when $m = 1$ and defined over

$$\mathbb{Z}[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}] = \mathbb{Z}[x^{\pm}, y^{\pm}, z^{\pm}]/(x^2, z^2)$$

when $m \geqslant 2$. These are extensions of the representations of $\{\mathbf{LB}_n\}_{n\geqslant 1}$ constructed in Theorem 4.24, after dividing the Laurent polynomial rings by the ideal (x^2).

5 Comparison with known representations

In this section, we present the various already existing families of representations that can be recovered as particular homological representations. Indeed for several families of groups such mapping class groups or classical and surface braid groups, many families of representations happen to be more or less explicitly applications of the general procedure encoded by the homological functors described in §2 for several families of groups such mapping class groups or classical and surfaces braid groups. Furthermore, we exhibit in §5.4 a connection between the construction of linear representations introduced in [Lon94] and generalised in [Sou18], called the *Long-Moody* construction, and a one of the homological functors of §2.

5.1 For classical braid groups

In [Big04a], Bigelow introduces a general method to construct a representation of the braid group $\mathbf{B}_n$ from a representation of the braid group $\mathbf{B}_m$ for two integers m and n. It builds the so-called families of *Lawrence-Bigelow representations*, first introduced by Lawrence [Law90] in a different

[10] Note that the unlink L does not come with an orientation, so this choice is purely arbitrary; however, we showed earlier (in the construction of the gamma quotients) that the resulting count (with sign) of passes through L is independent of this choice mod 2.

way to here, namely as representations of Hecke algebras. The most famous family among them is the one known as the *Lawrence-Krammer-Bigelow* representations that Bigelow [Big01] and Krammer [Kra02] independently proved to be faithful. We review here this construction and then show that the representations constructed in §4.1 are the Lawrence-Bigelow representations.

Bigelow's construction: Introducing Bigelow's construction requires the following tools. We recall that $\mathbb{D}_n$ denotes the unit 2-disc where n distinct smaller 2-discs are removed from the interior and that $C_m(\mathbb{D}_n)$ is the configuration space of m unordered points in $\mathbb{D}_n$. We fix m distinct points $\{z_i\}_{i\in\{1,\ldots,m\}}$ in the boundary of $\mathbb{D}_n$ and the configuration $c_0 = (z_1,\ldots,z_m)$ as the basepoint of $C_m(\mathbb{D}_n)$ and assume that $\mathbb{D}^2$ is the unit disc in the complex plane centered at 0.

Let $f : C_m(\mathbb{D}_n) \to \mathbb{C}^*$ be the map defined by

$$(x_1,\ldots,x_m) \longmapsto \prod_{i\in\{1,\ldots,m\}}\prod_{j\in\{1,\ldots,n\}}(x_i - p_j)$$

and recall that $W : \pi_1(C_m(\mathbb{D}_n), c_0) \to \mathbb{Z}$ is the map induced by sending a loop γ to the winding number of $f\circ\gamma$ around 0. Finally, recall that the inclusion $\mathbb{D}_n \hookrightarrow \mathbb{D}^2$ defined by gluing discs on all the interior boundary components induces an inclusion map $i : C_m(\mathbb{D}_n) \hookrightarrow C_m(\mathbb{D}^2)$, which induces surjective homomorphism in homotopy $i_* : \pi_1(C_m(\mathbb{D}_n), c_0) \to \mathbf{B}_m$.

Bigelow's construction starts with a representation $\rho : \mathbf{B}_m = C_m(\mathbb{D}) \to GL_{\mathbb{K}}(V)$ where $\mathbb{K}$ is an integral domain and we fix q a unit in $\mathbb{K}$. The first key point of Bigelow's construction is to consider the following deformation of ρ.

Definition 5.1 Let $\rho'_{q,n} : \pi_1(C_m(\mathbb{D}_n), c_0) \to GL_{\mathbb{K}}(V)$ be the representation defined by

$$\rho'_{q,n}(g)(v) = q^{w(g)}\rho(i_*(g))(v)$$

for all $g \in \pi_1(C_m(\mathbb{D}_n), c_0)$ and all $v \in V$.

We denote by $\widetilde{C_m(\mathbb{D}_n)}$ the universal covering space of $C_m(\mathbb{D}_n)$. We consider the quotient space

$$\mathcal{L}_\rho := \left(\widetilde{C_m(\mathbb{D}_n)} \times V\right)/\pi_1(C_m(\mathbb{D}_n), c_0),$$

where the diagonal action of $\pi_1(C_m(\mathbb{D}_n), c_0)$ on $\widetilde{C_m(\mathbb{D}_n)}$ is given by deck transformations and the one on V is given by $\rho'_{q,n}$. It is the canonical bundle (also known as Borel construction) associated with the action $\rho'_{q,n}$ of $C_m(\mathbb{D}_n)$ on V.

Then, Bigelow's construction consists in the natural action of the braid group on the (ordinary) homology group $H_*(C_m(\mathbb{D}_n), \mathcal{L}_\rho)$ or the Borel-Moore homology group $H_*^{BM}(C_m(\mathbb{D}_n), \mathcal{L}_\rho)$ of the configuration space $C_m(\mathbb{D}_n)$ with local coefficients $\mathcal{L}_\rho$:

Theorem 5.2 *[Big04a, Section 2] From a representation $\rho : \mathbf{B}_m = C_m(\mathbb{D}) \to GL_{\mathbb{K}}(V)$, the braid group $\mathbf{B}_n$ acts on $H_*(C_m(\mathbb{D}_n), \mathcal{L}_\rho)$ and $H_*^{BM}(C_m(\mathbb{D}_n), \mathcal{L}_\rho)$, thus defining $\mathfrak{Big}_{k,n}(\rho) : \mathbf{B}_n \to H_k(C_m(\mathbb{D}_n), \mathcal{L}_\rho)$ and $\mathfrak{Big}^{BM}_{k,n}(\rho) : \mathbf{B}_n \to H_k^{BM}(C_m(\mathbb{D}_n), \mathcal{L}_\rho)$ for all natural numbers k and n. They are called Bigelow's representation and Borel-Moore Bigelow's representation.*

Proof. Any $f \in \mathrm{Diff}(\mathbb{D}_n)$ that acts as the identity on the boundary defines an element of $\mathrm{Diff}_{c_0}(C_m(\mathbb{D}_n))$ and then induces an automorphism f_* of $\pi_1(C_m(\mathbb{D}_n), c_0)$. Since f fixes the m points in $C_m(\mathbb{D}_n)$, we deduce that $f_*(g) = g$ for all $g \in \pi_1(C_m(\mathbb{D}_n), c_0)$. Then, there exists a unique lift $\tilde{f}_* : \mathcal{L}_\rho \to \mathcal{L}_\rho$ of f_* acting as the identity on the fiber over c_0, which depends only on the homotopy class of f: this induces the action of $\mathbf{B}_n$ on the homology groups. □

Lawrence-Bigelow representations: Using the previous notations, we assign $V = \mathbb{K} = \mathbb{Z}[A_m]$ where $A_1 = \langle q^{\pm1}\rangle$ and $A_m = \langle q^{\pm1}, t^{\pm1}\rangle$ if $m \geqslant 2$. Let $\mathfrak{X}_m : \mathbf{B}_m \to \mathbb{K}^*$ be the morphism defined by sending each Artin generator σ_i to the multiplication by t if $m \geqslant 2$ and the trivial representation if $m = 1$. The Lawrence-Bigelow representations of braid groups are the applications of Bigelow's construction to these representations:

Definition 5.3 For all natural numbers n, $\mathfrak{Big}_{m,n}(\mathfrak{X}_m)$ and $\mathfrak{Big}^{BM}_{m,n}(\mathfrak{X}_m)$ are respectively called *Lawrence-Bigelow representations* and *Borel-Moore Lawrence-Bigelow representations*.

Remark 5.4 For $m = 1$ and each natural number n, $\mathfrak{Big}_{1,n}(\mathfrak{X}_1)$ and $\mathfrak{Big}^{BM}_{1,n}(\mathfrak{X}_1)$ both are the well-known reduced Burau representation (see [KT08, Section 3.3] for more details about the associated family of representations). Also, for $m = 2$ and each natural number n, $\mathfrak{Big}_{2,n}(\mathfrak{X}_2)$ and $\mathfrak{Big}^{BM}_{2,n}(\mathfrak{X}_2)$ both are the Lawrence-Krammer-Bigelow representation [Big03; Big01] introduced to prove that braid groups are linear. In addition, using [PP02, Theorem 1.2], the tensor product with the field of fractions $\mathfrak{Big}_{2,n}(\mathfrak{X}_2)\otimes_{\mathbb{Z}}\mathbb{Q}(q,t)$ is isomorphic to the Lawrence-Krammer representation [Kra02] for all natural numbers n.

Recovering: The representations which form the functors constructed in §4.1 are actually equivalent to the Lawrence-Bigelow representations:

Proposition 5.5 *As representations of the braid group $\mathbf{B}_n$, $L_m\left(F_{\theta_m,\phi_{m,n}}\right)$ and $L^{BM}_m\left(F_{\theta,\phi}\right)_n$ are respectively equivalent to $\mathfrak{Big}_{m,n}(\mathfrak{X}_m)$ and $\mathfrak{Big}^{BM}_{m,n}(\mathfrak{X}_m)$.*

Proof. Recall that the local coefficient system $\mathcal{L}_{\mathfrak{X}_m}$ is the canonical fiber bundle

$$\left(\widetilde{C_m(\mathbb{D}_n)}\times\mathbb{Z}[A_m]\right)/\pi_1\left(C_m(\mathbb{D}_n),c_0\right)\xrightarrow{\xi}C_m(\mathbb{D}_n)$$

so that $\xi^{-1}(c)\cong\mathbb{Z}[A_m]$ for all $c\in C_m(\mathbb{D}_n)$. Denoting $C_\bullet(X)$ the singular chain complex of the topological space X, there is an isomorphism of abelian groups:

$$H_m\left(C_m(\mathbb{D}_n),\mathcal{L}_{\mathfrak{X}_m}\right)\quad\cong\quad H_m\left(C_\bullet\left(\widetilde{C_m(\mathbb{D}_n)}\right)\underset{\mathbb{Z}[\pi_1(C_m(\mathbb{D}_n),c_0)]}{\otimes}\xi^{-1}(c)\right).$$

Recall that $\phi_{1,n}$ is the composite $\gamma_2\circ\Sigma$ (where $\gamma_2:\mathbf{F}_n\to\mathbb{Z}^n$ is the abelianisation map and Σ is the sum map) and that, for $m\geqslant 2$, $\phi_{m,n}:\pi_1\left(C_m(\mathbb{D}_n),c_0\right)\to\mathbb{Z}^2$ is defined by $\gamma\mapsto(T(\gamma),W(\gamma))$, where $T(\gamma)$ counts the total number of half-twists for a curve representing γ. In any case, the morphism $\phi_{m,n}$ induces a representation

$$\pi_1\left(C_m(\mathbb{D}_n),c_0\right)\xrightarrow{\phi_{m,n}}A_m\hookrightarrow\mathrm{Aut}_{\mathbb{Z}[A_m]}\left(\mathbb{Z}[A_m]\right)$$

where the second arrow is the morphism induced by the multiplication by the elements of A_m: this is exactly the morphism $(\mathfrak{X}_m)'_q$ of Definition 5.1 induced by $\mathfrak{X}_m$. Using Shapiro's lemma, we deduce that as abelian groups:

$$H_m\left(C_m(\mathbb{D}_n),\mathcal{L}_{\mathfrak{X}_m}\right)\cong H_m\left(C_m(\mathbb{D}_n)^{\phi_{m,n}},\mathbb{Z}\right)$$

and the action of $\mathfrak{Big}_{m,n}(\mathfrak{X}_m)$ for an element σ of $\mathbf{B}_n$ on $H_*\left(C_m(\mathbb{D}_n)^{\phi_{m,n}},\mathbb{Z}\right)$ through this isomorphism is given by the unique lift of σ for the covering space $C_m(\mathbb{D}_n)^{\phi_{m,n}}$. Hence, $L_m\left(F_{\theta_m,\phi_{m,n}}\right)\cong\mathfrak{Big}_{m,n}(\mathfrak{X}_m)$ as representations of $\mathbf{B}_n$.

The proof that $L^{BM}_m\left(F_{\theta,\phi}\right)_n\cong\mathfrak{Big}^{BM}_{m,n}(\mathfrak{X}_m)$ follows repeating mutatis mutandis the previous one, using Borel-Moore homology instead of standard homology. □

Proposition 5.5 justifies the notation $\mathfrak{LB}_m$ for the functor defined by the representations $L_m\left(F_{\theta_m,\phi_{m,n}}\right)$: this functor actually encodes the Lawrence-Bigelow representations.

Remark 5.6 In [Sou19, Section 1.2], it is proved that the family of reduced Burau and Lawrence-Krammer representations define functors over the category $\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle$. More precisely, they respectively form functors $\overline{\mathfrak{Bur}}:\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle\to\mathbb{C}\left[t^{\pm1}\right]$-Mod and $\mathfrak{LK}:\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle\to\mathbb{C}\left[t^{\pm1},q^{\pm1}\right]$-Mod. By Remark 5.4, these functors are actually respectively equivalent to the functors $\mathfrak{LB}_1$ and $\mathfrak{LB}_2$ tensored (and their respective alternative using Borel-Moore homology) by $\mathbb{C}$.

5.2 For surface braid groups

The application of the general construction of homological representations to surface braid groups in §4.2.2 provides several families of representations of these groups. Actually one of them is already defined by An and Ko in [AK10], where they describe an extension of homological representations from the classical braid groups to the surface braid groups. Some of the technical key tools of their construction are reinterpreted by Bellingeri, Godelle and Guaschi in [BGG17] using metabelian quotients of the surface braid groups. We review here the An-Ko representations with this approach.

We use the framework and notations of §4.2.2 where we introduced the representation

$$L_m\left(F_{\theta_m,\phi^{\Sigma_{g,1}}_{3,m,n}}\right) : \mathbf{B}_n\left(\Sigma_{g,1}\right) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}\left(H_k\left(C_m\left(\Sigma^{(n)}_{g,1}\right)^{\phi^{\Sigma_{g,1}}_{3,m,n}},\mathbb{Z}\right)\right).$$

Furthermore, any morphism $\psi_{\mathcal{Q}_{m,n}} : \mathbf{B}_{m,n}\left(\Sigma_{g,1}\right)/\varGamma_3\left(\mathbf{B}_{m,n}\left(\Sigma_{g,1}\right)\right) \to \mathcal{Q}_{m,n}$ induces a $\mathbf{B}_n\left(\Sigma_{g,1}\right)$-$A_{3,m,n}$-bimodule structure. We denote by $\psi^{\mathcal{Q}}_n$ the action of $\mathbf{B}_n\left(\Sigma_{g,1}\right)$ on $\mathbb{Z}\left[\mathcal{Q}_{m,n}\right]$ induced by the multiplication. Since the space of the representation is naturally endowed with a $\mathbb{Z}\left[A_{3,m,n}\right]$-module structure, then the tensor product of $\psi^{\mathcal{Q}}_n$ and $L_m\left(F_{\theta_m,\phi^{\Sigma_{g,1}}_{3,m,n}}\right)$ defines a morphism

$$\mathbf{B}_n\left(\Sigma_{g,1}\right) \longrightarrow \mathrm{Aut}_{\mathbb{Z}[\mathcal{Q}_{m,n}]}\left(\mathbb{Z}\left[\mathcal{Q}_{m,n}\right] \underset{\mathbb{Z}[A_{3,m,n}]}{\otimes} H_k\left(C_m\left(\Sigma^{(n)}_{g,1}\right)^{\phi^{\Sigma_{g,1}}_{3,m,n}},\mathbb{Z}\right)\right)$$

In [AK10], the authors introduce a particular H_Σ in an had hoc way. [BGG17, Section 4] actually prove that H_Σ is a quotient of the third lower central quotient group $\mathbf{B}_{m,n}\left(\Sigma_{g,1}\right)/\varGamma_3\left(\mathbf{B}_{m,n}\left(\Sigma_{g,1}\right)\right)$. Then the An-Ko representations are $\psi^{H_\Sigma}_n \underset{\mathbb{Z}[A_{3,m,n}]}{\otimes} L_m\left(F_{\theta_m,\phi^{\Sigma_{g,1}}_{3,m,n}}\right)$.

Remark 5.7 The groups $A_{3,m,n}$ and H_Σ are defined abstractly in [AK10] in terms of group presentation to satisfy certain technical homological constraints, without any connection to the third lower central quotient. The method applied in §4.2.2 underlines the mainspring of these groups. Also the use of the third lower central quotient is a valuable tool to define the homological representations: as it is done in Proposition 4.6, it allows to straightforwardly prove that the key Assumption 2.13 is satisfied. Hence it gives an alternative to the ad hoc technical [AK10, Lemma 3.1].

5.3 For mapping class groups

Some of the homological representations for mapping class groups of §4.3 have already been introduced in a different way and studied. We review here these cases. Recall that, for g a natural number, we consider the mapping class group $\mathbf{\Gamma}_{g,1}$ of the compact surface $\Sigma_{g,1}$ and that p_0 is a basepoint on the boundary of $\Sigma_{g,1}$.

5.3.1 The Magnus representations for surfaces

The Magnus representations of mapping class groups compact of connected oriented smooth surfaces and have been a fundamental tool in combinatorial group theory for many years. They were originally defined using the Fox free differential calculus. We refer the reader to [Bir74; Sak12] for further details on this definition. However, Suzuki [Suz05] introduced an equivalent topological definition that we present in this section: this interpretation shows that Magnus representations are a particular case of homological representations introduced in §4.3.2.

Recall that γ_2 denotes the abelianisation of $\pi_1\left(\Sigma_{g,1},p_0\right)$. Let $\xi_{\gamma_2} : \Sigma^{\gamma_2}_{g,1} \to \Sigma_{g,1}$ regular covering space associated with the abelianisation and we fix a lift $p^{\gamma_2}_0 \in \xi^{-1}_{\gamma_2}\left(p_0\right)$ in $\Sigma^{\gamma_2}_{g,1}$. Since the commutator subgroup $\left[\pi_1\left(\Sigma_{g,1},p_0\right),\pi_1\left(\Sigma_{g,1},p_0\right)\right]$ is a characteristic subgroup of the fundamental group, Assumption 2.13 is satisfied. Therefore, it follows from Proposition 2.14 that there is a well-defined action of the mapping class group on the reduced homology groups on the covering $\Sigma_{g,1}^{\gamma_2}$: this is the *Magnus representation* of the mapping class group

$$\mathfrak{Mag}_{\mathbf{\Gamma}}(g) : \mathbf{\Gamma}_{g,1} \to \mathrm{Aut}\left(H_1\left(\Sigma^{\gamma_2}, \xi_{\gamma_2}^{-1}(p_0) ; \mathbb{Z}\right)\right).$$

Note that Assumption 2.16 is satisfied if and only if we restrict $\mathfrak{Mag}_{\mathbf{\Gamma}}$ to the Torelli group of the surface. This restriction defines the *Magnus representation* of the Torelli group

$$\mathfrak{Mag}_{\mathcal{I}}(g) : \mathcal{I}_{g,1} \to \mathrm{Aut}_{\mathbb{Z}[H_1(\Sigma_{g,1};\mathbb{Z})]}\left(H_1\left(\Sigma^{\gamma_2}, \xi_{\gamma_2}^{-1}(p_0) ; \mathbb{Z}\right)\right).$$

5.3.2 The Moriyama representations

In [Mor07], Moriyama studies the natural action of the mapping class group $\mathbf{\Gamma}_{g,1}$ of the surface $\Sigma_{g,1}$ on some relative homology groups of the configuration spaces of n-points on a surface $\Sigma_{g,1}$. The main result of his work is that the kernel of the action of $\mathbf{\Gamma}_{g,1}$ coincides with the kernel of the natural action on the nth lower central quotient group of the fundamental group of $\Sigma_{g,1}$. We prove here that this family of representations is a particular case of the general construction introduced in §2.

For any diffeomorphim φ of $\Sigma_{g,1}$ which fixes pointwise the boundary component, the diagonal action of φ on $\Sigma_{g,1}^{\times m}$ preserves the subsets $\mathcal{D}_m(\Sigma_{g,1})$ and $\mathcal{A}_m(\Sigma_{g,1}, p_0)$ (introduced in §4.3.1) of $\Sigma_{g,1}^{\times m}$. Then the relative homology group

$$H_m\left(\Sigma_{g,1}^{\times m}, \left(\mathcal{D}_m(\Sigma_{g,1}) \cup \mathcal{A}_m(\Sigma_{g,1}, p_0)\right) ; \mathbb{Z}\right)$$

is equipped with a $\mathbf{\Gamma}_{g,1}$-module structure induced from the diagonal action. We call this structure the mth *Moriyama* representation of $\mathbf{\Gamma}_{g,1}$ and denote it by $\mathfrak{Mor}_m(g)$. Let $\mathcal{M}_2^{+,gen}$ be the full subcategory of $\mathcal{M}_2^{+}$ of the orientable surface with no marked points. The monoidal structure $\left(\mathcal{M}_2^{+}, \natural, 0\right)$ restricts to a braided monoidal structure both on the subgroupoids $\mathcal{M}_2^{+}$. Hence the representations considered in [Mor07] define a functor $\mathfrak{Mor}_m : \mathcal{M}_2^{+,gen} \to \mathbb{Z}\text{-Mod}$.

Recall that $F_m\left(\Sigma'_{g,1}\right)$ denotes the ordered configuration space of m points on the surface $\Sigma_{g,1}\backslash\{p_0\}$. Note that $F_m\left(\Sigma'_{g,1}\right)$ is homeomorphic to the complement of the set $\left(\mathcal{D}_m(\Sigma_{g,1}) \cup \mathcal{A}_m(\Sigma_{g,1}, p_0)\right)$ in $\Sigma_{g,1}^{\times m}$. Hence, as the $\mathbf{\Gamma}_{g,1}$-module structure is induced from the diagonal action in both situations, it follows from the definition of Borel-Moore homology that we have the following $\mathbf{\Gamma}_{g,1}$-modules isomorphism

$$H_m^{BM}\left(F_m\left(\Sigma'_{g,1}\right), \mathbb{Z}\right) \cong \mathfrak{Mor}_m(g)$$

for all natural numbers m and g. Hence the representation $L_m\left(F_{\theta,\gamma_0}\right)_n$ introduced in 4.3.1 associated with the universal covering induced by $\gamma_0 = id_{\pi_1\left(F_m\left(\Sigma'_{g,1}\right), c_0\right)}$ is equivalent to the mth Moriyama representation.

5.4 The Long-Moody constructions

In [Lon94], Long and Moody consider in a very general recipee for constructing homological representations for braid groups. This method and its variants have been studied with a functorial point of view in [Sou19] and then generalised in [Sou18] for general families of groups. We first review here these construction and then gives their connections to the homological functors of §2.

Let $(\mathcal{G}, \natural, 0)$ be a strict monoidal groupoid and $(\mathcal{M}, \natural)$ be a left-module over $\mathcal{G}$. We consider a functor $\mathcal{A} : \langle\mathcal{G}, \mathcal{M}\rangle \to \mathfrak{Gr}$. We also assume that $\mathrm{Obj}(\mathcal{G})$ and $\mathrm{Obj}(\mathcal{M})$ are both isomorphic to the natural numbers $\mathbb{N}$ and that there exist two objects $1 \in \mathrm{Obj}(\mathcal{G})$ and $\hat{0} \in \mathrm{Obj}(\mathcal{G})$ so that any object X of $\mathcal{M}$ is isomorphic to the monoidal product $\underline{n} := 1^{\natural n}\natural\mathbf{0}$. We denote by G_n the isomorphism group $\mathrm{Iso}_{\langle\mathcal{G},\mathcal{M}\rangle}(\underline{n})$.

For R a commutative ring, let R-Alg be the category of unital R-algebras. For all groups G, the group rings $R[G]$ and the augmentation ideals $\mathcal{I}_{R[G]}$ respectively assemble to define the group algebra functor $R[-]: \mathrm{Gr} \to R\text{-Alg}$ and the augmentation ideal functor $\mathcal{I}_{R[-]}: \mathrm{Gr} \to R\text{-Alg}$. We respectively denote by $R[\mathcal{A}]$ and $\mathcal{I}_{\mathcal{A}}$ the composite functors $R[-] \circ \mathcal{A}$ and $\mathcal{I}_{R[-]} \circ \mathcal{A}$.

Then $R[\mathcal{A}]$ is a monoid object in $\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, R\text{-Mod})$. Hence pointwise tensor product of functors automatically induce the *tensor product functor over* $R[\mathcal{A}]$ (see [Sou18, Definition 2.1])

$$- \underset{R[\mathcal{A}]}{\otimes} - : \text{Mod-}R[\mathcal{A}] \times R[\mathcal{A}]\text{-Mod} \to \mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, R\text{-Mod})$$

where $R[\mathcal{A}]\text{-Mod}$ and $\text{Mod-}R[\mathcal{A}]$ respectively denote the categories of left and right modules $R[\mathcal{A}]$. Moreover $\mathcal{I}_{\mathcal{A}}$ is a right $R[\mathcal{A}]$-module and therefore defines a functor $\mathcal{I}_{\mathcal{A}} \otimes_{R[\mathcal{A}]} - : R[\mathcal{A}]\text{-Mod} \to \mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, R\text{-Mod})$.

Recall from [MM94, Chapter 1, Section 5] that the *Grothendieck construction over* $\mathcal{A}$ denoted by $\int \mathcal{A}$, is the category of pairs $(\cdot_c, c)$, where $c \in Obj(\mathcal{M})$ and $\cdot_c$ is a group a group (viewed as a category with one object); a morphism in $\mathrm{Hom}_{\int \mathcal{A}}((\cdot_c, c), (\cdot_{c'}, c'))$ is a pair (α, f), where $f \in \mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M} \rangle}(c, c')$ and $\alpha \in \cdot_{c'}$. For $(\alpha, f) : (\cdot_c, c) \to (\cdot_{c'}, c')$ and $(\beta, g) : (\cdot_{c'}, c') \to (\cdot_{c''}, c'')$, the composition is defined by $(\beta, g) \circ (\alpha, f) = (\beta \circ F(g)(\alpha), g \circ f)$. Note that this definition is dual to the one used in §2.2 for a contravariant functor. There is a canonical projection functor $\int \mathcal{A} \to \langle \mathcal{G}, \mathcal{M} \rangle$, given by sending an object $(\cdot_c, c)$ to c. A section $\mathfrak{s}_{\mathcal{A}} : \langle \mathcal{G}, \mathcal{M} \rangle = \int 0 \hookrightarrow \int \mathcal{A}$ to this projection functor is induced by the trivial natural transformation $0 \to \mathcal{A}$ (where 0 is the trivial functor $\langle \mathcal{G}, \mathcal{M} \rangle \to \mathrm{Gr}$). We recall from [Sou18, Proposition 2.4] that the precomposition by $\mathfrak{s}_{\mathcal{A}}$ defines a natural equivalence $\mathbf{Fct}\left(\int \mathcal{A}, R\text{-Mod}\right) \cong R[\mathcal{A}]\text{-Mod}$.

Finally the key to define a functor which describes a Long-Moody construction is to consider a functor $\varsigma : \int \mathcal{A} \to \langle \mathcal{G}, \mathcal{M} \rangle$, so that the following diagram is commutative:

$$\begin{tikzcd}
\langle \mathcal{G}, \mathcal{M} \rangle \arrow[r, hook, "\mathfrak{s}_{\mathcal{A}}"] \arrow[dr, "1\natural-"'] & \int \mathcal{A} \arrow[d, "\varsigma"] \\
& \langle \mathcal{G}, \mathcal{M} \rangle ,
\end{tikzcd}$$

where $1\natural- : \langle \mathcal{G}, \mathcal{M} \rangle \to \langle \mathcal{G}, \mathcal{M} \rangle$ denotes the functor defined by $(1\natural-)(X) = 1\natural X$ for all $X \in Obj(\mathcal{M})$ and $(1\natural-)(\phi) = id_1 \natural \phi$ for all morphisms ϕ of $\langle \mathcal{G}, \mathcal{M} \rangle$.

Definition 5.8 [Sou18, Section 2] The Long-Moody functor $\mathbf{LM}_{\mathcal{A},\varsigma}$ associated with the functors $\mathcal{A}$ and ς is the

$$\mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, R\text{-Mod}) \xrightarrow{\mathfrak{s}_{\mathcal{A}}^{*} \circ \varsigma^{*}} R[\mathcal{A}]\text{-Mod} \xrightarrow{\mathcal{I}_{\mathcal{A}} \underset{R[\mathcal{A}]}{\otimes} -} \mathbf{Fct}(\langle \mathcal{G}, \mathcal{M} \rangle, R\text{-Mod}),$$

where $\mathfrak{s}_{\mathcal{A}}^{*}$ and ς^{*} respectively denote the precomposition functors $\mathfrak{s}_{\mathcal{A}}$ and ς.

The appropriate data $\mathcal{A}$ and ς naturally arise for many families of groups in connection with topology. We refer the reader to [Sou18, Section 3] for the introduction of non-trivial and natural such functors for the families of braid groups$\{\mathbf{B}_n\}_{n \in \mathbb{N}}$, surface braid groups $\{\mathbf{B}_n(\Sigma_{g,1})\}_{n \in \mathbb{N}}$, mapping class groups $\{\mathbf{\Gamma}_{n,1}\}_{n \in \mathbb{N}}$ and $\left\{\mathbf{\Gamma}_{g,1}^{n}\right\}_{n \in \mathbb{N}}$.

Recovering: We assume that there exists a topological category $\langle \mathcal{G}, \mathcal{M} \rangle^{\mathrm{t}}$ so that $\pi_0 \langle \mathcal{G}, \mathcal{M} \rangle^{\mathrm{t}} = \langle \mathcal{G}, \mathcal{M} \rangle$ (which is the case for all the aforementioned examples). To recover the Long-Moody constructions encoded by the functors, we restrict to the full subcategory $\langle \mathcal{G}, \mathcal{M} \rangle$ on the objects isomorphic to $\underline{n}$ for some natural number n. We denote it by $\cdot_{\underline{n}}$ since its skeleton is the group G_n viewed as a category, and by $\mathcal{A}_n$ the restriction of $\mathcal{A}$ to $\cdot_{\underline{n}}$. We pick a topological lift of $\cdot_{\underline{n}}$ in $\langle \mathcal{G}, \mathcal{M} \rangle^{\mathrm{t}}$, denoted by X_n. Recall from §2 the category $\widetilde{\mathrm{Cov}}_Q$ of based path-connected spaces X with a surjection $\pi_1(X) \twoheadrightarrow Q$. Hence $\widetilde{\mathrm{Cov}}_{\pi_1(X_n)}$ encodes the universal cover of $\pi_1(X_n)$. We denote by $U : \widetilde{\mathrm{Cov}}_{\pi_1(X_n)} \to \mathrm{Top}_*$ the functor which forgets the surjections. We require the following mild property:

Assumption 5.9 There exists a functor $\widehat{\mathcal{A}}_n : X_n \to \widetilde{\mathrm{Cov}}_Q$ which image is a topological space with a boundary component and so that the following diagram is commutative

$$\begin{tikzcd}
X_n \arrow[r, "\widehat{\mathcal{A}}_n"] \arrow[d, "\pi_0"'] & \widetilde{\mathrm{Cov}}_{\pi_1(X_n)} \arrow[r, "U"] & \mathrm{Top}_* \arrow[d, "\pi_1"] \\
\cdot_{\underline{n}} \arrow[rr, "\mathcal{A}_n"] & & \mathrm{Gr}.
\end{tikzcd}$$

In all the examples of [Sou18], the various functors $\mathcal{A}$ for braid groups and mapping class groups are induced by a geomtrical construction and a fortiori Assumption 5.9 is satisfied. We abuse the notation $\mathbf{LM}_{\mathcal{A},\varsigma}$ to denote the restriction of a Long-Moody functor to the category $\mathbf{Fct}\left(\cdot_{\underline{n}}, R\text{-Mod}\right)$. Then:

Proposition 5.10 *Let M be a R-module and $\rho : G_{n+1} \to GL_R(M)$ be a representation. Then there is an equivalence of representations $\mathbf{LM}_{\mathcal{A},\varsigma}(\rho) \cong L_1^r\left(\widehat{\mathcal{A}}_n; M\right)$.*

Proof. For the topological space, we denote by $C_\bullet(X,*)$ the singular chain complex of X relative to a point p on the boundary and by $\widetilde{\widehat{\mathcal{A}}_n(X_n)}$ the universal cover of $\widehat{\mathcal{A}}_n(X_n)$. Note that the functor ς induces the $\left(R\left[\pi_1\left(\widetilde{\mathcal{A}}_n(X_n)\right)\right], R\right)$-bimodule structure of M. There is an isomorphism of abelian groups:

$$H_1\left(C_\bullet\left(\widetilde{\widehat{\mathcal{A}}_n(X_n)}, \widetilde{p}\right) \underset{\pi_1\left(\widetilde{\mathcal{A}}_n(X_n)\right)}{\otimes} M\right) \cong \mathcal{I}_{\mathcal{A}_n} \underset{\pi_1\left(\widetilde{\mathcal{A}}_n(X_n)\right)}{\otimes} M.$$

The result then follows directly from the fact that the actions of both representations on the left hand sides are induced by $\mathcal{A}_n$ and the ones on M are defined by ρ. $\square$

6 Free generating sets for representations

Most of the homological representations described in the previous sections are defined via actions on the homology of some configuration spaces of points of a topological space. Instead of using ordinary homology, we may instead use Borel-Moore homology of these configuration spaces. These alternative homology groups have useful properties (see §6.1) which allow one to compute free generating sets of the representations under consideration. We formulate this property in §6.1 and discuss applications in §6.2.

6.1 A general lemma for Borel-Moore homology

The following lemma gives a criterion for an inclusion of spaces to induce isomorphisms on the (possibly twisted) Borel-Moore homology of their unordered configuration spaces. It abstracts the essential ideas of Lemma 3.1 of [Big04b] and Lemma 3.3 of [AK10].

Lemma 6.1 *Let M be a compact metric space, and let*

$$\varnothing \neq P \subseteq \Gamma \subseteq M$$

be subspaces, where P is finite and Γ is closed. Assume also that M and Γ are locally compact. Then the following conditions ensure that, for all $k \in \mathbb{N}$, the inclusion

$$C_k(\Gamma \smallsetminus P) \hookrightarrow C_k(M \smallsetminus P)$$

induces isomorphisms on Borel-Moore homology with any twisted coefficients.

- *There is a retraction $\pi\colon M \to \Gamma$.*
- *There is a continuous map $h\colon [0,\infty) \to \mathrm{Emb}_\pi^{\mathrm{ne}}(M,M)$, where $\mathrm{Emb}_\pi^{\mathrm{ne}}(M,M)$ is the space, with the compact-open topology, of self-embeddings of M that commute with the projection π and are* ***non-expanding****, i.e. do not increase distances between points, such that:*
  - *$h_0 = \mathrm{id}$,*
  - *h_t fixes Γ pointwise,*
  - *for all s and t, we have $h_s(h_t(M)) \subseteq h_t(M)$,*
  - *for all t, the subset $h_t(M)$ contains a neighbourhood of P,*
  - *for all sufficiently small $\epsilon > 0$ there is a value of t such that the fibres of the projection*

$$\pi_t = \pi|_{h_t(M)} \colon h_t(M) \longrightarrow \Gamma$$

have diameter smaller than ϵ. Moreover, its restriction to

$$\pi_t^\circ = \pi|_{h_t(M)\cap\pi^{-1}(\Gamma\smallsetminus P)} \colon h_t(M) \cap \pi^{-1}(\Gamma \smallsetminus P) \longrightarrow \Gamma \smallsetminus P$$

is non-expanding and admits a homotopy $\mathrm{incl}\circ\pi_t^\circ \simeq \mathrm{id}_{h_t(M)\cap\pi^{-1}(\Gamma\smallsetminus P)}$ through non-expanding maps that preserve the fibres of π_t°.

Sketch of proof. Consider the following commutative diagram of inclusions:

$$\begin{tikzcd}
\bar{C} \arrow[r, hook] & \hat{C}^t \arrow[r, hook] & C^t \arrow[r, hook] & C \\
\bar{C} \smallsetminus A_{\epsilon,t} \arrow[r, hook] \arrow[u, hook] \arrow[ur, phantom, "\text{(B)}"] & \hat{C}^t \smallsetminus A_{\epsilon,t} \arrow[r, hook] \arrow[u, hook] \arrow[ur, phantom, "\text{(C)}"] & C^t \smallsetminus A_{\epsilon,t} \arrow[r, hook] \arrow[u, hook] \arrow[ur, phantom, "\text{(A)}"] & C \smallsetminus A_{\epsilon,t} \arrow[u, hook]
\end{tikzcd} \tag{6.1}$$

where:

- $C = C_k(M \smallsetminus P)$,
- $\bar{C} = C_k(\Gamma \smallsetminus P)$,
- $C^t = C_k(h_t(M) \smallsetminus P)$,
- $\hat{C}^t \subseteq C^t$ is the subset of those configurations $\{x_1, \ldots, x_k\}$ such that $\pi(x_i) \notin P$ for all i and $\pi(x_i) \neq \pi(x_j)$ for all $i \neq j$,
- $A_{\epsilon,t} \subseteq C$ is the subset of those $\{x_1, \ldots, x_k\} \in C$ such that $d(x_i, x_j) \geqslant \epsilon$ for all $i \neq j$ and, for each i, we have either $x_i \notin h_t(M)$ or $d(\pi(x_i), P) \geqslant \epsilon$.

It suffices to show that the inclusion of pairs

$$(\bar{C}, \bar{C} \smallsetminus A_{\epsilon,t}) \longhookrightarrow (C, C \smallsetminus A_{\epsilon,t})$$

induces isomorphisms on twisted relative homology, for all sufficiently small $\epsilon > 0$ and sufficiently large t (where the lower bound on "sufficiently large t" is permitted to depend on ϵ), since the conditions imply that the map on twisted Borel-Moore homology induced by $\bar{C} \hookrightarrow C$ is the inverse limit of these maps. One sees this as follows:

(A) The horizontal inclusions in square (A) of (6.1) are homotopy equivalences, for all t and ϵ. This uses the homotopy given by h.

(B) For given ϵ, we may choose t sufficiently large such that the horizontal inclusions in square (B) of (6.1) are also homotopy equivalences. This uses the homotopy from $\mathrm{incl} \circ \pi_t^\circ$ to the identity assumed in the last property of the hypotheses.

(C) For given ϵ, we may choose t sufficiently large such that square (C) of (6.1) is *excisive*. The non-trivial thing to check for this is that $C^t \cap A_{\epsilon,t} \subseteq \hat{C}^t$, which is ensured by our assumption about the diameter of the fibres of π_t. □

Remark 6.2 If M is a connected, compact surface with one boundary-component and P is a finite subset of its interior, we may take Γ to be an embedded, connected graph in the interior of M, with $P = \{p_1, \ldots, p_n\}$ as its vertices, with $n-1$ edges between p_i and p_{i+1} for $i \in \{1, \ldots, n-1\}$ and with $1 - \chi(M)$ edges from p_1 to itself that generate $H_1(M)$. We thus see that the twisted Borel-Moore homology of unordered configuration spaces on the punctured surface $M \smallsetminus P$ is isomorphic to the

Borel-Moore homology of unordered configuration spaces on a disjoint union of $n - \chi(M)$ open intervals, which is free and concentrated in degree k, if we are considering the configuration space of k unordered points. Its dimension is:

$$\binom{n+k-\chi(M)-1}{k}.$$

Thus we recover Lemma 3.1 of [Big04b] and Lemma 3.3 of [AK10].[11] One may also deduce similar results if some of the points of P lie on the boundary of the surface M, again taking Γ to be an embedded, connected graph. See also §6.2, where some of these cases are discussed in more detail, considering also the induced actions of the braid group resp. mapping class group. One can also consider higher-dimensional applications of Lemma 6.1. Let $W_1 = (\mathbb{S}^n \times \mathbb{S}^n) \smallsetminus \mathrm{int}(\mathbb{D}^{2n})$ and take $M = W_g = W_1 \natural \cdots \natural W_1$ the g-fold boundary connected sum, for $g \geqslant 1$, and P a finite subset of its interior. We may then take Γ to be an embedded CW-complex whose 0-cells are $P = \{p_1, \ldots, p_n\}$ and which has $n-1$ one-cells joining p_i to p_{i+1} for $i \in \{1, \ldots, n-1\}$ and $2g$ distinct n-cells, each attached trivially to p_1. We then deduce that the twisted Borel-Moore homology of unordered configuration spaces on the punctured manifold $W_g \smallsetminus P$ is isomorphic to the twisted Borel-Moore homology of unordered configuration spaces on a disjoint union of $n-1$ open intervals together with $2g$ open n-discs.

6.2 Applications

The key result of Lemma 6.1 allows one to prove further properties for the Borel-Moore version of the Lawrence-Bigelow representations and the Moriyama representations. More precisely, we gain an explicit description of the spaces of representations and prove that these families of representations form functors over appropriate source categories.

6.2.1 Action the homological functors on morphisms

The isomorphism of Lemma 6.1 introduces a convenient free generating set for Borel-Moore homology of configuration spaces. In particular, this gives a useful description and additional properties of the Lawrence-Bigelow functors (see §4.1). This new description is a key point to prove the polynomiality results of §8.1.

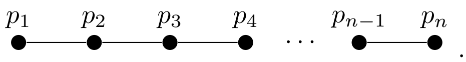
We fix two natural numbers $n \geqslant 0$ and $m \geqslant 1$. Let $\mathcal{P}_m(n)$ be the set of partitions of m into n numbers:

$$\mathcal{P}_m(n) = \left\{ (\omega_1, \ldots, \omega_n) \mid \omega_i \in \mathbb{N} \text{ and } \sum_{1 \leqslant i \leqslant n} \omega_i = m \right\} \text{ if } n \geqslant 1 \text{ and } \mathcal{P}_m(0) = \emptyset.$$

Considering the mth Lawrence-Bigelow functor using Borel-Moore homology $\mathfrak{LB}_m^{BM}$, we denote by (p_i, p_{i+1}) the open interval joining the punctures p_i and p_{i+1} for all $i \in \{1, \ldots, n-1\}$. The disjoint union

$$\mathbb{I}_n := \coprod_{1 \leqslant i \leqslant n-1} (p_i, p_{i+1}) \text{ if } n \geqslant 1 \text{ and } \mathbb{I}_0 = \mathbb{I}_1 = \emptyset$$

is the convenient subset of the punctured disc $\mathbb{D}_n$ to apply Lemma 6.1. We represent this subset by the following picture:



Then the configuration space $C_m(\mathbb{I}_n)$ is homeomorphic to a disjoint union of

$$\binom{m+n-2}{m} = \mathrm{Card}\left(\mathcal{P}_m(n-1)\right)$$

[11] There is a small typo in the statement of Lemma 3.1 of [Big04b]: the top line of the binomial coefficient should be $n+m-2$ rather than $n+m-1$.

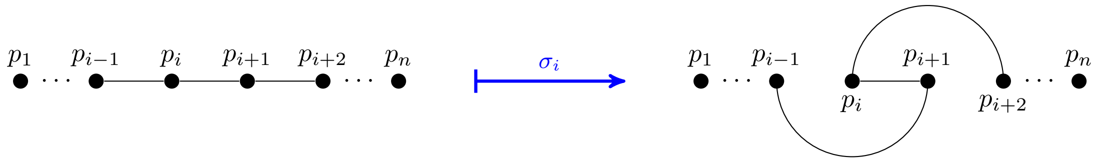
open m-balls which are parameterised by $(n-1)$-tuples $(\omega_1,\ldots,\omega_{n-1})$ of natural numbers so that the i-th interval (p_i,p_{i+1}) contains ω_i points from the configuration and $\omega_1+\omega_2+\cdots+\omega_{n-1}=m$. We deduce that we have an isomorphism of $\mathbb{Z}[A_m]$-modules

$$\mathfrak{LB}_m^{BM}(n)\cong\bigoplus_{\omega\in\mathcal{P}_m(n-1)}\mathbb{Z}[A_m]_\omega \tag{6.2}$$

if $n\geqslant 1$ and $\mathfrak{LB}_m^{BM}(0)=0$.

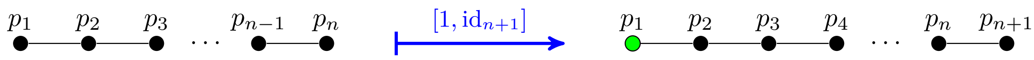
Moreover, the restriction to these subsets allows to fully understand the action of the morphisms of $\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle$ on the configuration spaces given by the Lawrence-Bigelow functors. Recall that the morphisms of the category $\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle$ are composite of automorphisms and some injections: it is thus enough to consider the action of the generators of the braid groups and the injection $[1,id_{1+n}]$ to fully describe the Lawrence-Bigelow functors on morphisms.

First, for each the Artin generator σ_i with $i\in\{1,\ldots,n-1\}$, the action of σ_i on each interval (p_i,p_{i+1}) is defined by the classical action of the braid group (seen as a mapping class group) on the fundamental group of the n-punctured disc. Hence this action is described for the whole subset $\mathbb{I}_n$ by the following picture.



Let $\theta_{m,n}(\sigma_i)$ be the induced homeomorphism of the configuration space $C_m(\mathbb{I}_n)$. Then $\mathfrak{LB}_m^{BM}(\sigma_i)$ is the morphism on Borel-Moore homology induced by the unique lift of this homeomorphism. Therefore the action of $\mathfrak{LB}_m^{BM}(\sigma_i)$ on each Borel-Moore homology class can be represented by the previous picture of the action of σ_i on the set $\mathbb{I}_n$ where each interval is labelled with the number of configurations points which sit inside.

On the other hand, the morphism $\mathfrak{LB}_m^{BM}([1,id_{n+1}])$ is the induced morphism on Borel-Moore homology of the unique lift $e_{m,n}^{\phi}$ of the configuration space map $e_{m,n}:C_m(\mathbb{D}_n)\to C_m(\mathbb{D}_{1+n})$ defined by adding a puncture on the left to the n-punctured surface $\mathbb{D}_n$. By Lemma 6.1, this is equivalent to the morphism on Borel-Moore homology induced by the embedding of $\mathbb{I}_n$ into $\mathbb{I}_{1+n}$ as the $n-1$ last intervals, i.e. sending the interval (p_i,p_{i+1}) of $\mathbb{I}_n$ to the interval (p_{i+1},p_{i+2}) of $\mathbb{I}_{n+1}$ for each $i\in\{1,\ldots,n-1\}$.



At the level of the sets of partitions, the map $e_{m,n}$ induces the injective map $\mathcal{P}_m(e_{m,n}):\mathcal{P}_m(n-1)\hookrightarrow\mathcal{P}_m(n)$ sending $(\omega_1,\ldots,\omega_{n-1})$ to $(0,\omega_1,\ldots,\omega_{n-1})$ for $n\geqslant 2$, and the trivial set maps $\mathcal{P}_m(e_{m,0}):\emptyset\to\emptyset$ and $\mathcal{P}_m(e_{m,1}):\emptyset\to\mathcal{P}_m(1)=\{(m)\}$. These induce injective morphisms (which is trivial for $n=0$)

$$\bigoplus_{\omega\in\mathcal{P}_m(n-1)}\mathbb{Z}[A_m]_\omega\hookrightarrow\bigoplus_{\omega\in\mathcal{P}_m(n)}\mathbb{Z}[A_m]_\omega$$

that we denote by $\iota_{\mathcal{P}_m(n)\setminus\mathcal{P}_m(n-1)}\oplus id_{\mathcal{P}_m(n-1)}$ for simplicity. Hence $\mathfrak{LB}_m^{BM}([1,id_{1+n}])$ is equivalent to this injection and its action on each Borel-Moore homology class can be represented by the above picture of the embedding of $\mathbb{I}_n$ in $\mathbb{I}_{n+1}$ where each interval is labelled with the number of configurations points which sit inside (0 being automatically the one of the interval (p_1,p_2) of the picture on the right-hand side).

6.2.2 Recovering of some results of [Mor07]:

Recall from §5.3.2 the representation $\mathfrak{Mor}_m(g)$ with fixed $g\geqslant 1$ of the mapping class group $\boldsymbol{\Gamma}_{g,1}$ introduced by Moriyama [Mor07] is isomorphic to an homological representation of §4.3.1 which representation space is $H_m^{BM}\left(F_m\left(\Sigma'_{g,1}\right),\mathbb{Z}\right)$, and defines a functor $\mathfrak{Mor}_m : \mathcal{M}_2^{+,gen} \to \mathbb{Z}\text{-Mod}$ with $m \geqslant 1$ a natural number. Lemma 6.1 then implies additional properties on the space of this representation. For $n \geqslant 0$ a natural number, let $\mathcal{Q}_m(n)$ be the set of arrangements of m into $2n$ numbers:

$$\mathcal{Q}_m(n) = \mathfrak{S}_m \times \mathcal{P}_m(2n).$$

Note that $m! \cdot \begin{pmatrix} m+n-1 \\ m \end{pmatrix} = \mathrm{Card}\left(\mathcal{Q}_m(n)\right)$.

Lemma 6.3 *For all natural numbers m and k, there is an abelian group isomorphism*

$$H_k^{BM}\left(F_m\left(\Sigma'_{g,1}\right),\mathbb{Z}\right) \cong \begin{cases} 0 & \text{if } k \neq m; \\ \bigoplus\limits_{\omega\in\mathcal{Q}_m(g)} \mathbb{Z}_\omega = \mathbb{Z}^{\oplus\frac{(2g+m-1)!}{(2g-1)!}} & \text{if } k = m. \end{cases}$$

Proof. First of all, we recall that for X a topological space homeomorphic to the complement of a closed subcomplex S in a finite CW-complex Y, the Borel-Moore homology group $H_*^{BM}(X,\mathcal{L})$ with local coefficient $\mathcal{L}$ is equivalent to the relative homology $H_*^{BM}(Y,S;\mathcal{L})$. The quotient map $\Sigma_{g,1}^{\times m} \to \Sigma_{g,1}^{\times m}/\mathfrak{S}_m$ defines a regular cover of $\Sigma_{g,1}^{\times m}/\mathfrak{S}_m$ path connected space. Hence it induces the locally trivial fibration $\mathfrak{S}_m \to \Sigma_{g,1}^{\times m} \to \Sigma_{g,1}^{\times m}/\mathfrak{S}_m$, together with the compatible locally trivial fibration of subspaces $\mathfrak{S}_m \to \mathcal{D}_m(\Sigma_{g,1}) \cup \mathcal{A}_m(\Sigma_{g,1},p_0) \to \mathcal{D}_m(\Sigma_{g,1}) \cup \mathcal{A}_m(\Sigma_{g,1},p_0)/\mathfrak{S}_m$. Then the associated Serre spectral sequence for relative homology (see for instance [DK01, Theorem 9.33]) has only one non-trivial row. Hence, for all natural numbers $p \geqslant 0$, we obtain the following isomorphism:

$$H_p^{BM}\left(F_m\left(\Sigma'_{g,1}\right),\mathbb{Z}\right) \cong H_p^{BM}\left(C_m\left(\Sigma'_{g,1}\right);\mathbb{Z}\left[\mathfrak{S}_m\right]\right).$$

Let W_{2g} be the wedge of $2g$ copies of the oriented circle $\mathbb{S}^1$ with the base point p_0 which define a free generating set of the fundamental group of $\Sigma_{g,1}$. Then applying Lemma 6.1 to the subset $W_{2g} \setminus \{p_0\}$ of $\Sigma'_{g,1}$ (which is homeomorphic to the disjoint union of $2g$ open intervals), we have:

$$H_m\left(C_m\left(\Sigma'_{g,1}\right),\mathbb{Z}\right) \cong \underset{\omega\in\mathcal{P}_m(n)}{\oplus} \mathbb{Z}\left[\mathfrak{S}_m\right]_\omega \cong \bigoplus_{\omega\in\mathcal{Q}_m(g)} \mathbb{Z}_\omega.$$ □

This result recovers [Mor07, Proposition 3.3, 4.2 and 4.3] using the $\mathbf{\Gamma}_{g,1}$-modules isomorphism $H_m^{BM}\left(F_m\left(\Sigma'_{g,1}\right),\mathbb{Z}\right) \cong \mathfrak{Mor}_m(g)$, the techniques used in [Mor07] being different from the ones presented here. This isomorphism is crucial to prove the polynomiality results of §8.2.

Remark 6.4 For each natural number m and g, the homology groups $H_m^{BM}\left(F_m\left(\Sigma'_{g,1}\right)/\mathfrak{S}_m,\mathbb{Z}\right)$ is isomorphic to $\mathrm{Sym}^n\left(H_1\left(\Sigma_{g,1},\mathbb{Z}\right)\right)$ where Sym^n denotes the nth symmetric tensor power.

Finally, Lemma 6.3 allows us to prove that the functor $\mathfrak{Mor}_m$ lifts to the category $\langle\mathcal{M}_2^{+,gen},\mathcal{M}_2^{+,gen}\rangle$:

Lemma 6.5 *The functor $\mathfrak{Mor}_m$ extends to $\mathfrak{Mor}_m : \langle\mathcal{M}_2^{+,gen},\mathcal{M}_2^{+,gen}\rangle \to \mathbb{Z}\text{-Mod}$ by assigning for all $\Sigma_{g,1},\Sigma_{g',1} \in Obj\left(\mathcal{M}_2^{+,gen}\right)$:*

$$\mathfrak{Mor}_m\left(\left[\Sigma_{g',1}, id_{\Sigma_{g',1}\natural\Sigma_{g,1}}\right]\right) = \iota_{\mathbb{Z}^{\oplus(\mathcal{Q}_m(g'+g)\setminus\mathcal{Q}_m(g))}} \oplus id_{\mathbb{Z}^{\oplus\mathcal{Q}_m(g)}}.$$

Proof. Relation (3.1) of Lemma 3.5 is trivially satisfied from the definition of $\mathfrak{Mor}_m\left(\left[\Sigma_{g',1}, id_{\Sigma_{g',1}\natural\Sigma_{g,1}}\right]\right)$. We consider $\varphi \in \mathbf{\Gamma}_{g,1}$ and $\varphi' \in \mathbf{\Gamma}_{g',1}$. It follows from Lemma 6.3 that $\mathfrak{Mor}_m(\varphi)$ is an automorphism of $\mathbb{Z}^{\oplus\mathcal{Q}_m(g)}$ in $\mathbb{Z}^{\oplus\left(\mathcal{Q}_m(g'+g)\setminus\mathcal{Q}_m(g)\right)} \oplus \mathbb{Z}^{\oplus\mathcal{Q}_m(g)} \cong \mathbb{Z}^{\oplus\mathcal{Q}_m(g'+g)}$ and that $\mathfrak{Mor}_m\left(id_{\Sigma_{g',1}}\natural\varphi\right) = id_{\mathbb{Z}^{\oplus(\mathcal{Q}_m(g'+g)\setminus\mathcal{Q}_m(g))}} \oplus \mathfrak{Mor}_m(\varphi)$. Hence

$$\mathfrak{Mor}_m\left(\left[\Sigma_{g',1}, id_{\Sigma_{g',1}\natural\Sigma_{g,1}}\right]\right) \circ \mathfrak{Mor}_m(\varphi) = \mathfrak{Mor}_m\left(id_{\Sigma_{g',1}}\natural\varphi\right) \circ \mathfrak{Mor}_m\left(\left[\Sigma_{g',1}, id_{\Sigma_{g',1}\natural\Sigma_{g,1}}\right]\right).$$

Also, it follows from Lemma 6.3 that $\mathfrak{Mor}_m\left(\varphi'\natural id_{\Sigma_{g,1}}\right)$ is an automorphism of $\mathbb{Z}^{\oplus\mathcal{Q}_m(g')}\hookrightarrow\mathbb{Z}^{\oplus\left(\mathcal{Q}_m(g'+g)\setminus\mathcal{Q}_m(g)\right)}$ in $\mathbb{Z}^{\oplus\left(\mathcal{Q}_m(g'+g)\setminus\mathcal{Q}_m(g)\right)}\oplus\mathbb{Z}^{\oplus\mathcal{Q}_m(g)}=\mathbb{Z}^{\oplus\mathcal{Q}_m(g'+g)}$. In particular, it follows from the definition of $\iota_{\mathbb{Z}^{\oplus(\mathcal{Q}_m(g'+g)\setminus\mathcal{Q}_m(g))}}$ that

$$\mathfrak{Mor}_m\left(\varphi'\natural id_{\Sigma_{g,1}}\right)\circ\mathfrak{Mor}_m\left(\left[\Sigma_{g',1},id_{\Sigma_{g',1}\natural\Sigma_{g,1}}\right]\right)=\mathfrak{Mor}_m\left(\left[\Sigma_{g',1},id_{\Sigma_{g',1}\natural\Sigma_{g,1}}\right]\right).$$

Hence, we deduce that

$$\mathfrak{Mor}_m\left(\varphi'\natural\varphi\right)\circ\mathfrak{Mor}_m\left(\left[\Sigma_{g',1},id_{\Sigma_{g',1}\natural\Sigma_{g,1}}\right]\right)=\mathfrak{Mor}_m\left(\left[\Sigma_{g',1},id_{\Sigma_{g',1}\natural\Sigma_{g,1}}\right]\right)\circ\mathfrak{Mor}_m\left(\varphi\right).$$

Hence Relation (3.2) of Lemma 3.5 is satisfied, which implies the desired result. □

7 Notions of polynomiality

In this section, we review the notions of *(very) strong* and *weak polynomial* functors with respect to the framework of the present paper. In [DV19, Section 1], Djament and Vespa introduce these notions in the context of a functor category $\mathbf{Fct}(\mathcal{M},\mathcal{A})$, where $\mathcal{M}$ is a for symmetric monoidal (small) categories where the unit is an initial object and $\mathcal{A}$ is a Grothendieck category. They define *strong* polynomial functors to extend the classical concept of polynomial functors, which were first defined using cross effects by Eilenberg and Mac Lane in [EM54] for functors on module categories. In particular, one reason for interesting in strong polynomial functors is their homological stability properties studied in [RW17]. Furthermore, the notion of weak polynomial functor is first introduced in [DV19, Section 1] and happens to be more appropriate to study the stable behavior for objects of the category $\mathbf{Fct}(\mathcal{M},\mathcal{A})$ (see [DV19, Section 5] and [Dja17]) and give a new tool for classifying the families of representations of families of groups (see §7.2 and §8.3.2). Then, the notions of strong and weak polynomial functors are extended in [Sou18, Section 4] to the larger setting where $\mathcal{M}$ is a full subcategory of a pre-braided monoidal category where the unit is an initial object. We also refer to [Pal17] for a comparison of the various instances of the notions of twisted coefficient system and polynomial functor.

This section thus recollects the definitions and properties of [Sou19, Section 3] and [Sou18, Section 4] to the present slightly larger framework, the various proofs being mutatis mutandis generalisations of these previous works.

For the remainder of §7, we fix a left-module $(\mathcal{M},\natural)$ over strict monoidal groupoid $(\mathcal{G},\natural,0)$, where we assume that $\mathcal{M}$ is a groupoid, $(\mathcal{G},\natural,0)$ has no zero divisors and $\mathrm{Aut}_{\mathcal{G}}(0)=\{id_0\}$. **We also fix a Grothendieck category $\mathcal{A}$.** We recall that therefore the functor category $\mathbf{Fct}(\mathcal{M},\mathcal{A})$ is a Grothendieck category.

7.1 Strong and very strong polynomial functors

Let X be an object of $\mathcal{G}$. Let $\tau_X:\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})\to\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ be the functor defined by $\tau_X(F)=F(n\natural-)$. It is called the *translation* functor. Let $i_X:Id\to\tau_X$ be the natural transformation of $\mathbf{Fct}(\mathcal{M},\mathcal{A})$ induced by precomposition with the morphisms $\{[X,id_{X\natural A}]:0\natural Y\to X\natural A\}_{A\in\mathrm{Ob}(\mathcal{M})}$. We define $\delta_X=\mathrm{coker}(i_X)$, called the *difference* functor, and $\kappa_X=\ker(i_X)$, called the *evanescence* functor. We recall the following elementary properties of the translation, evanescence and difference functors:

Proposition 7.1 *The translation functor τ_X is exact and induces the following exact sequence of endofunctors of* $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$*:*

$$0\longrightarrow\kappa_X\xrightarrow{\Omega_X}Id\xrightarrow{i_X}\tau_X\xrightarrow{\Delta_X}\delta_X\longrightarrow0.\tag{7.1}$$

Moreover, for a short exact sequence $0\longrightarrow F\longrightarrow G\longrightarrow H\longrightarrow0$ in the category $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$*, there is a natural exact sequence in the category* $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$*:*

$$0\longrightarrow\kappa_X(F)\longrightarrow\kappa_X(G)\longrightarrow\kappa_X(H)\longrightarrow\delta_X(F)\longrightarrow\delta_X(G)\longrightarrow\delta_X(H)\longrightarrow0.\tag{7.2}$$

In addition, for Y another object of $\mathcal{G}$, the functors τ_X and τ_Y commute up to natural isomorphism and they commute with limits and colimits; the difference functors δ_X and δ_Y commute up to natural isomorphism and they commute with colimits; the functors κ_X and κ_Y commute up to natural isomorphism and they commute with limits; the functor τ_X commute with the functors δ_X and κ_X up to natural isomorphism.

Notation 7.2 We respectively denote the iterations $\underbrace{\tau_X \cdots \tau_X \tau_X}_{k \text{ times}}$ and $\underbrace{\delta_X \cdots \delta_X \delta_X}_{k \text{ times}}$ by τ_X^k and δ_X^k.

Then, we can define the notions of strong and very strong polynomial functors using Proposition 7.1. Namely:

Definition 7.3 We recursively define on $d \in \mathbb{N}$ the categories of strong polynomial functors $\mathcal{P}ol_d^{str}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ and very strong polynomial functors $\mathcal{VP}ol_d\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$, both of degree less than or equal to d, to be the full subcategories of $\mathbf{Fct}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ as follows:

1. If $d < 0$, $\mathcal{P}ol_d^{strong}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right) = \mathcal{VP}ol_d\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right) = \{0\}$;
2. if $d \geqslant 0$, the objects of $\mathcal{P}ol_d^{str}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ are the functors F such that the functor $\delta_X(F)$ is an object of $\mathcal{P}ol_{d-1}^{strong}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$; the objects of $\mathcal{VP}ol_d\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ are the objects F of $\mathcal{P}ol_d\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ such that $\kappa_1(F) = 0$ and the functor $\delta_1(F)$ is an object of $\mathcal{VP}ol_{d-1}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$.

For an object F of $\mathbf{Fct}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ which is strong (respectively very strong) polynomial of degree less than or equal to $n \in \mathbb{N}$, the smallest natural number $d \leqslant n$ for which F is an object of $\mathcal{P}ol_d^{str}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ (respectively $\mathcal{VP}ol_d\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$) is called the strong (respectively very strong) degree of F.

Very strong polynomial functors turn out to be very useful for homological stability problems: for instance Randal-Williams and Wahl [RW17] prove homological stability results for several families of groups, including surface braid groups, loop braid groups and mapping class groups of surfaces, given by very strong polynomial functors. We develop this point in §8.3.1.

Finally, we recall useful properties of the categories associated with strong and very strong polynomial functors.

Proposition 7.4 *Let d be a natural number. The category $\mathcal{P}ol_d^{str}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ is closed under the translation functor, under quotient, under extension and under colimits. The category $\mathcal{VP}ol_d\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ is closed under the translation functor, under normal subobjects and under extension.*

Let F be an object of $\mathbf{Fct}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$. Then, F is an object of $\mathcal{P}ol_0^{strong}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ if and only if it is the quotient of a constant object of $\mathbf{Fct}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$. Moreover, F is an object of $\mathcal{VP}ol_0\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ if and only if it is isomorphic to a constant object of $\mathbf{Fct}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$.

Finally, we assume that there exists a finite set E of objects of the category $\langle \mathcal{G}, \mathcal{M} \rangle$ such that for all objects m of $\langle \mathcal{G}, \mathcal{M} \rangle$, m is isomorphic to a finite monoidal product of objects of E. Then, an object F of $\mathbf{Fct}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ belongs to $\mathcal{P}ol_d^{str}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ (respectively to $\mathcal{VP}ol_d\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$) if and only if $\delta_e(F)$ is an object of $\mathcal{P}ol_{d-1}^{strong}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ (respectively $\kappa_e(F) = 0$ and $\delta_e(F)$ is an object of $\mathcal{VP}ol_{n-1}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$), for all objects e of E.

7.2 Weak polynomial functors

We deal here with the concept of weak polynomial functor, introduced in [DV19, Section 2] for the category $\mathbf{Fct}(S, A)$ where S is a symmetric monoidal category where the unit is an initial object and A is a Grothendieck category, and extended in [Sou18, Section 4] when S is pre-braided monoidal. We review the definition and properties of weak polynomial functors, which extends verbatim to the present larger setting from those of [Sou18, Section 4].

Let F be an object of $\mathbf{Fct}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$. The subfunctor $\sum\limits_{n \in \mathrm{Ob}(\beta)} \kappa_X F$ of F is denoted by $\kappa(F)$. Let $\mathcal{S}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ be the full subcategory of $\mathbf{Fct}\left(\langle \mathcal{G}, \mathcal{M} \rangle, \mathcal{A}\right)$ of the objects F such that $\kappa(F) = F$. We have the following fundamental properties:

Proposition 7.5 *The category $\mathcal{S}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ is a thick subcategory of $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ and it is closed under colimits.*

The thickness property of Proposition 7.5 ensures that we can consider:

Definition 7.6 Let $\mathbf{St}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ be the quotient category

$$\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})/\mathcal{S}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A}).$$

The canonical functor associated with this quotient is denoted by $\pi_{\langle\mathcal{G},\mathcal{M}\rangle}:\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})\to\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})/\mathcal{S}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$, it is exact, essentially surjective and commutes with all colimits (see [Gab62, Chapter 3]). The right adjoint functor of $\pi_{\langle\mathcal{G},\mathcal{M}\rangle}$ is denoted by

$$s_{\langle\mathcal{G},\mathcal{M}\rangle}:\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})/\mathcal{S}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})\to\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$$

and called the section functor (see [Gab62, Section 3.1]).

The following proposition recalls the induced translation and difference functors on the category $\mathbf{St}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$.

Proposition 7.7 *Let x be an object of X. The translation functor τ_X and the difference functor δ_X of $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ respectively induce an exact endofunctor of $\mathbf{St}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ which commute with colimits, respectively again called the translation functor τ_X and the difference functor δ_X. In addition:*

1. *The following relations hold: $\delta_X\circ\pi_{\langle\mathcal{G},\mathcal{M}\rangle}=\pi_{\langle\mathcal{G},\mathcal{M}\rangle}\circ\delta_X$ and $\tau_X\circ\pi_{\langle\mathcal{G},\mathcal{M}\rangle}=\pi_{\langle\mathcal{G},\mathcal{M}\rangle}\circ\tau_X$.*
2. *The exact sequence ((7.1)) induces a short exact sequence of endofunctors of $\mathbf{St}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$:*
$$0\longrightarrow Id\xrightarrow{i_X}\tau_X\xrightarrow{\Delta_n}\delta_X\longrightarrow 0. \tag{7.3}$$
3. *For another object X', the endofunctors δ_X, $\delta_{X'}$, τ_X and $\tau_{X'}$ of $\mathbf{St}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ pairwise commute up to natural isomorphism.*

Definition 7.8 We recursively define on $d\in\mathbb{N}$ the category of polynomial functors of degree less than or equal to d, denoted by $\mathcal{P}ol_d(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$, to be the full subcategory of $\mathbf{St}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ as follows:

1. If $d<0$, $\mathcal{P}ol_d(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})=\{0\}$;
2. if $d\geqslant 0$, the objects of $\mathcal{P}ol_d(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ are the functors F such that the functor $\delta_1(F)$ is an object of $\mathcal{P}ol_{d-1}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$.

For an object F of $\mathbf{St}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ which is polynomial of degree less than or equal to $d\in\mathbb{N}$, the smallest natural number $n\leqslant d$ for which F is an object of $\mathcal{P}ol_d(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ is called the degree of F. An object F of $\mathbf{Fct}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ is weak polynomial of degree at most d if its image $\pi_{\langle\mathcal{G},\mathcal{M}\rangle}(F)$ is an object of $\mathcal{P}ol_d(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$. The degree of polynomiality of $\pi_{\langle\mathcal{G},\mathcal{M}\rangle}(F)$ is called the (weak) degree of F.

Finally, let us recall some useful properties of the categories of weak polynomial functors.

Proposition 7.9 *Let d be a natural number. As a subcategory of $\mathbf{St}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$, the category $\mathcal{P}ol_d(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ is thick and closed under limits and colimits. Furthermore, there is an equivalence of categories $\mathcal{A}\simeq\mathcal{P}ol_0(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$.*

Finally, we assume that there exists a finite set E of objects of the category $\langle\mathcal{G},\mathcal{M}\rangle$ such that for all objects m of $\langle\mathcal{G},\mathcal{M}\rangle$, m is isomorphic to a finite monoidal product of objects of E. Let F be an object of $\mathbf{St}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$. Then, the functor F is an object of $\mathcal{P}ol_d(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ if and only if the functor $\delta_e(F)$ is an object of $\mathcal{P}ol_{d-1}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ for all objects e of E.

Quotient categories: A fundamental reason for the notion of of weak polynomial functors to be introduced in [DV19] is that, contrary to the category $\mathcal{P}ol_d^{strong}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$, the category $\mathcal{P}ol_d(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ is *localizing*. This allows to define the quotient categories

$$\mathcal{P}ol_{d+1}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})/\mathcal{P}ol_d(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A}).$$

A refined description of the category $\mathcal{P}ol_d^{strong}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ is out of reach generally speaking even for small d. On the contrary, understanding the quotient categories $\mathcal{P}ol_{d+1}(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})/\mathcal{P}ol_d(\langle\mathcal{G},\mathcal{M}\rangle,\mathcal{A})$ is more attainable: for example, when $\mathcal{G}=\mathcal{M}=FB$ (the category of finite sets and bijections) [DV19, Proposition 5.9] gives a general equivalence of these quotients in terms of module categories.

Hence these quotients provide a new classifying tool to study polynomial functors and therefore representations of families of groups: this will be illustrated in §8.3.2.

8 Polynomiality of some homological functors

In this section, we study the (very) strong and weak polynomiality of some homological functors. Indeed, we prove that the Lawrence-Bigelow functors introduced in §4.1 and the Moriyama functors defined in §4.3.2 are both very strong and weak polynomial. The use of Borel-Moore homology and of the free generating sets studied in §6 are fundamental tools to establish these properties. Consequently, we prove homological stability results for braid groups with coefficients in the Lawrence-Bigelow representations and for mapping class groups with coefficients in the Moriyama representations. Also we gain a better understanding of the quotient categories for weak polynomial functors associated to these families of groups.

8.1 The Lawrence-Bigelow functors

The construction of §4.1 and identifications of §5.1 introduced the mth Borel-Moore Lawrence-Bigelow functor

$$\mathfrak{LB}_m^{BM}:\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle\to\mathbb{Z}[A_m]\text{-Mod}$$

for each natural numbers $m\geqslant 1$, where $A_1=\mathbb{Z}$ and $A_m=\mathbb{Z}^2$ if $m\geqslant 2$. This functor encodes the Lawrence-Bigelow representations (using Borel-Moore homology) of the braid groups $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$. **For the remainder of §8.1, we fix the natural numbers** $m\geqslant 1$**.** The aim of this section is to prove the following polynomiality results for this functor.

Theorem 8.1 *The Lawrence-Bigelow functor $\mathfrak{LB}_1^{BM}:\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle\to\mathbb{Z}[A_1]$-Mod is strong polynomial of degree 2 and weak polynomial of degree 1. If $m\geqslant 2$ then the Lawrence-Bigelow functor $\mathfrak{LB}_m^{BM}$: $\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle\to\mathbb{Z}[A_m]$-Mod is both very strong and weak polynomial of degree m.*

Remark 8.2 Recall from Remark 5.6 that $\mathfrak{LB}_1^{BM}\otimes\mathbb{C}$ is the reduced Burau functor $\overline{\mathfrak{Bur}}$ and that $\mathfrak{LB}_2^{BM}\otimes\mathbb{C}$ is the Lawrence-Krammer functor $\mathfrak{LK}$. Hence the result of Theorem 8.1 for strong polynomiality recovers those of [Sou19, Section 3.3], i.e. that $\overline{\mathfrak{Bur}}$ is strong polynomial of degree 2 and Lawrence-Krammer is very strong polynomial of degree 1.

The monoidal product $\natural:\boldsymbol{\beta}\times\boldsymbol{\beta}\to\boldsymbol{\beta}$ being defined by the usual addition for the objects, any object of braid groupoid $\boldsymbol{\beta}$ is isomorphic to a finite sum of 1. Hence by Propositions 7.4 and 7.9, it is enough to study the natural transformation i_1 to prove the polynomiality results. Our goal is to study the kernel and cokernel of the natural transformation $i_1\mathfrak{LB}_m^{BM}$.

Following Proposition 7.1, the natural transformation $i_1\mathfrak{LB}_m^{BM}$ is defined for all natural numbers n by the morphism

$$\left(i_1\mathfrak{LB}_m^{BM}\right)_n=\mathfrak{LB}_m^{BM}(\iota_1\natural id_n)=\mathfrak{LB}_m^{BM}([1,id_{n+1}]),$$

and fits into the following exact sequence in the category $\mathbf{Fct}(\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle,\mathbb{Z}[A_m]\text{-Mod})$:

$$0\longrightarrow\kappa_1\mathfrak{LB}_m^{BM}\longrightarrow\mathfrak{LB}_m^{BM}\xrightarrow{i_1\mathfrak{LB}_m^{BM}}\tau_1\mathfrak{LB}_m^{BM}\xrightarrow{\Delta_1\mathfrak{LB}_m^{BM}}\delta_1\mathfrak{LB}_m^{BM}\longrightarrow 0\,. \tag{8.1}$$

Following §6.2, the use of Borel-Moore homology allows to describe the morphism $\mathfrak{LB}_m^{BM}([1, id_{n+1}])$ in terms of sets of partitions of copies of the group ring $\mathbb{Z}[A_m]$. More precisely, recall from §6.2.1 that for all natural numbers $n \geqslant 1$

$$\mathfrak{LB}_m^{BM}(n) \cong \bigoplus_{\omega \in \mathcal{P}_m(n-1)} \mathbb{Z}[A_m]_\omega \text{ and } \mathfrak{LB}_m^{BM}(0) = 0$$

and therefore that the morphism $\mathfrak{LB}_m^{BM}([1, id_{n+1}])$ is equivalent to the injection of $\mathbb{Z}[A_m]$-modules

$$\iota_{\mathcal{P}_m(n) \setminus \mathcal{P}_m(n-1)} \oplus id_{\mathcal{P}_m(n-1)} : \bigoplus_{\omega \in \mathcal{P}_m(n-1)} \mathbb{Z}[A_m]_\omega \hookrightarrow \bigoplus_{\omega \in \mathcal{P}_m(n)} \mathbb{Z}[A_m]_\omega .$$

induced by the injective map $\mathcal{P}_m(n-1) \hookrightarrow \mathcal{P}_m(n)$ sending $(\omega_1, \ldots, \omega_{n-1})$ to $(0, \omega_1, \ldots, \omega_{n-1})$ if $n \geqslant 2$ and the trivial morphism if $n = 0$ and $n = 1$.

There are then two possible recursive ways to prove Theorem 8.1: either we can work on the dimensions of the difference functor with respect to the sets of partitions as detailed in §8.1.1, or we can establish a key original relation between the difference functor of the mth Lawrence-Bigelow functor and a translation of the $(m-1)$th one as shown in §8.1.2.

8.1.1 First proof: dimensional argument on the sets of partitions

This first proof consists in studying the partitions subsets which index the number of copies of $\mathbb{Z}[A_m]$ in the direct sum of the successive difference functors of $\mathfrak{LB}_m^{BM}$. Let k be a natural number. For each natural number n, we consider the set

$$\mathcal{P}_m^{\delta^k}(n) := \{(\omega_1, \ldots, \omega_{k+n-1}) \in \mathcal{P}_m(k+n-1) \mid \forall i \in \{1, \ldots, k\}, 1 \leqslant \omega_i \leqslant m\},$$

with the convention that it is the empty set if $n + k \leqslant 1$. Then there is a canonical injection $\mathcal{P}_m^{\delta^k}(n) \hookrightarrow \mathcal{P}_m^{\delta^k}(n+1)$ defined by sending $(\omega_1, \ldots, \omega_{k+n-1})$ to $(0, \omega_1, \ldots, \omega_{k+n-1})$ which induces a canonical bijection

$$\mathcal{P}_m^{\delta^{k+1}}(n) \cong \mathcal{P}_m^{\delta^k}(n+1) \setminus \mathcal{P}_m^{\delta^k}(n) . \tag{8.2}$$

This injective set map also defines an injective $\mathbb{Z}[A_m]$-module morphism

$$\mathcal{P}_{m,n}^k : \bigoplus_{\omega \in \mathcal{P}_m^{\delta^k}(n)} \mathbb{Z}[A_m]_\omega \hookrightarrow \bigoplus_{\omega \in \mathcal{P}_m^{\delta^k}(n+1)} \mathbb{Z}[A_m]_\omega$$

for $n \geqslant 1$ and the trivial morphism $\mathcal{P}_{m,0}^k$. These sets describe the successive difference functors of the mth Lawrence-Bigelow functor:

Proposition 8.3 *For all natural numbers n, there is an isomorphism of $\mathbb{Z}[A_m]$-modules*

$$\delta_1^k \mathfrak{LB}_m^{BM}(n) \cong \bigoplus_{\omega \in \mathcal{P}_m^{\delta^k}(n)} \mathbb{Z}[A_m]_\omega$$

and the morphism $\left(i_1\left(\delta_1^{k-1}\mathfrak{LB}_m^{BM}\right)\right)_n$ is equivalent to the injection $\mathcal{P}_{m,n}^{k-1}$ for all natural numbers $1 \leqslant k \leqslant m+1$ if $m \geqslant 2$ and for $k = 1$ if $m = 1$.

Proof. We proceed by induction on k. For $k = 1$, we already know from §6.2.1 that $\left(i_1 \mathfrak{LB}_m^{BM}\right)_n$ is equivalent to the injection $\mathcal{P}_{m,n}^0$ and it follows from the universal property of the cokernel that

$$\delta_1 \mathfrak{LB}_m^{BM}(n) \cong \bigoplus_{\omega \in \mathcal{P}_m^{\delta^1}(n)} \mathbb{Z}[A_m]_\omega .$$

Now we assume that the results of Proposition 8.3 are true for some fixed $k \geqslant 1$. Recall from Proposition 7.1 that, by the definition of the difference functor, the morphism $\left(i_1\left(\delta_1^k \mathfrak{LB}_m^{BM}\right)\right)_n$ is canonically induced by the morphism $\tau_1\delta_1^{k-1}\mathfrak{LB}_m^{BM}([1,id_{1+n}])$. The inductive hypothesis gives the equivalence between $\delta_1^{k-1}\mathfrak{LB}_m^{BM}([1,id_{1+n}])$ and $\mathcal{P}_{m,n}^{k-1}$. Using the bijection (8.2), the image of $\mathcal{P}_{m,n}^{k-1}$ induced by the canonical surjections $\mathcal{P}_m^{\delta^{k-1}}(n+1)\twoheadrightarrow\mathcal{P}_m^{\delta^k}(n)$ and $\mathcal{P}_m^{\delta^{k-1}}(n+2)\twoheadrightarrow\mathcal{P}_m^{\delta^k}(n+1)$ is the morphism $\mathcal{P}_{m,n}^k$: hence $\left(i_1\left(\delta_1^k\mathfrak{LB}_m^{BM}\right)\right)_n$ is equivalent to the injection $\mathcal{P}_{m,n}^k$.

Then the following diagram is commutative:

$$\begin{tikzcd}
\kappa_1\delta_1^k\mathfrak{LB}_m^{BM}(n) \arrow[r, hook] & \delta_1^k\mathfrak{LB}_m^{BM}(n) \arrow[r, "\left(i_1\left(\delta_1^k\mathfrak{LB}_m^{BM}\right)\right)_n"] \arrow[d, "\sim"] & \tau_1\delta_1^k\mathfrak{LB}_m^{BM}(n) \arrow[r, two heads] \arrow[d, "\sim"] & \delta_1^{k+1}\mathfrak{LB}_m^{BM}(n). \\
& \bigoplus\limits_{\omega\in\mathcal{P}_m^{\delta^k}(n)}\mathbb{Z}[A_m]_\omega \arrow[r, hook, "\mathcal{P}_{m,n}^k"] & \bigoplus\limits_{\omega\in\mathcal{P}_m^{\delta^k}(n+1)}\mathbb{Z}[A_m]_\omega &
\end{tikzcd}$$

We deduce that
$$\delta_1^{k+1}\mathfrak{LB}_m^{BM}(n)\cong\bigoplus_{\omega\in\mathcal{P}_m^{\delta^{k+1}}(n)}\mathbb{Z}[A_m]_\omega$$
using the universal property of the cokernel and the bijection (8.2). $\square$

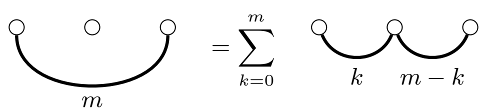
We are now ready to prove Theorem 8.1. First we deduce from Proposition 8.3 that $\delta_1\mathfrak{LB}_1^{BM}(n)=\mathbb{Z}[A_1]$ for all $n\geqslant 1$ and $\delta_1\mathfrak{LB}_1^{BM}(0)=0$: a fortiori $\delta_1^2\mathfrak{LB}_1^{BM}(n)=0$ for all $n\geqslant 1$ and $\delta_1^2\mathfrak{LB}_1^{BM}(0)=\mathbb{Z}[A_1]$. Hence: $\delta_1^3\mathfrak{LB}_1^{BM}=0$ and therefore $\mathfrak{LB}_1^{BM}$ is strong polynomial of degree two; $\delta_1^2\mathfrak{LB}_1^{BM}$ is a stably null object of the category $\mathbf{Fct}(\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle,\mathbb{Z}[A_1]\text{-Mod})$ and a fortiori $\mathfrak{LB}_1^{BM}$ is weak polynomial of degree one since $\delta_1\circ\pi_{\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle}=\pi_{\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle}\circ\delta_1$ by Proposition 7.7.

Let us now fix $m\geqslant 2$. We deduce from Proposition 8.3 that the morphisms $\left(i_1\left(\delta_1^m\mathfrak{LB}_m^{BM}\right)\right)_n$ is equivalent to the injection $\mathcal{P}_{m,n}^m$ for all natural numbers n. Note that the sets $\mathcal{P}_m^{\delta^m}(n)$ and $\mathcal{P}_m^{\delta^m}(n+1)$ are actually isomorphic: the injection $\mathcal{P}_{m,n}^m$ is therefore a bijection. Hence the natural transformation $i_1\left(\delta_1^m\mathfrak{LB}_m^{BM}\right)$ is a natural equivalence between non-null objects. Again the fact that $\delta_1\circ\pi_{\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle}=\pi_{\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle}\circ\delta_1$ implies that the functor $\mathfrak{LB}_m^{BM}:\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle\to\mathbb{Z}[A_m]\text{-Mod}$ is weak polynomial of degree m. Also, for all natural numbers k such that $1\leqslant k\leqslant m$, the evanescence functor $\kappa_1\delta_1^k\mathfrak{LB}_m^{BM}$ is null since $\left(i_1\left(\delta_1^k\mathfrak{LB}_m^{BM}\right)\right)_n$ is an injection by Proposition 8.3. Then the functor $\delta_1^k\mathfrak{LB}_m^{BM}$ is very strong polynomial polynomial of degree $m-k$ which ends the proof.

8.1.2 Second proof: a key relation between Lawrence-Bigelow functors

The key of this second method is the result of Theorem 8.5: the difference functor of the mth Lawrence-Bigelow functor is isomorphic a translation of the $(m-1)$th one. The proof ot Theorem 8.1 is then a straightforward induction on the degree of polynomiality. In addition this isomorphism gives an original relation between two different Lawrence-Bigelow functors.

Using the relations on Borel-Moore homology classes detailed in §9, we have



(8.3)

For all natural numbers $n\geqslant 1$, we define $\mathfrak{p}_m(n):\mathcal{P}_m^{\delta^1}(n)\cong\mathcal{P}_{m-1}(n)$ to be the set isomorphism
$$(\omega_1,\ldots,\omega_n)\longmapsto(\omega_1-1,\ldots,\omega_n)$$
and $(\mathfrak{p}_m)_0$ to be the trivial set map.

Notation 8.4 For consistency of the exposition we denote by $\mathfrak{LB}_0^{BM}:\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle\to\mathbb{Z}\left[A_1\right]$-Mod the subobject of the constant functor at $\mathbb{Z}\left[A_1\right]$ such that $\mathfrak{LB}_0^{BM}(0)=\mathfrak{LB}_0^{BM}(1)=0$. For convenience, we abuse the notation and write $\tau_1\mathfrak{LB}_i^{BM}$ for the functor $\tau_1\mathfrak{LB}_i^{BM}\underset{\mathbb{Z}[A_1]}{\otimes}\mathbb{Z}\left[A_2\right]$ for $i=0,1$.

It follows from §6.2.1 that the morphisms $\left\{\mathfrak{p}_m(n)\right\}_{n\in\mathbb{N}}$ induce $\mathbb{Z}\left[A_m\right]$-modules isomorphisms

$$\left\{\widehat{\mathfrak{p}}_m(n):\delta_1\mathfrak{LB}_m^{BM}(n)\overset{\sim}{\to}\tau_1\mathfrak{LB}_{m-1}^{BM}(n)\right\}_{n\in\mathbb{N}}.$$

induced by forgetting a point of the configuration space in the first interval of $\mathbb{I}_{1+n}$. These isomorphisms allow to uncover the following key relation between the difference functor of the mth Lawrence-Bigelow functor and the $(m-1)$th Lawrence-Bigelow functor.

Theorem 8.5 *The isomorphisms* $\left\{\widehat{\mathfrak{p}}_m(n)\right\}_{n\in\mathbb{N}}$ *define an isomorphism* $\widehat{\mathfrak{p}}_m:\delta_1\mathfrak{LB}_m^{BM}\overset{\sim}{\to}\tau_1\mathfrak{LB}_{m-1}^{BM}$ *in the category* $\mathbf{Fct}\left(\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle,\mathbb{Z}\left[A_m\right]\text{-Mod}\right)$.

Proof. First, we prove that the morphisms $\left\{\widehat{\mathfrak{p}}_m(n)\right\}_{n\in\mathbb{N}}$ define an isomorphism in the category $\mathbf{Fct}\left(\boldsymbol{\beta},\mathbb{Z}\left[A_m\right]\text{-Mod}\right)$. Recall from Proposition 7.1 that the definition of the difference functor $\delta_1\mathfrak{LB}_m^{BM}$ on the morphisms of $\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle$ of is formally induced by $\tau_1\mathfrak{LB}_m^{BM}$. We fix a natural number $n\geqslant 2$ (the proof being trivial for $n=0$ and $n=1$). We consider the Artin generator σ_i of $\mathbf{B}_n$ for some $i\in\{1,\ldots,n-1\}$. Using the work of §6.2.1, the morphism $\tau_1\mathfrak{LB}_m^{BM}(\sigma_i)$ (and a fortiori $\delta_1\mathfrak{LB}_m^{BM}(\sigma_i)$) is the morphism in Borel-Moore homology induced by the action of σ_{i+1} on the union of intervals $\mathbb{I}_{1+n}$. Hence it is enough to prove that forgetting a point of the configuration in the first interval (p_1,p_2) before or after the twisting by σ_{i+1} is exactly the same operation at the level of the Borel-Moore homology classes. Actually, a quick investigation shows that the only non-trivial case is for $i=1$: the property is indeed clear if $i\geqslant 2$ since σ_{i+1} acts trivially on (p_1,p_2) in this case. The pictures of Figure 8.1 represent the action of σ_2 by the Lawrence-Bigelow functor on a Borel-Moore homology class on the covering of $C_m\left(\mathbb{I}_{1+n}\right)$ for those on the left-hand side and of $C_{m-1}\left(\mathbb{I}_{1+n}\right)$ for those on the right-hand side. The red cross in the first interval denotes an additional configuration point of the first interval in $C_m\left(\mathbb{D}_{n+1}\right)$ compared to $C_{m-1}\left(\mathbb{D}_{n+1}\right)$: the morphism $\widehat{\mathfrak{p}}_m(n)$ corresponds to forgetting this additional point. It follows from Relation (8.3) that the diagram of Figure 8.1 is commutative (note that one of the partitions (corresponding to $k=0$) misses since we are in δ_1). Hence $\widehat{\mathfrak{p}}_m(n)\circ\delta_1\mathfrak{LB}_m^{BM}(\sigma_i)=\tau_1\mathcal{LB}_{m-1}^{BM}(\sigma_i)\circ\widehat{\mathfrak{p}}_m(n)$ for all $i\in\{1,\ldots,n-1\}$.

Now we prove that $\widehat{\mathfrak{p}}_m$ is a natural transformation in $\mathbf{Fct}\left(\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle,\mathbb{Z}\left[A_m\right]\text{-Mod}\right)$ by using Lemma 3.6. We fix a natural number $n\geqslant 1$ (the proof being trivial for $n=0$) and recall that

$$\tau_1\mathfrak{LB}_m^{BM}\left(\left[1,id_{n+1}\right]\right)=\mathfrak{LB}_m^{BM}\left(\sigma_1^{-1}\right)\circ\mathfrak{LB}_m^{BM}\left(\left[1,id_{n+2}\right]\right).$$

Recall from §6.2.1 that $\mathfrak{LB}_m^{BM}\left(\left[1,id_{n+2}\right]\right)$ is the morphism on Borel-Moore homology induced by the embedding of $\mathbb{I}_{1+n}$ into $\mathbb{I}_{2+n}$ as the $n-1$ last intervals: this amounts to sending the interval (p_i,p_{i+1}) of $\mathbb{I}_{n+1}$ to the interval (p_{i+1},p_{i+2}) of $\mathbb{I}_{n+2}$ for each $i\in\{1,\ldots,n\}$. Therefore there is no configuration point on the first interval (p_1,p_2) of the image of $\mathfrak{LB}_m^{BM}\left(\left[1,id_{n+2}\right]\right)$. Moreover, $\mathfrak{LB}_m^{BM}\left(\sigma_1^{-1}\right)$ is the morphism in Borel-Moore homology induced by the action of σ_2 on $\mathbb{I}_{2+n}$. The pictures of Figure 8.2 represent the composite of these two morphisms by the Lawrence-Bigelow functor on a Borel-Moore homology class on the covering of $C_m\left(\mathbb{I}_{1+n}\right)$ for those on the left-hand side and of $C_{m-1}\left(\mathbb{I}_{1+n}\right)$ for those on the right-hand side. Again the red cross denotes an additional configuration point, the morphism $\widehat{\mathfrak{p}}_m(n)$ corresponding to forgetting it. Then we have to prove that the composite $\mathfrak{LB}_m^{BM}\left(\sigma_1^{-1}\right)\circ\mathfrak{LB}_m^{BM}\left(\left[1,id_{n+2}\right]\right)$ commutes with the operation of forgetting a point of the configuration space in the interval (p_1,p_2) at the level of the Borel-Moore homology classes. Again Relation (8.3) gives the commutativity of the diagram of Figure 8.2 (note again that one of the partitions (corresponding to $k=0$) misses since we are in δ_1). Hence a straightforward induction on the natural number $k\geqslant 1$ gives that

$$\widehat{\mathfrak{p}}_m(n)\circ\delta_1\mathfrak{LB}_m^{BM}\left(\left[k,id_{n+1}\right]\right)=\tau_1\mathcal{LB}_{m-1}^{BM}\left(\left[k,id_{n+1}\right]\right)\circ\widehat{\mathfrak{p}}_m(n).$$

Hence Relation (3.3) of Lemma 3.6 is satisfied for all natural numbers n: this ends the proof. $\square$

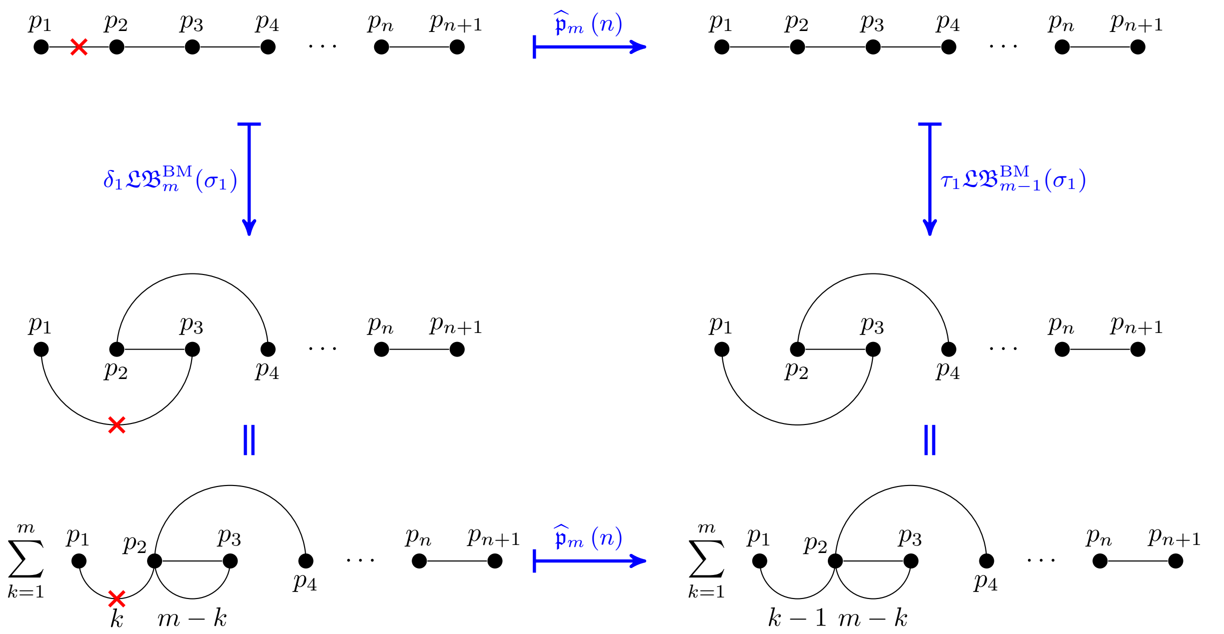


Figure 8.1

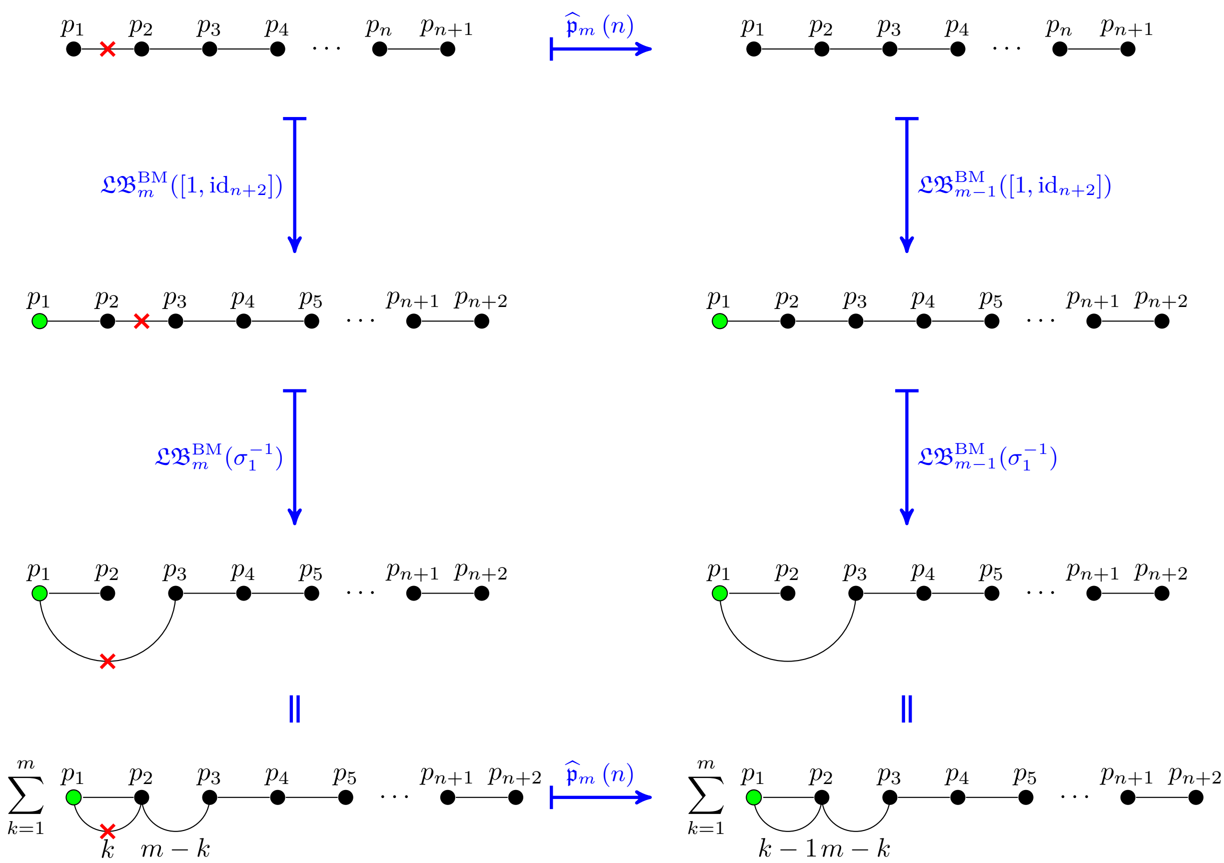


Figure 8.2

Let us now prove Theorem 8.1. Using the commutation property of δ_1 and τ_1 of Proposition 7.1, we deduce from Theorem 8.5 that for all natural numbers $0 \leqslant k \leqslant m$

$$\delta_1^k \left(\mathfrak{LB}_m^{BM} \right) \cong \tau_1^k \left(\mathfrak{LB}_{m-k}^{BM} \right). \tag{8.4}$$

For $m = 1$, note that $\tau_1 \mathfrak{LB}_0^{BM}$ is the subobject of the constant functor at $\mathbb{Z}[A_1]$ such that $\tau_1 \mathfrak{LB}_0^{BM}(0) = 0$. Hence $\mathfrak{LB}_1^{BM}$ is weak polynomial of degree 1 and $\delta_1^2 \mathfrak{LB}_1^{BM}$ is the functor which unique non-null value is $\delta_1^2 \mathfrak{LB}_1^{BM}(0) = \mathbb{Z}[A_1]$. A fortiori $\mathfrak{LB}_1^{BM}$ is strong polynomial of degree 2.

For $m \geqslant 2$, first note that for all natural numbers $l \geqslant 2$, $\tau_1^2 \mathfrak{LB}_0^{BM} = \tau_1^l \mathfrak{LB}_0^{BM}$ is a constant functor. It thus follows from the isomorphism (8.4) with $k = m$ that $\mathfrak{LB}_m^{BM}$ is both strong and weak polynomial of degree m.

From the definition of the morphisms $\mathfrak{LB}_m^{BM}([1, id_{n+1}])$ for all natural numbers n, we know that $\kappa_1 \left(\left(\mathfrak{LB}_m^{BM} \right) \right) = 0$. Using the isomorphism (8.4), the commutation property of κ_1 and τ_1 of Proposition 7.1 implies that $\kappa_1 \left(\delta_1^k \left(\mathfrak{LB}_m^{BM} \right) \right) = 0$ for all natural numbers $1 \leqslant k \leqslant m$: this proves that $\mathfrak{LB}_m^{BM}$ is very strong polynomial of degree m.

8.2 The Moriyama functors

We recall that the construction of §4.3.2, identification of §5.3.2 and properties of §6.2.2 prove that the representations of the mapping class groups considered in [Mor07] define the mth Moriyama functor

$$\mathfrak{Mor}_m : \langle \mathcal{M}_2^{+,gen}, \mathcal{M}_2^{+,gen} \rangle \to \mathbb{Z}\text{-Mod}.$$

for each natural numbers $m \geqslant 1$. **For the remainder of §8.2, we fix the natural numbers $m \geqslant 1$.** We prove in this section that this functor satisfies polynomial properties.

Let k be a natural number. For each natural number g, we consider the set

$$\mathcal{Q}_m^{\delta^k}(g) := \mathfrak{S}_m \times \{(\omega_1, \ldots, \omega_{2k+2g-1}) \in \mathcal{P}_m(2k+2g-1) \mid \forall i \in \{1, \ldots, k\}, (\omega_{2i-1}, \omega_{2i}) \neq (0,0)\},$$

with the convention that it is the empty set if $2k + 2g \leqslant 1$. Then there is a canonical injection $\mathcal{Q}_m^{\delta^k}(g) \hookrightarrow \mathcal{Q}_m^{\delta^k}(g+1)$ defined by sending $(\omega_1, \omega_2, \ldots, \omega_{2k+2g-1})$ to $(0, 0, \omega_1, \omega_2, \ldots, \omega_{k+n-1})$. It induces a canonical bijection

$$\mathcal{Q}_m^{\delta^{k+1}}(g) \cong \mathcal{Q}_m^{\delta^k}(g+1) \setminus \mathcal{Q}_m^{\delta^k}(g). \tag{8.5}$$

This injective set map also defines an injective $\mathbb{Z}$-module morphism

$$\mathcal{Q}_{m,g}^k : \bigoplus_{\omega \in \mathcal{Q}_m^{\delta^k}(g)} \mathbb{Z}_\omega \hookrightarrow \bigoplus_{\omega \in \mathcal{Q}_m^{\delta^k}(g+1)} \mathbb{Z}_\omega$$

for $g \geqslant 1$ and the trivial morphism $\mathcal{Q}_{m,0}^k$. These sets describe the successive difference functors of the mth Moriyama functor:

Proposition 8.6 *For all natural numbers g, there is an isomorphism of $\mathbb{Z}$-modules*

$$\delta_1^k \mathfrak{Mor}_m(g) \cong \bigoplus_{\omega \in \mathcal{Q}_m^{\delta^k}(g)} \mathbb{Z}_\omega$$

and the morphism $\left(i_1 \left(\delta_1^{k-1} \mathfrak{Mor}_m\right)\right)_n$ is equivalent to the injection $\mathcal{Q}_{m,n}^{k-1}$ for all natural numbers $1 \leqslant k \leqslant m+1$.

Proof. We proceed by induction on k. For $k = 1$, we already know from §6.2.2 that $(i_1 \mathfrak{Mor}_m)_g$ is equivalent to the injection $\mathcal{Q}_{m,g}^0$ and it follows from the universal property of the cokernel that

$$\delta_1 \mathfrak{Mor}_m(g) \cong \bigoplus_{\omega \in \mathcal{Q}_m^{\delta^1}(g)} \mathbb{Z}_\omega.$$

Now we assume that the results of Proposition 8.6 are true for some fixed $k \geqslant 1$. Recall from Proposition 7.1 that, by the definition of the difference functor, the morphism $\left(i_1\left(\delta_1^k\mathfrak{Mor}_m\right)\right)_g$ is canonically induced by the morphism $\tau_1\delta_1^{k-1}\mathfrak{Mor}_m\left([1, id_{1+g}]\right)$. The inductive hypothesis gives the equivalence between $\delta_1^{k-1}\mathfrak{Mor}_m\left([1, id_{1+g}]\right)$ and $\mathcal{Q}_{m,g}^{k-1}$. Using the bijection (8.5), the image of $\mathcal{Q}_{m,g}^{k-1}$ induced by the canonical surjections $\mathcal{Q}_m^{\delta^{k-1}}(g+1) \twoheadrightarrow \mathcal{Q}_m^{\delta^k}(g)$ and $\mathcal{Q}_m^{\delta^{k-1}}(g+2) \twoheadrightarrow \mathcal{Q}_m^{\delta^k}(g+1)$ is the morphism $\mathcal{Q}_{m,g}^k$: hence $\left(i_1\left(\delta_1^k\mathfrak{Mor}_m\right)\right)_g$ is equivalent to the injection $\mathcal{Q}_{m,g}^k$.

Then the following diagram is commutative:

$$\begin{tikzcd}
\kappa_1\delta_1^k\mathfrak{Mor}_m(g) \arrow[r, hook] & \delta_1^k\mathfrak{Mor}_m(g) \arrow[r, "\left(i_1\left(\delta_1^k\mathfrak{Mor}_m\right)\right)_g"] \arrow[d, "\sim"] & \tau_1\delta_1^k\mathfrak{Mor}_m(g) \arrow[r, two heads] \arrow[d, "\sim"] & \delta_1^{k+1}\mathfrak{Mor}_m(g). \\
 & \bigoplus\limits_{\omega\in\mathcal{Q}_m^{\delta^k}(g)}\mathbb{Z}_\omega \arrow[r, hook, "\mathcal{Q}_{m,g}^k"] & \bigoplus\limits_{\omega\in\mathcal{Q}_m^{\delta^k}(g+1)}\mathbb{Z}_\omega &
\end{tikzcd}$$

We deduce that

$$\delta_1^{k+1}\mathfrak{Mor}_m(g) \cong \bigoplus_{\omega\,\in\,\mathcal{Q}_m^{\delta^{k+1}}(g)}\mathbb{Z}_\omega$$

using the universal property of the cokernel and the bijection (8.5). □

From the previous properties, we deduce the following polynomiality results for the Moriyama functors:

Theorem 8.7 *The Moriyama functor $\mathfrak{Mor}_m : \left\langle\mathcal{M}_2^{+,gen}, \mathcal{M}_2^{+,gen}\right\rangle \to \mathbb{Z}\text{-Mod}$ is both very strong and weak polynomial of degree m.*

Proof. Note that the sets $\mathcal{Q}_m^{\delta^m}(n)$ and $\mathcal{Q}_m^{\delta^m}(n+1)$ are isomorphic: then it from Proposition 8.6 that the morphisms $\left(i_1\left(\delta_1^m\mathfrak{Mor}_m\right)\right)_n$ is a bijection. Hence the natural transformation $i_1\left(\delta_1^m\mathfrak{Mor}_m\right)$ is a natural equivalence between non-null objects. Since $\delta_1 \circ \pi_{\left\langle\mathcal{M}_2^{+,gen}, \mathcal{M}_2^{+,gen}\right\rangle} = \pi_{\langle\beta,\beta\rangle} \circ \delta_1$, we deduce that the functor $\mathfrak{Mor}_m : \left\langle\mathcal{M}_2^{+,gen}, \mathcal{M}_2^{+,gen}\right\rangle \to \mathbb{Z}\text{-Mod}$ is weak polynomial of degree m.

Also, for all natural numbers k such that $1 \leqslant k \leqslant m$, the evanescence functor $\kappa_1\delta_1^k\mathfrak{Mor}_m$ is null since $\left(i_1\left(\delta_1^k\mathfrak{Mor}_m\right)\right)_n$ is an injection by Proposition 8.6. Then the functor $\delta_1^k\mathfrak{Mor}_m$ is very strong polynomial polynomial of degree $m-k$ which ends the proof. □

8.3 Applications

Finally, we detail here some uses of the polynomiality results stated in §8.1 and §8.2.

8.3.1 Homological stability

A first application of the previous polynomial results is their homological stability properties. Indeed Randal-Williams and Wahl [RW17] prove homological stability for several families of groups (including in particular most of those considered in §3.4) with twisted coefficients given by very strong polynomial functors. Namely, fixing a strict monoidal groupoid $(\mathcal{G}, \natural, 0)$, an object X of $\mathcal{G}$, a left-module $(\mathcal{M}, \natural)$, an object A of $\mathcal{M}$, we denote by $\langle\mathcal{G}, \mathcal{M}\rangle_{X,A}$ the full subcategory of $\langle\mathcal{G}, \mathcal{M}\rangle$ with objects $\left\{X^{\natural n}\natural A\right\}_{n\in\mathbb{N}}$. We also denote by G_n the automorphism group $\mathrm{Aut}_{\langle\mathcal{G},\mathcal{M}\rangle}\left(X^{\natural n}\natural A\right)$ for all natural numbers n. Then:

Definition 8.8 The family of groups $\{G_n\}_{n\in\mathbb{N}}$ is said to satisfy homological stability (with twisted coefficients) if for any very strong polynomial functor $F : \langle\mathcal{G}, \mathcal{M}\rangle_{X,A} \to \mathbb{Z}\text{-}\mathfrak{Mod}$ of degree d, the natural maps

$$H_*\left(G_n, F\left(X^{\natural n}\natural A\right)\right) \to H_*\left(G_n, F\left(X^{\natural 1+n}\natural A\right)\right)$$

are isomorphisms for $N(*, d) \leqslant n$ with $N(*, d) \in \mathbb{N}$ depending on $*$ and d.

Theorem 8.9 *[RW17, Theorem A] Homological stability with twisted coefficients is satisfied for:*

- *Classical braid groups of surfaces* $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$ *using* $\mathcal{G} = \mathcal{M} = \boldsymbol{\beta}$*;*
- *Braid groups on surfaces* $\{\mathbf{B}_n(\Sigma_{g,1})\}_{n\in\mathbb{N}}$ *and* $\{\mathbf{B}_n(\mathcal{N}_{c,1})\}_{n\in\mathbb{N}}$*, using* $\mathcal{G} = \boldsymbol{\beta}$ *and respectively* $\mathcal{M} = \mathcal{B}_2^{g,+}$ *and* $\mathcal{M} = \mathcal{B}_2^{c,-}$*;*
- *Mapping class groups* $\{\boldsymbol{\Gamma}_{n,1}\}_{n\in\mathbb{N}}$ *and* $\{\boldsymbol{\mathcal{N}}_{c,1}\}_{n\in\mathbb{N}}$*, using* $\mathcal{G} = \mathcal{M} = \mathcal{M}_2^{+}$ *and* $\mathcal{G} = \mathcal{M} = \mathcal{M}_2^{-}$ *respectively;*
- *Extended and non-extended loop braid groups* $\left\{\mathbf{LB}_n^{ext}\right\}_{n\in\mathbb{N}}$ *and* $\{\mathbf{LB}_n\}_{n\in\mathbb{N}}$*, using* $\mathcal{G} = \mathcal{M} = \mathcal{L}\boldsymbol{\beta}^{ext}$ *and* $\mathcal{G} = \mathcal{M} = \mathcal{L}\boldsymbol{\beta}$ *respectively.*

The above framework is generalised in [Kra17] to a topological setting: more precisely, the homological stability results of [RW17] are extended to families of spaces that are not necessarily classifying spaces of discrete groups. [Pal18] also proves homological stability stability for configuration spaces of path-connected subspaces on an open connected manifold, with twisted coefficients given by polynomial-type functors. Homological stability is thus satisfied by the Lawrence-Bigelow functors and Moriyama functors by Theorem 8.9. As the representation theory of braid groups and mapping class groups of surfaces is wild and an active research topic (see for example [BB05], [FLM01], [Fun99], [Kor02], [Mas08] or [MR12]), there are very few known examples of very strong polynomial functors over the asssociated categories. Hence the result of §8.1 and §8.2 allow to gain a better understanding of polynomial functors for braid groups of surfaces and mapping class groups, and therefore extends the scope of twisted homological stability to more sophisticated sequences of representations.

8.3.2 Classification

A first matter of interest in the notion of weak polynomiality is that it reflects more accurately than the strong polynomiality the behaviour of functors for large values. As Theorem 8.1 shows, the first Lawrence-Bigelow functor $\mathfrak{LB}_1^{BM}$ is strong polynomial of degree two whereas it is weak polynomial of degree one. On the contrary the further Lawrence-Bigelow functors $\mathfrak{LB}_m^{BM}$ are strong and weak polynomial of degree m. We also refer the reader to [DV19, Section 6] and [Dja17] for examples of such phenomena.

Also, the quotient categories for weak polynomial functors shed a light onto what is going on with the successive Lawrence-Bigelow functors (see §7.2). Indeed, denoting for convenience the category of weak polynomial functors $\mathcal{P}ol_d(\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle, \mathbb{Z}\text{-Mod})$ by $\mathcal{P}ol_d(\boldsymbol{\beta})$, we have as a consequence of Theorem 8.1:

Corollary 8.10 *The sequence of the quotient categories induced by the difference functors:*

$$\cdots \xleftarrow{\delta_1} \mathcal{P}ol_m(\boldsymbol{\beta})/\mathcal{P}ol_{m-1}(\boldsymbol{\beta}) \xleftarrow{\delta_1} \mathcal{P}ol_{m+1}(\boldsymbol{\beta})/\mathcal{P}ol_m(\boldsymbol{\beta}) \xleftarrow{\delta_1} \cdots$$

gives a classification of the family of Lawrence-Bigelow functors $\left\{\mathfrak{LB}_m^{BM}\right\}_{m\geqslant 1}$ *and the connection between two functors of this family.*

Proof. Each Lawrence-Bigelow functor $\mathfrak{LB}_m^{BM}$ is both weak polynomial and very strong polynomial of degree d. Therefore $\mathfrak{LB}_m^{BM} \cong \tau_1\mathfrak{LB}_m^{BM}$ in the quotient categories. □

9 Appendix: A relation in Borel-Moore homology

In this appendix, we give a quick proof of the relation (8.3) among Borel-Moore homology classes. In §9.1 we recall the basic facts about fundamental classes in Borel-Moore homology, and we use this in §9.2 to prove the relation (8.3).

9.1 Fundamental classes in Borel-Moore homology

Let X be a manifold (possibly with boundary) and let M be an orientable k-manifold, without boundary and with finitely many components. Then any proper embedding

$$e\colon M \hookrightarrow X \tag{9.1}$$

determines an element $[e] \in H_k^{\mathrm{BM}}(X)$. More precisely: the homology group $H_k^{\mathrm{BM}}(M)$ is $\mathbb{Z}^r$, where r is the number of components of M. An orientation of M is a choice of generator for the summand corresponding to each component, which determines a fundamental class

$$[M] \in H_k^{\mathrm{BM}}(M).$$

Borel-Moore homology is functorial with respect to proper embeddings, so the image of this class is the element $[e]$ mentioned above. So we should say more precisely that an orient**ed** k-manifold M (without boundary and with finitely many components) and a proper embedding (9.1) determine an element $[e] \in H_k^{\mathrm{BM}}(X)$. This is additive for finite disjoint unions, i.e., if M is the disjoint union of $M_1, \ldots, M_j$, then

$$[e] = \sum_{i=1}^{j} [e|_{M_i}]$$

in the group $H_k^{\mathrm{BM}}(X)$.

Now let N be an oriented $(k+1)$-manifold with finitely many boundary-components and let

$$f\colon N \hookrightarrow X$$

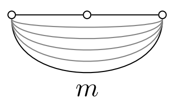
be a proper embedding. Then we have the relation

$$[f|_{\partial N}] = 0 \tag{9.2}$$

in the group $H_k^{\mathrm{BM}}(X)$.

9.2 The relation.

By §9.1, we simply need to exhibit an oriented manifold with boundary, embedded in the (cover of the) m-point configuration space, such that the image of its boundary (considered with its induced orientation) is the difference of the two sides of the relation (8.3). We may define such a manifold as follows:



This is the space of unordered configurations of m points in the plane, such that all m points lie on one of the lines depicted in the closed half-disc above, and do *not* intersect any of the three punctures. In particular, note that, if they lie on the top (straight) line, they must be partitioned into two subsets of size k and $m-k$ by the two little intervals between the three punctures. It is then not hard to see that this has the desired (oriented) boundary.

Remark 9.1 We note that [Ito13, Section 4] and [Ike18, Section 3.4] prove analogous relations among Borel-Moore homology classes.

10 Appendix: Notations and tools

We take the convention that the set of natural numbers $\mathbb{N}$ is the set of non-negative integers $\{0, 1, 2, \ldots\}$.

We denote by Gr the category of groups and by $*$ the coproduct in this category. The trivial group is denoted by 0_{Gr}. For a group G, the lower central series of G is the descending chain of subgroups $\{\Gamma_l(G)\}_{l\geqslant 0}$ defined by $\Gamma_0(G) := 0_{\mathrm{Gr}}$, $\Gamma_1(G) := G$ and $\Gamma_{l+1}(G) := [G, \Gamma_l(G)]$ the subgroup of G generated by all commutators $[x,y] := xyx^{-1}y^{-1}$ with x in G and y in $\Gamma_l(G)$. The induced canonical projection on the quotient by the lth lower central term is denoted by $\gamma_l(G) : G \twoheadrightarrow G/\Gamma_l(G)$. In particular $\gamma_2(G)$ denotes the abelianisation map of the group G. When there is no ambiguity, we omit G from the notations.

For a (commutative) ring R, we denote by R-Mod the category of R-modules. For M a G-module, we denote by $\mathrm{Aut}_G(M)$ the group of G-module automorphisms of M. If $G = \mathbb{Z}$, the we omit it from the notation if there is no ambiguity.

We denote by Top the category of topological spaces and by Top_* the category of based topological spaces with based maps. For X a topological space, we denote by $\mathrm{Homeo}(X)$ the group of self-homeomorphisms of X. If X is a differentiable manifold, then we denote by $\mathrm{Diff}(X)$ the self-diffeomorphism group of X.

Let Cat denote the category of small categories. Let $\mathcal{C}$ be an object of Cat. We use the abbreviation $\mathrm{Ob}(\mathcal{C})$ to denote the set of objects of $\mathcal{C}$. If there exists an initial object $\varnothing$ in the category $\mathcal{C}$, then we denote by $\iota_A : \varnothing \to A$ the unique morphism from $\varnothing$ to A. If T is a terminal object in the category $\mathcal{C}$, then we denote by $t_A : A \to T$ the unique morphism from A to T. Let Grpd denote the subcategory of Cat of small groupoids. The maximal subgroupoid $\mathcal{G}\mathrm{r}(\mathcal{C})$ is the subcategory of $\mathcal{C}$ which has the same objects as $\mathcal{C}$ and of which the morphisms are the isomorphisms of $\mathcal{C}$. We denote by $\mathcal{G}\mathrm{r} : \mathrm{Cat} \to \mathrm{Grpd}$ the functor which associates to a category its maximal subgroupoid. For $\mathcal{D}$ a category and $\mathcal{C}$ a small category, we denote by $\mathbf{Fct}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$.

We take this opportunity to recall some terminology about (strict) monoidal categories and modules over them. We refer to [Mac98] for a complete introduction to these notions. A strict monoidal category $(\mathcal{C}, \natural, 0)$, where $\mathcal{C}$ is a category, $\natural$ is the monoidal product and 0 is the monoidal unit. If it is braided, then its braiding is denoted by $b^{\mathcal{C}}_{A,B} : A\natural B \xrightarrow{\sim} B\natural A$ for all objects A and B of $\mathcal{C}$. A left-module $(\mathcal{M}, \sharp)$ over a (strict) monoidal category $(\mathcal{C}, \natural, 0)$ is a category $\mathcal{M}$ with a functor $\sharp : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ that is unital (i.e. $\sharp \circ (\iota_{\mathcal{C}} \times id_{\mathcal{M}}) = id_{\mathcal{M}}$ where $\iota_{\mathcal{C}} : 0_{\mathrm{Cat}} \to \mathcal{C}$ takes the trivial category 0_{Cat} to the unit object 0 of $\mathcal{C}$) and associative (i.e. $\sharp \circ (id_{\mathcal{C}} \times \sharp) = \sharp \circ (\sharp \times id_{\mathcal{M}})$). For instance, a monoidal category $(\mathcal{C}, \natural, 0)$ is automatically equipped with a left-module structure over itself, induced by the monoidal product $\natural : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

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