ON HOMOLOGICAL STABILITY FOR CONFIGURATION SPACES ON CLOSED BACKGROUND MANIFOLDS

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Abstract. We introduce new maps between configuration spaces of points in a background manifold, the *extrinsic* and *intrinsic replication maps*, and prove that they are homology isomorphisms in a range with certain coefficients. The other natural map between configuration spaces is the classical stabilisation map, which is an integral homology isomorphism in a stable range. The fact that the latter map exists only for open manifolds leaves unanswered the question of homological stability in the closed case. In fact the 2-sphere provides a counterexample to homological stability for many coefficients. However, some partial stability results are known. We improve upon stability results of Bendersky and Miller on the torsion in the homology of configuration spaces on closed manifolds, and using the replication maps we also improve results of Randal-Williams on their homology with field coefficients.

1. Introduction

Let M be a smooth, connected manifold without boundary and denote by $C_k(M)$ the unordered configuration space of k points in M:

$$C_k(M) := \{\mathbf{q} \subset M \mid |\mathbf{q}| = k\},$$

which is topologised as a quotient space of a subspace of M^n. After removing a point $*$ from M one can define a map

$$C_k(M \smallsetminus \{*\}) \longrightarrow C_{k+1}(M \smallsetminus \{*\}),$$

called the *stabilisation map*, which expands the configuration away from $*$ and adds a new point near to it. More generally, one can define such a stabilisation map $C_k(M) \to C_{k+1}(M)$ using any properly embedded ray in M to bring in a point from infinity (such a ray exists if and only if M is non-compact).

Definition 1.1 Given a smooth, connected, non-compact manifold M, a properly embedded ray in M and an abelian group A, the function $sr_{\mathrm{stab}} \colon \mathbb{N} \to \mathbb{N}$ is defined to be the (pointwise) maximum $f \colon \mathbb{N} \to \mathbb{N}$ such that the stabilisation map $C_k(M) \to C_{k+1}(M)$ induces isomorphisms on $H_*(-;A)$ in the range $* \leqslant f(k)$. This is called the *stable range of the stabilisation map.*

Homological stability is the phenomenon that sr_{stab} diverges. More precisely, we have the following known lower bounds for sr_{stab}:

- $sr_{\mathrm{stab}}(k) \geqslant \frac{k}{2}$ if $A = \mathbb{Z}$ and $\dim(M) \geqslant 2$, by [McD75, Seg79, RW13].
- $sr_{\mathrm{stab}}(k) \geqslant k$ if $A = \mathbb{Q}$ and either $\dim(M) \geqslant 3$ or M is non-orientable, by [RW13, Knu14].
- $sr_{\mathrm{stab}}(k) \geqslant k-1$ if $A = \mathbb{Q}$ and M is orientable, by [Chu12, Knu14].
- $sr_{\mathrm{stab}}(k) \geqslant k$ if $A = \mathbb{Z}[\frac{1}{2}]$ and $\dim(M) \geqslant 3$, by [KM14b].

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See also Propositions A.2 and A.9. Further improvements to the lower bound are possible under extra hypotheses [KM14b, Remark 4.4].

For closed manifolds (i.e. if M is closed and we do not remove a point) such stabilisation maps do not exist. Nevertheless, it is possible to prove that the rank of the homology groups with field coefficients stabilises in many cases [BCT89, Chu12, RW13] (if the dimension of M is odd it stabilises with all field coefficients, otherwise it stabilises with $\mathbb{F}_2$ and $\mathbb{Q}$ coefficients).

Let us assume from now on that the manifold is endowed with a Riemannian metric with injectivity radius bounded below by $\delta > 0$. Define $C_k^\delta(M) \subset C_k(M) \times (0, \delta)$ to be the space of pairs $(\mathbf{q}, \epsilon)$, where $\mathbf{q}$ is a configuration whose points are pairwise at distance at least 2ϵ. The projection to $C_k(M)$ is a fibre bundle with contractible fibres, hence a homotopy equivalence. The main theorem in [McD75] concerns the *scanning map*

$$\mathscr{S} \colon C_k^\delta(M) \longrightarrow \Gamma_c(\dot{T}M)_k$$

which takes values in the space of degree-k compactly-supported sections of the fibrewise one-point compactification of TM.

Definition 1.2 Given a smooth, connected, Riemannian manifold M with injectivity radius bounded below by $\delta > 0$ and an abelian group A, the function $sr_{\mathrm{scan}} \colon \mathbb{N} \to \mathbb{N}$ is defined to be the (pointwise) maximum $f \colon \mathbb{N} \to \mathbb{N}$ such that the scanning map $\mathscr{S} \colon C_k^\delta(M) \to \Gamma_c(\dot{T}M)_k$ induces isomorphisms on $H_*(-;A)$ in the range $* \leqslant f(k)$. This is called the *stable range of the scanning map*, or simply the *stable range*.

McDuff's theorem states that the scanning map is an isomorphism in a certain range, precisely:

$$sr_{\mathrm{scan}}[M](k) = \min_{j \geqslant k}(sr_{\mathrm{stab}}[M \smallsetminus \{*\}](j)).$$

Stability for p-torsion. By the main result in [Møl87], the localisation $\Gamma_c(\dot{T}M)_{(p)}$ at a prime p is homotopy equivalent to the space of sections $\Gamma_c(\dot{T}M_{(p)})$ of the fibrewise localisation of $\dot{T}M$. In [BM13] Bendersky and Miller proved the existence of homotopy equivalences

$$\Gamma_c(\dot{T}M_{(p)})_k \longrightarrow \Gamma_c(\dot{T}M_{(p)})_j$$

whenever $p \geqslant \frac{n+3}{2}$ and either M is odd-dimensional or $\frac{2k-\chi}{2j-\chi}$ is a unit in $\mathbb{Z}_{(p)}$, where χ is the Euler characteristic of M. If $\dot{T}M$ is trivial, then the result holds for all primes. Using McDuff's theorem one obtains a zig-zag of $\mathbb{Z}_{(p)}$-homology isomorphisms in the stable range

$$C_k(M) \longrightarrow \Gamma_c(\dot{T}M)_k \longrightarrow \Gamma_c(\dot{T}M)_j \longleftarrow C_j(M).$$

We will show that pairs of linearly independent sections of $TM \oplus \epsilon$ give rise to families of fibrewise homotopy equivalences of $\dot{T}M$ after localisation, from which we are able to extend the results of Bendersky and Miller to all odd primes and under certain conditions to the prime 2. For a number $k \in \mathbb{Z}$, we denote by $(k)_p$ the p-adic valuation of k, and observe that $\frac{j}{k}$ is a unit in $\mathbb{Z}_{(p)}$ if and only if $(k)_p = (j)_p$. If ℓ is a collection of primes, the ℓ-adic valuation is the sequence of all p-adic valuations with $p \in \ell$.

Theorem A *Let M be a closed, connected, smooth manifold. If M is odd-dimensional there are zig-zags of maps inducing isomorphisms*

$$H_*(C_k(M);\mathbb{Z}_{(\ell)}) \cong H_*(C_j(M);\mathbb{Z}_{(\ell)}) \quad \text{if } 2 \notin \ell \text{ or } k-j \text{ is even} \tag{1.1}$$

in the stable range. In particular, there are isomorphisms in the stable range

$$H_*(C_k(M);\mathbb{Z}) \cong H_*(C_{k+2}(M);\mathbb{Z})$$
$$H_*(C_k(M);\mathbb{Z}[\tfrac{1}{2}]) \cong H_*(C_{k+1}(M);\mathbb{Z}[\tfrac{1}{2}]).$$

If M is even-dimensional and its Euler characteristic χ is even (resp. odd), then for each set ℓ of primes (resp. odd primes) there are zig-zags of maps inducing isomorphisms

$$H_*(C_k(M);\mathbb{Z}_{(\ell)}) \cong H_*(C_j(M);\mathbb{Z}_{(\ell)}) \quad \text{if } (2k-\chi)_\ell = (2j-\chi)_\ell \tag{1.2}$$

in the stable range. In particular, when $j = \chi - k$ we have $2j - \chi = -(2k-\chi)$ and may take $\ell = \mathrm{Spec}(\mathbb{Z})$, so we have integral homology isomorphisms between $C_k(M)$ and $C_{\chi-k}(M)$.

Observe that since these isomorphisms are induced by zig-zags of maps, they also give isomorphisms between the cohomology rings of configuration spaces.

Replication maps. Our next result gives a map between configuration spaces inducing some of the homology isomorphisms of Theorem A. This provides a new map between configuration spaces, different from the stabilisation map. This is especially interesting when M is closed, in which case the stabilisation map is not defined. However, the map is also useful when M is open; it will be used later to prove Theorem D below.

Let v be a non-vanishing vector field on M of norm 1. Define the *r-replication map* $\rho_r = \rho_r[v]\colon C_k^\delta(M) \to C_{rk}^\delta(M)$ by adding $r-1$ points near each point in the configuration, in the direction of the vector field v:

$$\rho_r[v](\mathbf{q} = \{q_1,\dots,q_k\},\epsilon) = \left(\left\{\exp(\tfrac{j\epsilon}{r}v(q_i)) \;\middle|\; {\scriptstyle i=1,\dots,k \atop j=0,\dots,r-1}\right\}, \tfrac{\varepsilon}{2r}\right).$$

Theorem B *If M admits a non-vanishing vector field v and p is a prime not dividing r, then the homomorphism induced by $\rho_r[v]$:*

$$H_*(C_k^\delta(M);\mathbb{Z}_{(p)}) \longrightarrow H_*(C_{rk}^\delta(M);\mathbb{Z}_{(p)})$$

is an isomorphism in the stable range. If M is not closed, then it is always injective. This gives a zig-zag of (at most two) maps of configuration spaces realising the isomorphisms (1.1) for a set of primes ℓ whenever k and j have the same ℓ-adic valuation.

In general, however, the isomorphisms (1.1) and (1.2) are only realised by zig-zags of maps of section spaces, together with the scanning maps.

Remark 1.3 Observe that the map ρ_r does not induce isomorphisms on r-torsion in general. For example take M to be simply-connected and of dimension at least 3. Then $\pi_1(C_k(M)) \cong \Sigma_k$ and $H_1(C_k(M)) \cong \mathbb{Z}/2$, given by the sign of the permutation. The map $\Sigma_k \to \Sigma_{2k}$ induced by ρ_2 on π_1 sends a permutation σ to the concatenation (σ,σ), whose sign is the square of the sign of σ, therefore zero. Hence the map induced on first homology by ρ_2 is zero. In particular this shows that ρ_2 cannot be homotopic to a composition of stabilisation maps.

Configurations with labels and the intrinsic replication map. Given a fibre bundle $\theta\colon E \to M$ with path-connected fibres, one can define a configuration space $C_k(M;\theta)$ with labels in θ by

$$C_k(M;\theta) = \{\{q_1,\ldots,q_k\} \subset E \mid \theta(q_i) \neq \theta(q_j) \text{ for } i \neq j\}.$$

The stabilisation map in this case may be defined similarly to the case of unlabelled configuration spaces; see Definition A.8 for a precise definition. To define the replication map it is more convenient to use the following alternative model:

$$C_k^\delta(M;\theta) = \{(\mathbf{q},\epsilon,s) \mid (\mathbf{q},\epsilon) \in C_k^\delta(M),\ s\colon B_{\epsilon/2}(\mathbf{q}) \to E \text{ a section of } \theta\},$$

where $B_{\epsilon/2}(\mathbf{q})$ means the (disjoint) union of the $(\epsilon/2)$-balls around q for each $q \in \mathbf{q}$. So a point in this space consists of a configuration $\mathbf{q}$ with prescribed pairwise separation, together with a choice of label on a small contractible neighbourhood of each configuration point. On the other hand, the pullback $\dot{C}_k^\delta(M;\theta)$ of $C_k^\delta(M)$ along the map $C_k(M;\theta) \to C_k(M)$ which forgets the labels consists of a configuration with prescribed pairwise separation, together with a choice of label just over each configuration point. Since $C_k^\delta(M) \to C_k(M)$ is a fibre bundle with contractible fibres, so is its pullback $\dot{C}_k^\delta(M;\theta) \to C_k(M;\theta)$, which is therefore a homotopy equivalence. There is a map $C_k^\delta(M;\theta) \to \dot{C}_k^\delta(M;\theta)$ which just remembers the label at the centre of each ball, which is also a fibre bundle with contractible fibres, so a homotopy equivalence. Hence the composition $C_k^\delta(M;\theta) \to C_k(M;\theta)$ which completely forgets the labels is a weak equivalence (homotopy equivalence if E is paracompact).

The replication map $\rho_r\colon C_k^\delta(M;\theta) \to C_{rk}^\delta(M;\theta)$ for a non-vanishing vector field v is defined similarly to the unlabelled version; the labels of the new configuration of rk points are simply a restriction of the labels of the original configuration of k points.

Just as for configuration spaces without labels, one can define the *stable range of the stabilisation map* as follows: $sr_{\mathrm{stab}}(k)$ is the largest d such that the stabilisation map $C_k(M;\theta) \to C_{k+1}(M;\theta)$ induces isomorphisms on $H_*(-;A)$ for all $* \leqslant d$. This now additionally depends on the bundle θ. See Proposition A.9 and Remark A.10 for some known lower bounds on sr_{stab} – these are either the same or one degree less than the lower bounds given on page 1 for unlabelled configuration spaces. The *stable range* (of the scanning map) is also defined analogously to the unlabelled version, and by the version of McDuff's theorem with labels in a bundle (Theorem B.7), we again have $sr_{\mathrm{scan}}[M](k) = \min_{j\geqslant k}(sr_{\mathrm{stab}}[M \smallsetminus \{*\}](j))$. In Section 4 we show that Theorem B extends to the configuration spaces $C_k^\delta(M;\theta)$ and the more general replication map (see Theorem B′).

If $\theta\colon S(TM) \to M$ is the unit sphere bundle of TM it is possible to define a new map which we call the *intrinsic replication map* $\mathfrak{z}_r\colon C_k^\delta(M;\theta) \longrightarrow C_{rk}^\delta(M;\theta)$. It sends the labelled configuration $(\mathbf{q} = \{q_1,\ldots,q_k\},\epsilon,s\colon B_{\epsilon/2}(\mathbf{q}) \to E)$ to the labelled configuration

$$\Big(\Big\{\exp(\tfrac{j\epsilon}{r}s(q_i)) \;\Big|\; {\scriptstyle i=1,\ldots,k \atop j=0,\ldots,r-1}\Big\}, \tfrac{\varepsilon}{2r},\ \text{restriction of } s\Big).$$

In contrast with the (extrinsic) replication map of Theorem B, this map is defined for every manifold M.

Theorem C *The map $\mathfrak{z}_r\colon C_k^\delta(M;\theta) \to C_{rk}^\delta(M;\theta)$ induces isomorphisms on homology with $\mathbb{Z}[\frac{1}{r}]$-coefficients in the stable range. Hence if $(k)_p = (j)_p$ the groups $H_*(C_k^\delta(M;\theta);\mathbb{Z}_{(p)})$ and $H_*(C_j^\delta(M;\theta);\mathbb{Z}_{(p)})$ are isomorphic in the stable range.*

This theorem can be generalised (Theorem C′) to any fibre bundle with path-connected fibres $\theta\colon E \to M$ which factors through the unit sphere bundle $S(TM)$, for instance the (oriented) frame bundle of TM.

An extension for field coefficients. The last part (§5) does not involve section spaces, but rather uses the result of Theorem B in the case of open manifolds M to push our results a bit further. If M is a closed, connected manifold one can choose a vector field on M which is non-vanishing away from a point $* \in M$. This vector field (suitably normalised) therefore induces an r-replication map for configuration spaces on $M \smallsetminus \{*\}$, which induces isomorphisms on homology with $\mathbb{Z}[\frac{1}{r}]$ coefficients in the stable range by Theorem B.

We then use a technique similar to that of [RW13, §9]. First, we can fit $C_k(M)$ into a cofibre sequence in which the other two spaces are suspensions of configuration spaces on $M \smallsetminus \{*\}$. We can then define stabilisation maps on the other two spaces using the r-replication map and the ordinary stabilisation map, which are isomorphisms in the stable range away from r. We will therefore have homological stability for $C_n(M)$, with field coefficients of characteristic coprime to r, as long as the square formed by this pair of stabilisation maps commutes. In fact it does *not* commute in general, but the obstruction to commutativity on homology is a single homology class whose divisibility we can calculate. Thus we obtain the following, where

$$\lambda_r(k) = \min\{sr_{\text{scan}}(k), sr_{\text{scan}}(k-1) + n - 1, sr_{\text{stab}}(rk - i) \mid i = 2, \ldots, r\}.$$

Theorem D *Let M be a closed, connected, smooth manifold with Euler characteristic χ. Choose a field $\mathbb{F}$ of positive characteristic p and let $r \geqslant 2$ be an integer coprime to p such that p divides $(\chi - 1)(r - 1)$. Alternatively choose a field $\mathbb{F}$ of characteristic $p = 0$, let $r \geqslant 2$ be any positive integer and assume that $\chi = 1$. Then there are isomorphisms*

$$H_*(C_k(M); \mathbb{F}) \;\cong\; H_*(C_{rk}(M); \mathbb{F})$$

in the range $* \leqslant \lambda_r(k)$. In particular when $\chi \in 1 + p\mathbb{Z}$ there are isomorphisms

$$H_*(C_k(M); \mathbb{F}) \;\cong\; H_*(C_j(M); \mathbb{F})$$

in the range $* \leqslant \lambda_r(k)$ whenever $(k)_p = (j)_p$ (which is a vacuous condition when $p = 0$).

See Remark 5.5 for an explanation of how the function λ_r arises, and the remark that if sr_{stab} is linear with slope $\leqslant \dim(M) - 1$ and $r, k \geqslant 2$, then $\lambda_r(k) = sr_{\text{stab}}(k)$.

This theorem also generalises to configuration spaces with labels in a fibre bundle over M with path-connected fibres. See §5.4 for the proof for configuration spaces without labels and §5.6 for a sketch of the generalisation to configuration spaces with labels (Theorem D′).

For odd-dimensional manifolds Theorem D follows from Theorem A: when $p \neq 2$ these isomorphisms exist more generally with $\mathbb{Z}[\frac{1}{2}]$ coefficients; when $p = 2$ the conditions imply that r is odd so that $rk - k$ is even so these isomorphisms exist with $\mathbb{Z}$ coefficients. Also, Theorem C′ implies Theorem D′ if the bundle $\theta\colon E \to M$ factors through the unit sphere bundle $S(TM)$ of the tangent bundle of M.

Note that the last sentence of Theorem D is in contrast with the even-dimensional part of Theorem A where the corresponding condition is instead that $(2k - \chi)_p = (2j - \chi)_p$. Also note that the requirement that $k \geqslant 2$ is essential: for example $H_1(C_1(S^2); \mathbb{F}_3) = 0 \ncong \mathbb{F}_3 \cong H_1(C_4(S^2); \mathbb{F}_3)$.

Combining Theorems A and D. Theorem A says that in odd dimensions there are at most two stable homologies, depending on the parity of the number of points k. On the other hand, in even dimensions – even when taking homology with $\mathbb{Z}_{(p)}$ coefficients – there may be infinitely many different stable homologies: one for each possible p-adic valuation of $2k-\chi$. In fact this is sharp, as the calculation of $H_1(C_k(S^2);\mathbb{Z}) \cong \mathbb{Z}/(2k-2)$ shows.

However, the situation is simpler when taking $\mathbb{F}_p$ coefficients: $H_1(C_k(S^2);\mathbb{F}_p)$ is either one- or zero-dimensional depending on whether or not p divides $2k-2$, so there are at most two stable homologies in this special case. One can combine Theorems A and D to prove that this phenomenon holds more generally: when p is odd and does not divide χ there are at most two stable homologies, one for those k such that p divides $2k-\chi$ and one for those k such that p does not divide $2k-\chi$. The following corollary summarises what can be deduced from Theorems A and D.

Corollary E *Let M be a closed, connected, smooth manifold with Euler characteristic χ and let $\mathbb{F}$ be a field of characteristic p. Then Table 1 shows how the homology group $H_*(C_k(M);\mathbb{F})$ depends on k in the stable range (resp. the range $* \leqslant \lambda_r(k)$ for lines 5–9).*

dimension	conditions		$H_*(C_k(M);\mathbb{F})$ depends only on	$\sharp$	
odd	$p \neq 2$	—	—	1	A
	$p = 2$	—	parity of k	2	A
even	$p = 0$	—	whether $2k = \chi$	1	A
	p odd	—	$(2k-\chi)_p$	∞	A
		$\chi \not\equiv 0 \bmod p$	whether p divides $2k-\chi$	2	A,D
		$\chi \equiv 1 \bmod p$	—	1	A,D
	$p = 2$	—	$(k)_2$	∞	D
		$(\chi)_2 \geqslant 1$	$\min\{(k)_2, (\chi)_2\}$	$(\chi)_2 + 1$	A,D
		$(\chi)_2 = 1$	parity of k	2	A,D

Table 1. Lines 4–6 and 7–9 are written in order of decreasing generality. The second-from-right column is the number of possible stable homologies and the rightmost column indicates which theorem each line follows from.

When $p = 2$ this only recovers part of the story. In fact homological stability holds with no extra conditions when taking coefficients in a field of charactersitic 2, by Theorem C of [RW13], so there is only one stable homology in this case. Together with the sixth line of the table, this tells us that if $\chi = 1$ then homological stability holds with coefficients in any field. Also, when $p = 0$ and $\dim(M)$ and χ are both even, Corollary E leaves unresolved whether $H_*(C_{\frac{\chi}{2}}(M);\mathbb{F})$ is an exception in the stable range. In fact it is not, also by Theorem C of [RW13].

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2. Homological stability via the scanning map

2.1. **Scanning maps.** Let T^1M denote the open unit disc bundle of the tangent bundle of a connected Riemannian manifold M, and let $\dot{T}^1M$ and $\dot{T}M$ denote the fibrewise one point compactifications of T^1M and TM. We denote by ∞ the point at infinity in each fibre. We denote by ι the section with value ∞ and by z the zero section. Let $\delta > 0$ be smaller than the injectivity radius of M. Define the *linear scanning map*

$$\mathscr{S} : C_k^\delta(M) \longrightarrow \Gamma_c(\dot{T}^1M)_k$$

to the space of degree k compactly supported sections of $\dot{T}^1M$ as

$$\mathscr{S}(\mathbf{q}, \epsilon)(x) = \begin{cases} \infty & \text{if } x \notin B_\epsilon(q)\ \forall q \in \mathbf{q}, \\ \frac{\exp_x^{-1}(q)}{\epsilon} & \text{if } x \in B_\epsilon(q), q \in \mathbf{q}. \end{cases}$$

The degree of a section s is the fibrewise intersection, counted with multiplicity, of s and the zero section (see also §2.3).

Theorem 2.1 ([McD75]) *The scanning map induces an isomorphism on homology groups in the stable range.*

We refer the reader to the introduction for the definition of the stable range. The strategy to prove Theorem A will be to construct $\mathbb{Z}_{(p)}$-equivalences ϕ_r

$$\begin{tikzcd} C_k(M) \arrow[r, "\mathscr{S}"] & \Gamma_c(\dot{T}^1M)_k \arrow[d, "\phi_r"] \\ C_{[\phi_r](k)}(M) \arrow[r, "\mathscr{S}"] & \Gamma_c(\dot{T}^1M)_{[\phi_r](k)}, \end{tikzcd} \tag{2.1}$$

where $[\phi_r] = \pi_0(\phi_r)$, which, together with McDuff's theorem, prove Theorem A.

Let D be the unit n-dimensional open disc, let $\dot{D}$ be its one point compactification and define $\psi^\delta(D)$ to be the quotient of $\bigcup_k C_k^\delta(\mathbb{R}^n)$, where two configurations $(\mathbf{q}, \epsilon)$ and $(\mathbf{q}', \epsilon')$ are identified if $\mathbf{q} \cap D = \mathbf{q}' \cap D$ and either $\epsilon = \epsilon'$ or $\mathbf{q} \cap D = \emptyset$. We write $\psi^\delta(T^1M)$ for the result of applying this construction fibrewise to T^1M.

Let γ be a number smaller than the injectivity radius of M. The *radius γ non-linear scanning map*

$$\mathfrak{s}^\gamma : C_k^\delta(M) \to \Gamma_c(\psi^\delta(T^1M))$$

sends a configuration $\mathbf{q}$ to $\frac{1}{\gamma}\exp_x^{-1}(\mathbf{q})$ — which may consist of more than one point.

There is an inclusion $i\colon \dot{D} \hookrightarrow \psi^\delta(D)$ given by $i(q) = (q, \delta/2)$ as the subspace of configurations with at most one point. This inclusion has a homotopy inverse

$$h(\mathbf{q}, \epsilon) = \frac{\mathbf{q}}{\mathbf{q}_{\text{second}}}$$

where $\mathbf{q}_{\text{first}}$ is the norm of a closest point in $\mathbf{q}$ to the origin, and $\mathbf{q}_{\text{second}}$ is defined to be 1 if $|\mathbf{q}| = 1$ and $(\mathbf{q}')_{\text{first}}$ otherwise, where $\mathbf{q}'$ is the result of removing a single closest point of $\mathbf{q}$ to the origin. The composite hi is the identity and $H_t(\mathbf{q}, \epsilon) = \left(\frac{\mathbf{q}}{(1-t)+t\mathbf{q}_{\text{second}}}, t\delta/2 + (1-t)\epsilon\right)$ gives a homotopy between the identity and ih.

Each of i, h and H_t is $O(n)$-equivariant, so they can be defined on the vector bundle TM, obtaining homotopy equivalences

$$i\colon \dot{T}^1M \longleftrightarrow \psi^\delta(T^1M)\colon h$$

which induce by composition homotopy equivalences

$$i\colon \Gamma_c(\dot{T}^1M) \longleftrightarrow \Gamma_c(\psi^\delta(T^1M))\colon h$$

that commute with the linear and non-linear scanning maps:

$$\begin{tikzcd} C_k^\delta(M) \arrow[r, "\mathfrak{s}^\gamma"] \arrow[dr, "\mathscr{S}"'] & \Gamma_c(\psi^\delta T^1 M) \arrow[d, bend left, "h"] \arrow[d, phantom, "\simeq"] \\ & \Gamma_c(\dot{T}^1 M). \arrow[u, bend left, "i"] \end{tikzcd} \tag{2.2}$$

2.2. **Sphere bundles, localisation and fibrewise homotopy equivalences.** Let M be a connected manifold and $E \to M$ a rank n inner product vector bundle. Let $\dot{E}$ be the fibrewise one point compactification of E. The topological bundle $\dot{E}$ is isomorphic to the unit sphere bundle $S(E \oplus \epsilon)$ of the Whitney sum of E and a trivial line bundle.

Since the fibre of $\dot{E} \to M$ is nilpotent and the base is homotopy equivalent to a finite complex, then by [Møl87, Theorem 4.1], each connected component of the space of sections is also nilpotent. We may therefore consider the localisation $\Gamma(\dot{E})_{(p)}$. We may also consider the fibrewise localisation $\dot{E} \to \dot{E}_{(p)}$, and [Møl87, Theorem 5.3] implies that the induced map $\Gamma(\dot{E})_{(p)} \to \Gamma(\dot{E}_{(p)})$ is a localisation in each component. Let $\{K_j\}_{j \in \mathbb{N}}$ be a nested covering of M by compact subsets. The restriction maps $\Gamma(\dot{E}) \longrightarrow \Gamma(\dot{E}_{|K_j})$ define a map $\Gamma(\dot{E}) \longrightarrow \operatorname{colim}_j \Gamma(\dot{E}_{|K_j})$ whose fibre is $\Gamma_c(\dot{E})$. Since localisation commutes with colimits, by Møller's theorem, all vertical maps but the leftmost in the following diagram

$$\begin{tikzcd} \Gamma_c(\dot{E}) \arrow[r] \arrow[d] & \Gamma(\dot{E}) \arrow[r] \arrow[d] & \operatorname{colim}_j \Gamma(\dot{E}_{|K_j}) \arrow[d] \\ \Gamma_c(\dot{E}_{(p)}) \arrow[r] & \Gamma(\dot{E}_{(p)}) \arrow[r] & \operatorname{colim}_j \Gamma(\dot{E}_{(p)|K_j}) \end{tikzcd}$$

are $\mathbb{Z}_{(p)}$-equivalences, and therefore the leftmost is a $\mathbb{Z}_{(p)}$-equivalence too.

A bundle endomorphism f of $\dot{E}_{(p)}$ is *compactly supported* if $f \circ \iota = \iota$ outside a compact subset of M. We denote by $\operatorname{End}_c^r(\dot{E}_{(p)})$ the space of compactly supported endomorphisms which induce on fibres maps of degree r. By Theorem 3.3 in [Dol63], if r is a unit in $\mathbb{Z}_{(p)}$, then any endomorphism in $\operatorname{End}_c^r(\dot{E}_{(p)})$ admits a fibrewise homotopy inverse. Postcomposition with it induces a homotopy equivalence between path-components

$$\Gamma_c(\dot{E}_{(p)})_k \longrightarrow \Gamma_c(\dot{E}_{(p)})_{[\phi](k)},$$

where $[\phi]$ denotes the map induced by ϕ on π_0.

2.3. **The degree of a section.** Let β be a compactly supported section of $\pi \colon \dot{T}M \to M$, and let $\operatorname{Th}(\beta)$ be the Thom class in $H^n(\dot{T}M; \pi^*\mathcal{O})$, where $\mathcal{O}$ is the orientation sheaf of M. The β*-degree* of a compactly supported section α is

$$\deg_\beta(\alpha) = \alpha^*(\operatorname{Th}(\beta))^\vee \in H_0(M; \mathbb{Z}),$$

the Poincare dual in M of $\alpha^*\operatorname{Th}(\beta) \in H_c^n(M; \mathcal{O})$. If M is orientable, then $\operatorname{Th}(\beta)$ is the Poincare dual of $\beta_*[M] \in H_n(\dot{T}M; \mathbb{Z})$, and $\deg_\beta(\alpha)$ is also equal to the intersection product of $\alpha_*[M]$ and $\beta_*[M]$. We will write deg for $\deg_z$, where z is the zero section of $\dot{T}M$. Observe that this definition also applies to the bundle $\dot{T}M_{(p)}$, and the degree of a section is then an element in $H_0(M; \mathbb{Z}_{(p)})$.

Assume now that M is closed and orientable. The Gysin sequence for the sphere bundle $S^n \xrightarrow{i} \dot{T}M \xrightarrow{\pi} M$ splits an exact sequence

$$0 \longrightarrow H_n(S^n;\mathbb{Z}) \xrightarrow{i_*} H_n(\dot{T}M;\mathbb{Z}) \xrightarrow{\pi_*} H_n(M;\mathbb{Z}) \longrightarrow 0.$$

The zero section $z\colon M \to \dot{T}M$ is an inverse of π, so the group $H_n(\dot{T}M) \cong \mathbb{Z}\oplus\mathbb{Z}$ is generated by $i_*[S^n]$ and $z_*[M]$. The fibres over two different points give two disjoint representatives of $i_*[S^n]$, therefore $i_*[S^n]\cap i_*[S^n]=0$. On the other hand, the intersection of the zero section with itself is the Euler characteristic χ of M. And it is also clear that the intersection of $i_*[S^n]$ and $z_*[M]$ consists of a single point. The intersection products of $4k$ (resp. $4k+2$) dimensional manifolds are symmetric (antisymmetric). Therefore we have:

Lemma 2.2 *If M is connected, closed, orientable and of dimension n, then the intersection pairing of $\dot{T}M$ with respect to the above basis is given by*

$$\begin{pmatrix} 0 & 1 \\ (-1)^n & \chi \end{pmatrix}.$$

If α is a section of π, then $\alpha_*[M] = (\deg(\alpha)-\chi, 1)$ in this basis.

For the second claim, observe that α is an inverse of π too, so the second component of $\alpha_*[M]$ is the same as the second component of $z_*[M]$. The first component is obtained from the following equation:

$$\deg(\alpha) = \alpha_*[M]\cap z_*[M] = (a,1)\begin{pmatrix} 0 & 1 \\ (-1)^n & \chi \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a+\chi. \tag{2.3}$$

2.4. **Fibrewise homotopy equivalences of many degrees.** Let $V_2(E\oplus\epsilon)$ be the fibrewise Stiefel manifold of $E\oplus\epsilon$. If σ is a section of $\Gamma(V_2(E\oplus\epsilon)_{(p)})$ we denote by σ_0 the image of σ under the localisation of the map that forgets the second vector:

$$\Gamma(V_2(E\oplus\epsilon)_{(p)}) \longrightarrow \Gamma(S(E\oplus\epsilon)_{(p)}).$$

We denote by $\Gamma_c(V_2(E\oplus\epsilon)_{(p)})$ the space of sections σ such that σ_0 is compactly supported.

Lemma 2.3 *Let E be a vector bundle over a manifold M and let p be either a prime or a unit. For each $r\notin p\mathbb{Z}$ there are maps*

$$\Phi_r^p\colon \Gamma_c(V_2(E\oplus\epsilon)_{(p)}) \longrightarrow \mathrm{End}_c^r(\dot{E}_{(p)})$$

which are natural with respect to pullback of bundles. If M is closed and $E=TM$, then $\Phi_r^p(\sigma)$ sends sections of degree k to sections of degree $rk-(r-1)\deg(\sigma_0)$.

Proof. Assume first that $p=1$. Any $\sigma\in\Gamma_c(V_2(E\oplus\epsilon))$ determines an isomorphism between $E\oplus\epsilon$ and $E'\oplus E''$, where $E'=M\times\mathbb{R}^2$, and therefore an isomorphism φ_σ from $S(E\oplus\epsilon)$ to the fibrewise union $S(E')*S(E'')$. Define the map

$$\Phi_r := \Phi_r^1\colon \Gamma_c(V_2(E\oplus\epsilon)) \longrightarrow \mathrm{End}_c^r(\dot{E}).$$

by sending a section σ to $\Phi_r(\sigma) = \varphi_\sigma^{-1}(f_r * \mathrm{Id})\varphi_\sigma$, with

$$f_r * \mathrm{Id}\colon S(E')*S(E'') \longrightarrow S(E')*S(E'')$$

and $f_r(x,v) = (x, e^{2\pi i r})$. Observe now that $\Phi_r(\sigma)$ preserves σ_0, because the maps $e^{2\pi i r}$ preserve the basepoint of S^1, which corresponds under the trivialization to σ_0. In particular, since σ_0 is compactly supported, the map $\Phi_r(\sigma)$ sends compactly supported sections to compactly supported sections.

If $\Gamma_c(V_2(E\oplus\epsilon))$ is non-empty, then Φ_r^p is the localisation of Φ_r. Otherwise, we choose a good cover of M. Then the space of sections $\Gamma_c(V_2(E\oplus\epsilon))$ is non-empty over each open subset of the cover, hence there are maps Φ_r^p over each open subset, which glue together to define a map Φ_r^p on the whole M.

By construction, $f^*(\Phi_r^p(\sigma)) = \Phi_r^p(f^*(\sigma))$, so these maps are natural. Similarly, observe that

$$\mathrm{End}_c^r(\dot{E}_{(p)}) \times \Gamma(\dot{E}_{(p)}) \longrightarrow \Gamma(\dot{E}_{(p)}) \tag{2.4}$$

is also natural with respect to pullback of bundles.

Now we describe the effect of $\phi_r := \Phi_r^p(\sigma)$ on components of $\Gamma(\dot{T}M_{(p)})$ when M is closed. Assume first that M is orientable, in which case $\dot{T}M$ is also orientable and Lemma 2.2 applies. First we identify the induced map $(\phi_r)_*\colon H_n(\dot{E}_{(p)}) \to H_n(\dot{E}_{(p)})$. Since $\phi_r(\sigma_0) = \sigma_0$, we have

$$(\phi_r)_*(\deg(\sigma_0) - \chi, 1) = (\deg(\sigma_0) - \chi, 1).$$

On the other hand, ϕ_r acts on the fibre over a point as a map of degree r, hence

$$(\phi_r)_*(1,0) = (r,0).$$

From this we deduce that $(\phi_r)_*$ has the form

$$\begin{pmatrix} r & -(r-1)(\deg(\sigma_0)-\chi) \\ 0 & 1 \end{pmatrix},$$

hence, for an arbitrary section α, we have that

$$\begin{aligned}(\deg(\phi_r(\alpha)_*[M]) - \chi, 1) &= \phi_r(\alpha)_*[M] = (\phi_r)_*(\alpha_*[M]) \\ &= (r(\deg(\alpha)-\chi) - (r-1)(\deg(\sigma_0)-\chi), 1)\end{aligned}$$

and so $\deg(\phi_r(\alpha)) = r\deg(\alpha) - (r-1)\deg(\sigma_0)$.

Assume now that M is non-orientable. We take then the orientation cover $f\colon \tilde{M} \to M$. If s is a section of $\dot{T}M$ and σ is a section of $V_2(TM\oplus\epsilon)$, we can pull back both sections along f to obtain a section f^*s of $\dot{T}\tilde{M}$ and a section $f^*\sigma$ of $V_2(T\tilde{M}\oplus\epsilon)$. Then, because f is a double cover, $\deg(f^*s) = 2\deg(s)$, and by the naturality of ϕ_r and (2.4) we have that $\Phi_r(f^*\sigma)(f^*s) = f^*(\Phi_r(\sigma)(s))$. On the other hand, since $\tilde{M}$ is orientable, by the previous paragraph we now that $\deg(\Phi_r(f^*\sigma)(s)) = r\deg(f^*s) - (r-1)\deg(f^*\sigma_0)$. As a consequence:

$$\begin{aligned}2\deg(\Phi_r(\sigma)(s)) &= \deg(f^*(\Phi_r(\sigma)(s))) \\ &= \deg(\Phi_r(f^*\sigma)(f^*(s)) \\ &= r\deg(f^*s) - (r-1)\deg(f^*\sigma_0) \\ &= 2r\deg(s) - (r-1)2\deg(\sigma_0).\end{aligned}$$

□

We now face the following lifting problem:

$$\begin{tikzcd} & V_2(TM\oplus\epsilon)_{(p)} \arrow[d, "\varpi", "(p)"'] \\ M \arrow[ur, dashed, "\sigma"] \arrow[r, "\sigma_0"] & S(TM\oplus\epsilon)_{(p)}. \end{tikzcd}$$

Proposition 2.4 *If M is closed, of dimension $n \geqslant 2$ and n is odd, then every diagram has a lift, whereas if n is even only sections of degree $\chi/2$ (when they exist) have a lift.*

Proof. The above problem is equivalent to find a section of the pullback $\eta_{(p)}$ of $\varpi_{(p)}$ along σ_0, which is an $S^{n-1}_{(p)}$-bundle over an n-dimensional manifold. If n is odd, $\eta_{(p)}$ has always a section, hence in that case every section σ_0 admits a lift. If n is even, the complete obstruction (if M is orientable) is the Euler class $e(\eta_{(p)})$ of $\eta_{(p)}$. We proceed to compute it:

Assume first that M is orientable and $p = 1$. The bundle η is the unit sphere bundle of $\sigma_0^* T^v(TM \oplus \epsilon)$, whose Euler number can be computed by taking the self-intersection of its zero section in the fibrewise one point compactification of $\sigma_0^* T^v(TM \oplus \epsilon)$, which is precisely $S(TM \oplus \epsilon)$. As the zero section of $\sigma_0^* T^v(TM \oplus \epsilon)$ is σ_0, we have that (we denote by $x^\vee$ the Poincaré dual of x)

$$e(\eta)^\vee = \sigma_0[M] \cap \sigma_0[M] \tag{2.5}$$

$$= (\deg(\sigma_0) - \chi, 1) \begin{pmatrix} 0 & 1 \\ 1 & \chi \end{pmatrix} \begin{pmatrix} \deg(\sigma_0) - \chi \\ 1 \end{pmatrix} = 2\deg(\sigma_0) - \chi. \tag{2.6}$$

Hence a section admits a lift if and only if $\deg(\sigma_0) = \chi/2$.

Let us assume now that M is orientable and p is a prime. In this case, the above computation is no longer valid, as it relies on a geometric interpretation of the Euler class. We will first compute the Euler class e of $\varpi_{(p)}$:

$$e(\eta_{(p)}) = \sigma_0^*(e)^\vee = e \frown \sigma_0[M] = e^\vee \cap \sigma_0[M],$$

and therefore, if $e^\vee = (a, b)$ in the basis described before, it holds that

$$e(\eta_{(p)})^\vee = \sigma_0^*(e)^\vee = (a, b) \begin{pmatrix} 0 & 1 \\ 1 & \chi \end{pmatrix} \begin{pmatrix} \deg(\sigma_0) - \chi \\ 1 \end{pmatrix} = a + b\deg(\sigma_0).$$

This, together with (2.5) (which holds for integral values), implies that $e^\vee = (-\chi, 2)$, and therefore that $\sigma_0^*(e)^\vee = 2\deg(\sigma_0) - \chi$. Hence, after localising we obtain that only sections of degree $\chi/2$ admit a lift.

Finally, let M be non-orientable and let $f\colon \tilde{M} \to M$ be the orientation cover of M. Then $\deg_{f^*\sigma_0}(f^*\sigma_0) = 2\deg_{\sigma_0}(\sigma_0)$ and the Euler characteristic of $\tilde{M}$ is 2χ, so $\deg_{\sigma_0}(\sigma_0) = 0$ if and only if $(2\chi)/2 = \deg(f^*\sigma_0) = 2\deg(\sigma_0)$. Hence only sections of degree $\chi/2$ have lifts. □

Proof of Theorem A. If the dimension of M is odd and ℓ is a set of primes, then by Proposition 2.4 and Lemma 2.3, there exist homotopy equivalences

$$\Gamma_c(\dot{T}M_{(\ell)})_k \longrightarrow \Gamma_c(\dot{T}M_{(\ell)})_{rk-(r-1)d}$$

for all integers r and d such that $r \notin \ell\mathbb{Z}$. Observe first that if r is odd, then k and $rk - (r-1)d$ have the same parity. Hence if k and j have different parity then a homotopy equivalence $\Gamma_c(\dot{T}M_{(\ell)})_k \longrightarrow \Gamma_c(\dot{T}M_{(\ell)})_j$ as above exists only if $2 \notin \ell$.

Taking $r = -1$ and d arbitrary we obtain maps which induce $\mathbb{Z}_{(\ell)}$-homotopy equivalences for all ℓ between every pair of components of $\Gamma_c(\dot{T}M)$ of sections with the same parity. Taking $r = 2$ and d arbitrary, we obtain $\mathbb{Z}_{(\ell)}$-homotopy equivalences for all ℓ between every pair of components of $\Gamma_c(\dot{T}M)$.

If the dimension of M is even and χ is even (resp. odd) and ℓ is a set of primes (resp. odd primes), we use Proposition 2.4 and Lemma 2.3, to construct, for each integer r with trivial ℓ-adic valuation, a homotopy equivalence

$$\Gamma_c(\dot{T}M_{(\ell)})_k \longrightarrow \Gamma_c(\dot{T}M_{(\ell)})_{rk-(r-1)\chi/2}.$$

It is clear that $2k - \chi$ and $2(rk - (r-1)\chi/2) - \chi = r(2k - \chi)$ have the same ℓ-adic valuation if $p \nmid r$ for each $p \in \ell$. If $2k - \chi$ and $2j - \chi$ have the same ℓ-adic valuation l, and $s = (2k - \chi)/\prod_{p\in\ell} p^{l(p)}$ and $r = (2j - \chi)/\prod_{p\in\ell} p^{l(p)}$, then $rk - (r-1)\chi/2 = sj + (s-1)\chi/2$, and therefore there is a zig-zag of $\mathbb{Z}_{(\ell)}$-homotopy equivalences for all ℓ such that $r, s \notin \ell\mathbb{Z}$ between $\Gamma_c(\dot{T}M_{(\ell)})_k$ and $\Gamma_c(\dot{T}M_{(\ell)})_j$.

Theorem A is then a consequence of Theorem 2.1 applied to diagram (2.1).

For the last claim, observe that if M has even dimension, taking $r = -1$ one obtains a homotopy equivalence $\Gamma_c(\dot{T}M) \to \Gamma_c(\dot{T}M)$ (without localising) that sends sections of degree k to sections of degree $\chi - k$. This homotopy equivalence may be obtained by postcomposition with the antipodal map $\dot{T}M \to \dot{T}M$. □

3. The extrinsic replication map

Theorem B. *Let M be a connected, smooth manifold and let v be a non-vanishing section of TM. Then there exists a map $\phi_r \in \mathrm{End}_c^r(\dot{T}^1M)$ that makes the following diagram commute up to homotopy:*

$$\begin{tikzcd} C_k^\delta(M) \arrow[r, "\mathscr{S}"] \arrow[d, "\rho_r[v]"] & \Gamma_c(\dot{T}^1M \to M)_k \arrow[d, "\phi_r"] \\ C_{rk}^\delta(M) \arrow[r, "\mathscr{S}"] & \Gamma_c(\dot{T}^1M \to M)_{rk}. \end{tikzcd}$$

Hence the r-replication map induces an isomorphism on $\mathbb{Z}_{(p)}$-homology in the stable range with $\mathbb{Z}_{(p)}$ coefficients for all primes $p \nmid r$.

Remark 3.1 It can be proven that the map ϕ_r is homotopic to $\Phi_r(\iota, v)$.

Proof. The proof has two steps. First, since $C_k^\delta(M)$ is independent of δ up to homotopy, we let 2δ be smaller than the injectivity radius of M. We claim that the following diagram commutes on the nose:

$$\begin{tikzcd} C_k^\delta(M) \arrow[r, "\mathfrak{s}^{2\delta}"] \arrow[d, "\rho_r[v]"] & \Gamma_c(\psi^\delta(T^1M)) \arrow[d, "\varsigma_r"] \\ C_{rk}^\delta(M) \arrow[r, "\mathfrak{s}^{\delta}"] & \Gamma_c(\psi^\delta(T^1M)) \end{tikzcd}$$

where ς_r is given by postcomposition with the bundle map $\rho_r[\exp_{2\delta}^*(v)] \colon \psi^\delta(T^1M) \to \psi^\delta(T^1M)$ followed by the expansion $\mathfrak{2} \colon \psi^\delta(T^1M) \to \psi^\delta(T^1M)$ that sends each point q in the configuration to $2q$. Observe that the bundle map $\rho_r[\exp_{2\delta}^*(v)]$ is not continuous but it becomes continuous after composing with $\mathfrak{2}$.

In order to understand this square, we check what happens with the adjoint of the scanning map $M \times C_k(M) \to \psi^\delta(T^1M)$ over each point $x \in M$:

$$\begin{tikzcd} \{x\} \times C_k^\delta(M) \arrow[r, "\mathfrak{s}_x^{2\delta}"] \arrow[d, "\rho_r[v]"] & \psi^\delta(T_x^1M) \arrow[d, "\varsigma_r"] \\ \{x\} \times C_{rk}^\delta(M) \arrow[r, "\mathfrak{s}_x^{2\delta}"] & \psi(T_x^1M). \end{tikzcd}$$

The square commutes on the nose unless there exists some $q \in \mathbf{q}$ such that

$$\varsigma_r(q) \cap B_\delta(x) \neq \emptyset, \text{and } q \notin B_{2\delta}(x).$$

But this is not possible, as $d(\varsigma_r(q), x) \geqslant d(q, x) - \max_{q' \in \varsigma_r(q)} d(q, q') \geqslant 2\delta - \epsilon \geqslant \delta$.

Second, observe that since the exponential map is homotopic to the projection $\pi \colon TM \to M$, the maps $\varsigma_r = \mathfrak{2}\rho_r[\exp_{2\delta}^*(v)]$ and $\sigma_r = \mathfrak{2}\rho_r[\pi^* v]$ are homotopic.

Third, consider now the diagram

$$\begin{tikzcd} \Gamma_c(\psi^\delta(T^1M)) \arrow[d, "\sigma_r"] & \Gamma_c(\dot{T}^1M) \arrow[l, "i"'] \arrow[d] \\ \Gamma_c(\psi^\delta(T^1M)) \arrow[r, "h"] & \Gamma_c(\dot{T}^1M) \end{tikzcd}$$

whose maps are induced by the fibrewise maps which on each fibre are

$$\begin{tikzcd} \psi^\delta(T^1_xM) \arrow[d, "\sigma_r"] & \dot{T}^1_xM \arrow[l, "i"'] \arrow[d] \\ \psi^\delta(T^1_xM) \arrow[r, "h"] & \dot{T}^1_xM \end{tikzcd}$$

Let us denote by v the value of the vector field v at the point x. Then $\sigma_r(q,1) = q \cup q+v \cup \ldots \cup q+(r-1)v$ and

$$h\sigma_r i(q) = \begin{cases} 2\frac{q+jv}{\|q+(j-1)v\|} & \text{if } q+jv \text{ is the closest point and } \langle v, q+jv\rangle > 0 \\ 2\frac{q+jv}{\|q+(j+1)v\|} & \text{if } q+jv \text{ is the closest point and } \langle v, q+jv\rangle < 0. \end{cases}$$

The inverse image of a point (for instance the origin) consists of r points ($\{-jv\}_{j=0}^{r-1}$), all of them oriented according to the sign of r. Hence $h\sigma_r i$ induces a map of degree r on fibres. □

Corollary 3.2 *If M is a connected open manifold of dimension at least 2, then the homomorphism induced in $\mathbb{Z}_{(p)}$-homology by the r-replication map for $r \notin p\mathbb{Z}$ is injective.*

Proof. The scanning map is injective in homology in all degrees as can be deduced from [McD75, p. 103], from the fact that the stabilisation map $C_k(M) \longrightarrow C_{k+1}(M)$ is injective in homology and that $\operatorname{colim}_k H_*(\Gamma_c(\dot{T}M)_k)$ ($\lim_k H_*(\Gamma_k(M, \partial M))$ in the notation of that paper) is constant.

Therefore, in the commutative square of the previous proposition the composite $\phi_r\mathscr{S}$ is injective in $\mathbb{Z}_{(p)}$-homology, hence $\mathscr{S}\rho_r$ is injective in $\mathbb{Z}_{(p)}$-homology too, so ρ_r is injective in $\mathbb{Z}_{(p)}$-homology. □

4. The intrinsic replication map

We first recall the statement of McDuff's theorem (Theorem 2.1) with labels: Let $\theta \colon X \to M$ be a fibration, and recall from the introduction the (weak) homotopy equivalent spaces

$$C_k(M;\theta) = \{(\mathbf{q}, f) \mid \mathbf{q} \in C_k(M), f \in \Gamma(\theta_{|\mathbf{q}})\}$$
$$C_k^\delta(M;\theta) = \{(\mathbf{q}, \epsilon, f) \mid (\mathbf{q}, \epsilon) \in C_k^\delta(M), f \in \Gamma(\theta_{|B_{\mathbf{q}}(\epsilon)}).$$

The pullback $\theta^*TM \to X$ is also fibred over M, and the fibres are vector bundles. We denote by $\dot{T}^\theta M$ the fibrewise Thom construction of θ^*TM viewed as a bundle over M. The inclusion of the points at infinity define a cofibre sequence over M

$$X \longrightarrow \theta^*\dot{T}M \longrightarrow \dot{T}^\theta M.$$

The pullback map $\theta^*\dot{T}M \to \dot{T}M$ factors through the bundle maps

$$\theta^*\dot{T}M \longrightarrow \dot{T}^\theta M \xrightarrow{\xi} \dot{T}M. \tag{4.1}$$

We define the *degree* of a section s as the degree of $\xi(s)$. If the fibres of θ are path connected, then the n-skeleton of the fibres of $\dot{T}^\theta M$ is homotopic to S^n, therefore the forgetful map

$$\Gamma_c(\dot{T}^\theta M) \longrightarrow \Gamma_c(\dot{T}M)$$

induces a bijection on connected components.

Theorem 4.1 (McDuff's with labels) *There is a scanning map*

$$\mathscr{S}^\theta \colon C_k^\delta(M;\theta) \longrightarrow \Gamma_c(\dot{T}^\theta M)_k$$

which is a homology equivalence in a range if the fibres of θ are path-connected. We refer to this range as the stable range with labels.

The fibrewise homotopy equivalences of Lemma 2.3 lift to fibrewise homotopy equivalences of $\theta^*\dot{T}M_{(p)}$, which in turn descend to fibrewise homotopy equivalences $\dot{T}^\theta M_{(p)}$ if and only if they fix the section at infinity. This implies that $\sigma_0 = \iota$ in that lemma, and therefore all of them send sections of degree k to sections of degree rk. Hence we only recover part of Theorem A:

Theorem A′ *If M is a closed, connected manifold with trivial Euler characteristic, then $H_*(C_k(M;\theta)) \cong H_*(C_j(M;\theta))$ in the stable range with labels if k and j have the same p-adic valuation.*

On the other hand, Theorem B generalizes in full generality:

Theorem B′ *If v is a non-vanishing vector field on a connected manifold M and $p \nmid r$, then the r-replication map ρ_r^θ with labels induces isomorphisms in $\mathbb{Z}_{(p)}$-homology in the stable range with labels.*

Let θ be the projection $S(TM) \to M$. Recall the definition of the intrinsic replication map $\mathfrak{z}_r \colon C_k^\delta(M;\theta) \longrightarrow C_{rk}^\delta(M;\theta)$ from page 4.

Theorem C *If M is a connected manifold, the map $\mathfrak{z}_r \colon C_k^\delta(M;\theta) \to C_{rk}^\delta(M;\theta)$ induces isomorphisms on homology with $\mathbb{Z}[\frac{1}{r}]$-coefficients in the stable range with labels. Hence if k and j have the same p-adic valuation, the groups $H_*(C_k^\delta(M;\theta);\mathbb{Z}_{(p)})$ and $H_*(C_j^\delta(M;\theta);\mathbb{Z}_{(p)})$ are isomorphic in the stable range with labels.*

Proof. Define $\sigma_r \colon \psi(TM;\theta) \to \psi(TM;\theta)$ to be the fibrewise version of $\mathfrak{z}_r$ composed with $\mathfrak{L}$ as in the proof of Theorem B. The first square in the following diagram

$$\begin{tikzcd}
C_k^\delta(M;\theta) \arrow[r] \arrow[d, "\mathfrak{z}_r"] & \Gamma_c(\psi(TM;\theta)) \arrow[d, "\sigma_r"] & \Gamma_c(\dot{T}^\theta M)_k \arrow[l, "i"'] \arrow[d, "h\sigma_r i"] \\
C_{rk}^\delta(M;\theta) \arrow[r] & \Gamma_c(\psi(TM;\theta)) \arrow[r, "h"] & \Gamma_c(\dot{T}^\theta M)_{rk}.
\end{tikzcd} \tag{4.2}$$

commutes, by the same argument as the first step the proof of Theorem B. Therefore we obtain a fibrewise map $h\sigma_r i$ on the right hand side. The map $h\sigma_r i$ is obtained by postcomposition with a fibrewise map $g \colon \dot{T}^\theta M \to \dot{T}^\theta M$. The map g_x on the fibre over the point x restricts to the identity on the points at infinity, therefore it extends to the following diagram of cofibre sequences

$$\begin{tikzcd}
S(T_xM) \times \{\infty\} \arrow[r] \arrow[d, "\mathrm{Id}"] & S(T_xM) \times \dot{T}_xM \arrow[r] \arrow[d, "f"] & \dot{T}_x^\theta M \arrow[d, "g_x"] \\
S(T_xM) \times \{\infty\} \arrow[r] & S(T_xM) \times \dot{T}_xM \arrow[r] & \dot{T}_x^\theta M
\end{tikzcd}$$

where the leftmost horizontal maps are the inclusion of the points at infinity. After localizing the diagram at a prime p not dividing r, the map $f_{(p)}$ is a map of sphere bundles that induces a map of degree r on fibres (as in the proof of Theorem B). Since r is a unit in $\mathbb{Z}_{(p)}$, it follows that $f_{(p)}$ is a homotopy equivalence, and therefore that $(g_x)_{(p)}$ is a homotopy equivalence .

Since the fibrewise localisation of $g_{(p)}$ induces a homotopy equivalence on fibres, it follows that g is a homotopy equivalence as well, and so is $(h\sigma_i r)_{(p)}$. $\square$

If the bundle $\theta\colon X \to M$ factors through $S(TM)$ and has path-connected fibres (for instance the oriented frame bundle of TM, if M is orientable), then Theorem C generalises to:

Theorem C′ *If $p \nmid r$, then the intrinsic replication map with labels*

$$\mathfrak{z}_r\colon C_k(M;\theta) \longrightarrow C_{rk}(M;\theta)$$

induces isomorphisms on homology with $\mathbb{Z}_{(p)}$ coefficients in the stable range with labels.

We can use now the zigzag $C_k^\delta(M) \leftarrow C_k^\delta(M;\theta)) \to C_{rk}^\delta(M;\theta) \to C_{rk}^\delta(M)$ of homology isomorphisms to try to deduce something more about the homology of $C_k(M)$ is low dimensions. The remainder of this section is a complicated way of proving part of the following trivial statement. The purpose of doing so is to say explicitly what can be done with this new approach.

Remark 4.2 The homology groups $H_*(C_k(M);A)$ and $H_*(C_{k+1}(M);A)$ are independent of k provided that $* < n-1$ and $*$ belongs to the stable range with labels with A-coefficients. This follows from the cofibre sequence (defined in (5.1))

$$C_k(M \setminus \{x\}) \longrightarrow C_k(M) \longrightarrow \Sigma^n(C_{k-1}(M \setminus \{x\})_+)$$

where $x \in M$, the left hand side satisfies homological stability and the right hand side is $(n-1)$-connected.

The fibre of the map that forgets the vectors

$$C_k^\delta(M;\theta) \longrightarrow C_k^\delta(M)$$

is homotopy equivalent to a product of k spheres of dimension $n-1$. The fundamental group of $C_k(M)$ acts on this product interchanging the factors (and also changing their orientation if M is not orientable). By the relative Hurewicz theorem, the first non-trivial relative homology group of this pair is

$$H_n(C_k^\delta(M), C_k^\delta(M;\theta)) \cong \frac{\pi_n(C_k^\delta(M), C_k^\delta(M;\theta))}{\pi_1(C_k^\delta(M))} \cong \begin{cases} \mathbb{Z} & \text{if } M \text{ is orientable,} \\ \mathbb{Z}/2 & \text{if } M \text{ is not orientable.} \end{cases}$$

Therefore if $* \leqslant n-2$ and $*$ is in the stable range with labels of k and j, there are isomorphisms

$$H_*(C_k(M);\mathbb{Z}_{(p)}) \cong H_*(C_j(M);\mathbb{Z}_{(p)})$$

induced by zigzag of maps whenever k and j have the same p-adic valuation. In addition if $n-1$ belongs to the stable range with labels, there are exact sequences

$$\begin{aligned} \mathbb{Z}_{(p)} \longrightarrow A_{p,k} \longrightarrow H_{n-1}(C_k(M);\mathbb{Z}_{(p)}) \longrightarrow 0 \quad & \text{if } M \text{ is orientable} \\ (\mathbb{Z}/2)_{(p)} \longrightarrow A_{p,k} \longrightarrow H_{n-1}(C_k(M);\mathbb{Z}_{(p)}) \longrightarrow 0 \quad & \text{if } M \text{ is not orientable,} \end{aligned}$$

where the $A_{p,k} = H_{n-1}(C_k(M;\theta);\mathbb{Z}_{(p)})$ depends only on the p-adic valuation of k.

From this together with Theorem A we deduce that if $* \leqslant n-2$ (or $* = n-1$, $p \neq 2$ and M is not orientable) there are isomorphisms

$$H_*(C_k(M);\mathbb{Z}_{(p)}) \;\cong\; H_*(C_j(M);\mathbb{Z}_{(p)})$$

in the stable range with labels under the following conditions (recall that $(k)_p$ denotes the p-adic valuation of k):

(1)	χ even,	if $(k)_p = (j)_p$ or $(k-\chi/2)_p = (j-\chi/2)_p$
(2)	χ odd, $p \neq 2$	if $(k)_p = (j)_p$ or $(k-\chi/2)_p = (j-\chi/2)_p$
(3)	χ odd, $p = 2$	if $(k)_2 = (j)_2$

Corollary 4.3 *Let M be a closed, connected, smooth n-manifold with n even and Euler characteristic $\chi \neq 0$. Assume that $* \leqslant n-2$ (or $* = n-1$, $p \neq 2$ and M non-orientable). Assume in addition that one of the following holds:*

- *χ is even and $(k)_p, (j)_p \geqslant (\frac{\chi}{2})_p$*
- *χ is odd, $p \neq 2$ and $(k)_p, (j)_p \geqslant (\frac{\chi}{2})_p$*
- *$(k)_p = (j)_p$*

Then $H_*(C_k(M);\mathbb{Z}_{(p)}) \;\cong\; H_*(C_j(M);\mathbb{Z}_{(p)})$ in the stable range with labels of the scanning map with labels. Therefore for any fixed j, k there are isomorphisms

$$H_*(C_k(M);\mathbb{Z}[\tfrac{2}{\chi}]) \;\cong\; H_*(C_j(M);\mathbb{Z}[\tfrac{2}{\chi}]) \qquad \textit{if } \chi \textit{ is even}$$
$$H_*(C_k(M);\mathbb{Z}[\tfrac{1}{2\chi}]) \;\cong\; H_*(C_j(M);\mathbb{Z}[\tfrac{1}{2\chi}]) \qquad \textit{if } \chi \textit{ is odd}$$

when $* \leqslant n-2$ and $*$ is in the stable range with labels for j and k. If $n-1$ belongs to the stable range with labels for j and k then we also have exact sequences

$$\mathbb{Z} \longrightarrow A \longrightarrow H_{n-1}(C_k(M);\mathbb{Z}) \longrightarrow 0 \qquad \textit{if } M \textit{ is orientable}$$
$$\mathbb{Z}/2 \longrightarrow A \longrightarrow H_{n-1}(C_k(M);\mathbb{Z}) \longrightarrow 0 \qquad \textit{if } M \textit{ is non-orientable}$$

where the p-torsion of $A = H_{n-1}(C_k(M;\theta);\mathbb{Z})$ depends only on $(k)_p$ and the p-torsion of $H_{n-1}(C_k(M);\mathbb{Z})$ depends only on $(k-\frac{\chi}{2})_p$ (unless $p = 2$ and χ is odd).

Proof. If $(k)_p = (j)_p$ we are done. If $(k)_p, (j)_p > (\chi/2)_p$, then $(k-\chi/2)_p = (\chi/2)_p = (j-\chi/2)_p$. If $(j)_p > (k)_p = (\chi/2)_p$, then pick $i \geqslant j, k$ with $(i)_p = (j)_p$ and set $l = i + \chi/2$. Then $(j)_p = (\chi/2)_p = (l)_p$ and $(l-\chi/2)_p = (\chi/2)_p = (j-\chi/2)_p$. Notice that in the last case the stable range with labels for l is bigger than the stable ranges with labels for k and j. If $(k)_p < (\chi/2)_p$, then $(k-\chi/2)_p = (k)_p$, so both conditions are equivalent. □

5. Homological stability via vector fields with exactly one zero

We now use some different techniques to extend our results a bit further for homology with field coefficients. Section spaces are not involved in this part; instead we apply Theorem B (homological stability with respect to the r-replication map) to $M \smallsetminus \{*\}$ and classical homological stability for $M \smallsetminus \{*\}$ to obtain Theorem D.

5.1. Vector fields.

Let M be a closed connected manifold with Euler characteristic χ.

Definition 5.1 Given a vector field $X \in \Gamma(TM)$ with an isolated zero $z \in M$, define the degree $\deg_X(z)$ of z as follows. Choose a coordinate chart $U \cong \mathbb{R}^n$ with $z \in U$ so that there are no other zeros of X in U and choose a trivialisation of $TM|_U$. The restriction of X to $U \smallsetminus \{z\}$ is in this way identified with a map $\mathbb{R}^n \smallsetminus \{0\} \to \mathbb{R}^n \smallsetminus \{0\}$. The degree of this map is by definition $\deg_X(z)$.

A simple observation is that M admits a vector field with at most one zero, which will therefore have index χ by the Poincaré-Hopf theorem. Moreover we can choose exactly what this zero looks like locally:

Lemma 5.2 *Suppose we are given a vector field X on a closed ball $B \subseteq M$ with exactly one zero which lies in its interior and has index χ. Then this extends to a vector field $\hat{X}$ on M which is non-vanishing on $M \smallsetminus B$.*

Proof. First choose a vector field X' on M which has only isolated (and therefore finitely many) zeros. Choose a larger closed ball $B' \supset B$ and a trivialisation of $TM|_{B'}$. Now homotope X' if necessary so that all its zeros lie in $\mathrm{int}(B') \smallsetminus B$. The restriction of X' to $\partial B'$ is a map $\partial B' \to \mathbb{R}^n \smallsetminus \{0\}$, whose degree is the sum of the degrees of all zeros of X', which is χ by the Poincaré-Hopf theorem. The restriction of the vector field X to ∂B is a map $\partial B \to \mathbb{R}^n \smallsetminus \{0\}$ which also has degree χ by assumption. Since any two maps $S^{n-1} \to \mathbb{R}^n \smallsetminus \{0\}$ of the same degree are homotopic, there is a map $X'' \colon B' \smallsetminus \mathrm{int}(B) \cong S^{n-1} \times [0,1] \to \mathbb{R}^n \smallsetminus \{0\}$ agreeing with X' on $\partial B'$ and with X on ∂B. We can therefore define $\hat{X}$ to be equal to X on B, X'' on $B' \smallsetminus B$ and X' on $M \smallsetminus B'$. □

5.2. **A cofibre sequence of configuration spaces.** Choose a Riemannian metric on M and an isometric embedding $D \hookrightarrow M$ of the closed unit disc $D \subseteq \mathbb{R}^n$. Following [RW13, §6] we define $U_k(M)$ to be the subspace of $C_k(M)$ of configurations which have a unique closest point in D to its centre $0 \in D$. There is an open cover of $C_k(M)$ given by the subsets $U_k(M)$ and $C_k(M \smallsetminus \{0\})$, so by excision the homotopy cofibre of the inclusion $C_k(M \smallsetminus \{0\}) \hookrightarrow C_k(M)$ is homology-equivalent to that of the inclusion $U_k(M \smallsetminus \{0\}) \hookrightarrow U_k(M)$. This map is homeomorphic to the inclusion $D \smallsetminus \{0\} \hookrightarrow D$ times the identity on $C_{k-1}(M \smallsetminus \{0\})$, so its homotopy cofibre is homotopy equivalent to $\Sigma^n(C_{k-1}(M \smallsetminus \{0\})_+)$.

By the above discussion we have a homology cofibre sequence:[1]

$$C_k(M \smallsetminus \{0\}) \longrightarrow C_k(M) \longrightarrow \Sigma^n(C_{k-1}(M \smallsetminus \{0\})_+). \tag{5.1}$$

The connecting homomorphism of the long exact sequence on homology induced by this can be described as follows. Define a map

$$t_{k-1} \colon S^{n-1} \times C_{k-1}(M \smallsetminus \{0\}) \longrightarrow C_k(M \smallsetminus \{0\})$$

which radially expands the configuration in $C_{k-1}(M \smallsetminus \{0\})$ away from 0 until it has no points in D, and then adds the point in $S^{n-1} = \partial D$ to the configuration. By the Künneth theorem we have a decomposition of the reduced homology of the domain of this map:

$$\begin{aligned} &\widetilde{H}_{*-1}(S^{n-1} \times C_{k-1}(M \smallsetminus \{0\})) \\ &\cong \widetilde{H}_{*-1}(C_{k-1}(M \smallsetminus \{0\})) \oplus H_{*-n}(C_{k-1}(M \smallsetminus \{0\})) \\ &\cong \widetilde{H}_{*-1}(C_{k-1}(M \smallsetminus \{0\})) \oplus \widetilde{H}_*(\Sigma^n(C_{k-1}(M \smallsetminus \{0\})_+)). \end{aligned} \tag{5.2}$$

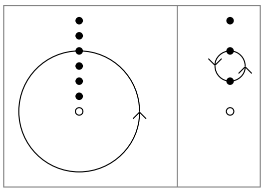
The restriction of the induced map $\widetilde{H}_{*-1}(t_{k-1})$ to the second direct summand above is the connecting homomorphism of the long exact sequence on homology induced by (5.1).

[1]In general, this means that there is a zig-zag of maps of diagrams which are objectwise homology equivalences between this diagram and one of the form $A \to B \to \mathit{hocofib}(A \to B)$. In this case the zig-zag can be taken to have length one and be the identity on the left and middle spaces of the diagram.

5.3. **Configuration spaces on cylinders.** For the remainder of this section $n = \dim(M)$ will always be assumed even. As pointed out in the introduction, Theorem D follows from Theorem A when M is odd-dimensional, so this is not a problem.

Some natural homology classes. We will need to do some calculations inside the homology group $H_{n-1}(C_k(\mathbb{R}^n \smallsetminus \{0\});\mathbb{Z})$ of punctured Euclidean space. There are certain natural elements of this group which one can write down. For example we have the following elements (see also Figure 5.1):

(a) For any $0 \leqslant i \leqslant k-1$ we have a map $\Delta_i \colon S^{n-1} \to C_k(\mathbb{R}^n \smallsetminus \{0\})$ which sends $v \in S^{n-1}$ to the configuration $\{v, p_1, \ldots, p_{k-1}\}$, where $p_1, \ldots, p_{k-1}$ are arbitrary fixed points in $\mathbb{R}^n \smallsetminus \{0\}$ with $|p_j| < 1$ for $j \leqslant i$ and $|p_j| > 1$ for $j > i$. By abuse of notation we denote the element $(\Delta_i)_*([S^{n-1}])$ simply by $\Delta_i \in H_{n-1}(C_k(\mathbb{R}^n \smallsetminus \{0\});\mathbb{Z})$. We will systematically use this abuse of notation for maps $S^{n-1} \to C_k(\mathbb{R}^n \smallsetminus \{0\})$.
(b) We also have a map $\pi \colon \mathbb{RP}^{n-1} \to C_k(\mathbb{R}^n \smallsetminus \{0\})$ which sends $\{v, -v\} \in \mathbb{RP}^{n-1} = S^{n-1}/\sim$ to the configuration $\{\underline{2}+v, \underline{2}-v, p_1, \ldots, p_{k-2}\}$, where $\underline{2} = (2, 0, \ldots, 0)$ and $p_1, \ldots, p_{k-2}$ are fixed points in $\mathbb{R}^n \smallsetminus B_1(\underline{2})$. This gives us an element $\pi \in H_{n-1}(C_k(\mathbb{R}^n \smallsetminus \{0\});\mathbb{Z})$.
(c) Composing this map with the double covering $S^{n-1} \to \mathbb{RP}^{n-1}$ gives a map representing 2π. This is homotopic to the map $\tau \colon S^{n-1} \to C_k(\mathbb{R}^n \smallsetminus \{0\})$ which sends $v \in S^{n-1}$ to the configuration $\{p_1, s(v), p_2, \ldots, p_{k-1}\}$, where $s \colon S^{n-1} \to \mathbb{R}^n \smallsetminus \{0\}$ is an embedding so that p_1 is in the interior of $s(S^{d-1})$ and $0, p_2, \ldots, p_{k-1}$ are in its exterior.
(d) More generally, for any $1 \leqslant i \leqslant k-1$ we can define a map $\tau_i \colon S^{n-1} \to \mathbb{R}^n \smallsetminus \{0\}$ which sends $v \in S^{n-1}$ to the configuration $\{p_1, \ldots, p_i, s(v), p_{i+1}, \ldots, p_{k-1}\}$, where $p_1, \ldots, p_i$ are in the interior of $s(S^{n-1})$ and $0, p_{i+1}, \ldots, p_{k-1}$ are in its exterior. So $\tau_1 = \tau = 2\pi$.



Δ_4 π

FIGURE 5.1. Examples of homology classes in $H_1(C_6(\mathbb{R}^2 \smallsetminus \{0\});\mathbb{Z})$ and $H_1(C_3(\mathbb{R}^2 \smallsetminus \{0\});\mathbb{Z})$ respectively. The small circle denotes the puncture 0 and bullets denote points of the configuration.

Relations between homology classes. Let P^n denote the closed n-dimensional disc $\mathbb{D}^n$ with two open subdiscs (whose closures are disjoint) removed; this is the n-dimensional pair-of-pants. Consider the map $r \colon P^n \to C_k(\mathbb{R}^n \smallsetminus \{0\})$ pictured in Figure 5.2. The image $r_*([\partial P^n])$ of the fundamental class of its boundary is the class $\Delta_{i+1} - \Delta_i - \tau_1$, which is therefore equal to zero in $H_{n-1}(C_k(\mathbb{R}^n \smallsetminus \{0\});\mathbb{Z})$. Similarly the map $r' \colon P^n \to C_k(\mathbb{R}^n \smallsetminus \{0\})$ pictured in Figure 5.2 shows that $\tau_{i+1} - \tau_i - \tau_1 = 0$. Hence by induction and the fact that $\tau_1 = 2\pi$ we have

$$\Delta_i = \Delta_0 + 2i\pi \qquad \text{and} \qquad \tau_i = 2i\pi. \tag{5.3}$$

Now let $\hat{\Delta}$ denote the image of the fundamental class under the map $S^{n-1} \to C_k(\mathbb{R}^n \smallsetminus \{0\})$ which sends v to $\{v, 2v, \ldots, kv\}$. This map can be homotoped to the

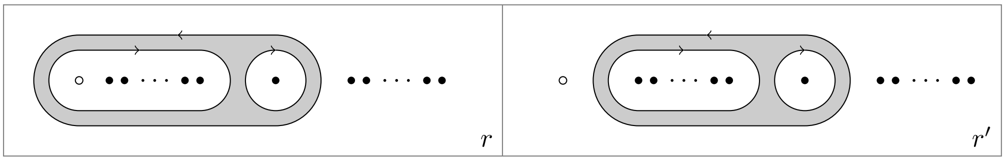


FIGURE 5.2. Pictures of maps $r, r' \colon P^n \to C_k(\mathbb{R}^n \smallsetminus \{0\})$ such that $r_*([\partial P^n]) = \Delta_{i+1} - \Delta_i - \tau_1$ and $r'_*([\partial P^n]) = \tau_{i+1} - \tau_i - \tau_1$. In each case there are $i+1$ fixed points in the bounded white region and $k-i-2$ fixed points in the unbounded white region. The remaining point is in the shaded region; its position is parametrised by P^n.

map
$$S^{n-1} \longrightarrow S^{n-1} \vee \cdots \vee S^{n-1} \longrightarrow C_k(\mathbb{R}^n \smallsetminus \{0\})$$
which collapses $k-1$ equators to get a wedge sum of k copies of S^{n-1} and then applies the maps $\Delta_0, \ldots, \Delta_{k-1}$ on these summands. Hence $\hat{\Delta} = \Delta_0 + \cdots + \Delta_{k-1}$ and so by (5.3),
$$\hat{\Delta} = k\Delta_0 + k(k-1)\pi. \tag{5.4}$$
Similarly we let $\hat{\tau}$ denote the image of the fundamental class under the map $S^{n-1} \to C_k(\mathbb{R}^n \smallsetminus \{0\})$ which sends v to $p_1 + \{0, v, 2v, \ldots, (k-1)v\}$, where p_1 is a fixed point in $\mathbb{R}^n$ with $|p_1| \geqslant k$. Just as above, we can homotope this to see that $\hat{\tau} = \tau_1 + \cdots \tau_{k-1}$ and so by (5.3),
$$\hat{\tau} = k(k-1)\pi. \tag{5.5}$$
One can see this very directly in the case $n = 2$. In this case we are talking about $H_1(C_k(\mathbb{R}^2); \mathbb{Z}) = \beta_k/[\beta_k, \beta_k] = \mathbb{Z}\{\pi\}$, where β_k denotes the braid group on k strands. Any one of the standard generators $\sigma_1, \ldots, \sigma_{k-1}$ of β_k, which interchange two consecutive strands, is sent to the generator π. The element $\hat{\tau}$ is the image of the full twist of all k strands, which can be written as a product of $k(k-1)$ generating elements, and so after abelianisation we have $\hat{\tau} = k(k-1)\pi$.

We now apply the above discussion to prove the following:

Lemma 5.3 *For any map $f \colon S^{n-1} \to S^{n-1}$ define $\sigma_f \colon S^{n-1} \to C_k(\mathbb{R}^n \smallsetminus \{0\})$ by sending v to $\{v, v + \frac{1}{k} f(v), \ldots, v + \frac{k-1}{k} f(v)\}$. Denoting the image of the fundamental class under this map also by σ_f we have*
$$\sigma_f = k\Delta_0 + \deg(f) k(k-1)\pi. \tag{5.6}$$

Proof. Note that if $\deg(f) = 1$ then $\sigma_f = \sigma_{\mathrm{id}} = \hat{\Delta}$ so this is just (5.4). In general this can be seen as follows. Write $d = \deg(f)$ and first assume that $d > 0$.

Denote the constant map to the basepoint by $* \colon S^{n-1} \to S^{n-1}$ and the map $S^{n-1} \to S^{n-1} \vee \cdots \vee S^{n-1}$ which collapses $d-1$ equators by c_d. Then σ_f can be homotoped to the map
$$v \mapsto \left\{ s(v), s(v) + \tfrac{1}{k} g(v), \ldots, s(v) + \tfrac{k-1}{k} g(v) \right\}$$
where $s = (\mathrm{id} + * + \cdots + *) \circ c_d$ and $g = (\mathrm{id} + \mathrm{id} + \cdots + \mathrm{id}) \circ c_d$, which is in turn homotopic to the map $(\hat{\Delta} + \hat{\tau} + \cdots + \hat{\tau}) \circ c_d \colon S^{n-1} \to C_k(\mathbb{R}^n \smallsetminus \{0\})$. Therefore the homology class σ_f is equal to $\hat{\Delta} + (d-1)\hat{\tau}$, which is the claimed formula by (5.4) and (5.5).

If $d \leqslant 0$ we can instead take $s = (\mathrm{id} + * + \cdots + *) \circ c_{2-d}$ and $g = (\mathrm{id} + r + \cdots + r) \circ c_{2-d}$, where r is a reflection of S^{n-1}, to see that σ_f is homotopic to the map $(\hat{\Delta} + \hat{\tau} \circ r + \cdots + \hat{\tau} \circ r) \circ c_{2-d}$. The image of the fundamental class $[S^{n-1}]$ under $\hat{\tau} \circ r$ is just $-\hat{\tau}$, so we again get that the homology class σ_f is equal to $\hat{\Delta} + (d-1)\hat{\tau}$. $\square$

Remark 5.4 Rationally, the $(n-1)$st homology of $C_k(\mathbb{R}^n \smallsetminus \{0\})$ is known to be two-dimensional by the presentation of the bigraded $\mathbb{Q}$-algebra $H_*(C_*(\mathbb{R}^n \smallsetminus \{0\});\mathbb{Q})$ given in Proposition 3.4 of [RW]. Specifically, it is generated by the elements Δ_0 and Δ_1, corresponding to $[k-1]\cdot\Delta$ and $[k-2]\cdot\Delta\cdot[1]$ in the notation of the cited paper.

5.4. **Proof of Theorem D.** Abbreviate $C_l(M \smallsetminus \{0\})$ by just C_l. Fix a field $\mathbb{F}$ of characteristic p and write $\widetilde{H}_*(-) = \widetilde{H}_*(-;\mathbb{F})$. Denote by $T_{k,*}$ the map

$$\widetilde{H}_{*+1}(\Sigma^n((C_{k-1})_+)) \longrightarrow \widetilde{H}_*(C_k)$$

from the long exact sequence on homology induced by (5.1). By exactness we have:

$$\begin{aligned}\dim(\widetilde{H}_*(C_k(M))) &= \dim(\mathrm{codomain}(T_{k,*})) + \dim(\mathrm{domain}(T_{k,*-1})) \\ &\quad - \mathrm{rank}(T_{k,*}) - \mathrm{rank}(T_{k,*-1}).\end{aligned} \tag{5.7}$$

Fix a positive integer $r \geqslant 2$ coprime to p (no condition if $p=0$). We will construct maps a, b and c such that the square

$$\begin{tikzcd} S^{n-1}\times C_{k-1} \arrow[r, "a"] \arrow[d, "t_{k-1}"'] & S^{n-1}\times C_{rk-r} \arrow[r, "b"] & S^{n-1}\times C_{rk-1} \arrow[d, "t_{rk-1}"] \\ C_k \arrow[rr, "c"'] & & C_{rk} \end{tikzcd} \tag{5.8}$$

commutes on homology with coefficients in $\mathbb{F}$. Applying $\widetilde{H}_*(-)$ and passing to the second direct summand in the Künneth decomposition (5.2) on the top row gives a commutative square

$$\begin{tikzcd} \widetilde{H}_{*+1}(\Sigma^n(C_{k-1})_+) \arrow[r, "\alpha"] \arrow[d, "T_{k,*}"'] & \widetilde{H}_{*+1}(\Sigma^n(C_{rk-r})_+) \arrow[r, "\beta"] & \widetilde{H}_{*+1}(\Sigma^n(C_{rk-1})_+) \arrow[d, "T_{rk,*}"] \\ \widetilde{H}_*(C_k) \arrow[rr, "c_*"'] & & \widetilde{H}_*(C_{rk}). \end{tikzcd} \tag{5.9}$$

Recall that we defined the function

$$\lambda_r(k) = \min\{sr_{\mathrm{scan}}(k), sr_{\mathrm{scan}}(k-1)+n-1, sr_{\mathrm{stab}}(rk-i) \mid i=2,\ldots,r\},$$

and the isomorphisms claimed by Theorem D are in the range $* \leqslant \lambda_r(k)$. We will show that α, β and c_* are isomorphisms in the range $* \leqslant \lambda_r(k)$, therefore identifying the maps $T_{k,*}$ and $T_{rk,*}$ in this range. Hence by (5.7) the vector spaces $\widetilde{H}_*(C_k(M))$ and $\widetilde{H}_*(C_{rk}(M))$ have the same dimension for $* \leqslant \lambda_r(k)$, which is Theorem D.

Remark 5.5 The first two terms of $\lambda_r(k)$ come from our use of the replication map and Theorem B, which tells us that the r-replication map induces isomorphisms in the stable range sr_{scan}. The remaining terms come from our use of the classical stabilisation map, which by definition induces isomorphisms in the range sr_{stab}. If we assume that sr_{stab} is non-decreasing (so $sr_{\mathrm{scan}} = sr_{\mathrm{stab}}$) and $r,k \geqslant 2$ then the range $* \leqslant \lambda_r(k)$ simplifies to

$$* \leqslant \min\{sr_{\mathrm{stab}}(k), sr_{\mathrm{stab}}(k-1)+n-1\}.$$

For example if $sr_{\mathrm{stab}}(k) = ak+b$ then this is

$$\begin{aligned} &* \leqslant ak+b && \text{if } \dim(M) \geqslant a+1 \\ &* \leqslant ak+b-(a+1-n) && \text{if } \dim(M) < a+1, \end{aligned}$$

i.e. the same as the stable range, except possibly shifted down by a constant if the manifold is low-dimensional compared to the slope of the stable range.

Constructing the maps. Fix a basepoint $0 \in M$. By Lemma 5.2 we can choose a vector field X on M which is non-vanishing except possibly at 0. This has an associated one-parameter family of diffeomorphisms ϕ_t. Define the r-replication map

$$\rho_{r,k} \colon C_k(M \smallsetminus \{0\}) \longrightarrow C_{rk}(M \smallsetminus \{0\})$$

to take a configuration $c = \{x_1, \ldots, x_k\}$ to the configuration

$$\{\phi_{it/r}(x_1), \ldots, \phi_{it/r}(x_k) \mid 0 \leqslant i \leqslant r-1\},$$

where $t = t(c) > 0$ is sufficiently small that $\phi_s(x_i) \neq \phi_u(x_j)$ for $s, u \in (0,t)$ unless $i = j$ and $s = u$. This agrees up to homotopy with the earlier definition of the r-replication map under the identifications $C_k^\delta(M \smallsetminus \{0\}) \simeq C_k(M \smallsetminus \{0\})$ and $C_{rk}^\delta(M \smallsetminus \{0\}) \simeq C_{rk}(M \smallsetminus \{0\})$. We now define

$$\begin{aligned} a &= \mathrm{id} \times \rho_{r,k-1} \\ b &= (\mathrm{pr}_1, t_{rk-2}) \circ \cdots \circ (\mathrm{pr}_1, t_{rk-r}) \\ c &= \rho_{r,k}. \end{aligned}$$

In other words a and c replace each point of the configuration by r copies in the direction determined by the vector field, whereas b adds $r-1$ new points near the missing point 0 in the direction determined by the vector in S^{n-1}.

Isomorphisms in a range. The vector field is non-vanishing on $M \smallsetminus \{0\}$, so if $p > 0$ Theorem B tells us that the r-replication map $\rho_{r,k}$ induces isomorphisms in the stable range on homology with $\mathbb{Z}_{(p)}$ coefficients, and hence also with $\mathbb{F}$ coefficients. When $p = 0$ choose any prime q not dividing r and apply Theorem B to get isomorphisms in the stable range with $\mathbb{Z}_{(q)}$ coefficients, which implies isomorphisms for coefficients in the field $\mathbb{F}$, since it has characteristic 0.

Hence c_* is an isomorphism in the stable range $* \leqslant sr_{\mathrm{scan}}(k)$. The map $\rho_{r,k-1}$ induces isomorphisms on $\mathbb{F}$-homology up to degree $sr_{\mathrm{scan}}(k-1)$, so its suspension $\Sigma^n((\rho_{r,k-1})_+)$ induces isomorphisms up to degree $sr_{\mathrm{scan}}(k-1)+n$. The map that this induces on $\widetilde{H}_{*+1}(-)$ is α, which is therefore an isomorphism in the range $* \leqslant sr_{\mathrm{scan}}(k-1)+n-1$. For the map β consider the map of (trivial) fibre bundles

$$\begin{tikzcd} S^{n-1} \times C_{rk-i} \arrow[rr, "(\mathrm{pr}_1, t_{rk-i})"] \arrow[dr] & & S^{n-1} \times C_{rk-i+1} \arrow[dl] \\ & S^{n-1} & \end{tikzcd}$$

for $i = 2, \ldots, r$. Its fibre over a point in S^{n-1} is the classical stabilisation map and therefore induces isomorphisms on $\mathbb{F}$-homology up to degree $sr_{\mathrm{stab}}(rk-i)$. Hence by the relative Serre spectral sequence the map $(\mathrm{pr}_1, t_{rk-i})$ also induces isomorphisms on $\mathbb{F}$-homology in this range. So the map b induces isomorphisms on $\mathbb{F}$-homology up to degree $\min\{sr_{\mathrm{stab}}(rk-i) \mid 2 \leqslant i \leqslant r\}$.

In general for a map $f \colon S^d \times A \to S^d \times B$ over S^d, the map on homology under the Künneth isomorphism $f_* \colon H_*(A) \oplus H_{*-d}(A) \to H_*(B) \oplus H_{*-d}(B)$ is triangular, more precisely the component $H_*(A) \to H_{*-d}(B)$ is zero. To see this note that a representing cycle c for an element in the $H_*(A)$ component can be taken to have support in a single fibre. Since f is a map over S^d the image $f_\sharp(c)$ will also have support in a single fibre, and therefore the image $f_*([c])$ will be in the $H_*(B)$

component of the Künneth decomposition of the right-hand side. Hence if f induces an isomorphism on $H_*(-)$, then it also restricts to isomorphisms $H_*(A) \to H_*(B)$ and $H_{*-d}(A) \to H_{*-d}(B)$.

Applying this fact to the map b we obtain that β is an isomorphism in the range $* \leqslant \min\{sr_{\mathrm{stab}}(rk-i) \mid 2 \leqslant i \leqslant r\}$. Hence, by definition of $\lambda_r(k)$, each of α, β and c_* are isomorphisms in the range $* \leqslant \lambda_r(k)$.

Commutativity. It therefore remains to show that the square (5.8) commutes on $\mathbb{F}$-homology. Choose a coordinate neighbourhood $U \cong \mathbb{R}^n$ of $0 \in M$ and define the map

$$\zeta\colon C_r(\mathbb{R}^n \smallsetminus \{0\}) \times C_{k-1}(M \smallsetminus \{0\}) \longrightarrow C_{rk}(M \smallsetminus \{0\}) \tag{5.10}$$

to first apply the map $\rho_{r,k-1}$ to the configuration in $M \smallsetminus \{0\}$, i.e. replace each point by r copies according to the vector field, then push the resulting configuration radially away from 0 so that it is disjoint from U, and finally insert the configuration of r points in $\mathbb{R}^n \smallsetminus \{0\} = U \smallsetminus \{0\}$ into the vacated space. Choosing a trivialisation of TM over $U \cong \mathbb{R}^n$, the vector field X restricts to a map $\mathbb{R}^n \to \mathbb{R}^n$ which is non-vanishing on S^{n-1}, so we may rescale it to obtain a map $f\colon S^{n-1} \to S^{n-1}$. Recall from Lemma 5.3 that such a map induces a map $\sigma_f\colon S^{n-1} \to C_r(\mathbb{R}^n \smallsetminus \{0\})$. One can then easily see that the two ways $c \circ t_{k-1}$ and $t_{rk-1} \circ b \circ a$ around the square (5.8) are homotopic to

$$\zeta \circ (\sigma_f \times \mathrm{id}) \text{ and } \zeta \circ (\sigma_{\mathrm{id}} \times \mathrm{id})\colon\ S^{n-1} \times C_{k-1}(M \smallsetminus \{0\}) \longrightarrow C_{rk}(M \smallsetminus \{0\})$$

respectively. It suffices to show that σ_f and $\sigma_{\mathrm{id}}\colon S^{n-1} \to C_r(\mathbb{R}^n \smallsetminus \{0\})$ induce the same map on $\mathbb{F}$-homology, and we only need to check this on the fundamental class. Using our abuse of notation from §5.3 this means that we just need to check that the homology classes σ_f and σ_{id} in $H_{n-1}(C_r(\mathbb{R}^n \smallsetminus \{0\}); \mathbb{F})$ are equal.

The degree of $f\colon S^{n-1} \to S^{n-1}$ is χ by the Poincaré-Hopf theorem (c.f. Definition 5.1) so by Lemma 5.3 we have

$$\begin{aligned} \sigma_f &= r\Delta_0 + \chi r(r-1)\pi \\ \sigma_{\mathrm{id}} &= r\Delta_0 + r(r-1)\pi \end{aligned}$$

in $H_{n-1}(C_r(\mathbb{R}^n \smallsetminus \{0\}); \mathbb{Z})$. Their difference is $(\chi-1)r(r-1)\pi$. If p is positive then it divides $(\chi-1)(r-1)$ by hypothesis, so the difference $\sigma_f - \sigma_{\mathrm{id}}$ is indeed zero in $H_{n-1}(C_r(\mathbb{R}^n \smallsetminus \{0\}); \mathbb{F})$, and therefore the square (5.8) commutes on homology with $\mathbb{F}$ coefficients. If $p = 0$ then we have assumed that $\chi = 1$ so in fact $f \simeq \mathrm{id}$, so $\sigma_f \simeq \sigma_{\mathrm{id}}$ and so the square (5.8) actually commutes up to homotopy in this case. Either way this completes the proof of Theorem D.

5.5. **The case of the two-sphere.** For $M = S^2$ we have the well-known calculation $H_1(C_k(S^2); \mathbb{Z}) \cong \mathbb{Z}/(2k-2)\mathbb{Z}$ for $k \geqslant 2$ obtained from a presentation for $\pi_1(C_k(S^2))$ (see [FVB62]). The degree-one $\mathbb{F}_p$-homology is therefore either one- or zero-dimensional, depending on whether $p \mid 2k-2$ or not. So the statement of Theorem D in degree 1 for $M = S^2$ for mod-p coefficients reduces to the following purely number-theoretic statement: if p is a prime and r is a positive integer such that $p \mid r-1$ then $p \mid 2k-2$ if and only if $p \mid 2rk-2$. This is of course obviously true: we have $r-1 = ap$ for some a, so $2rk-2 = 2k+2kap-2 \equiv 2k-2 \pmod{p}$.

5.6. **Generalisation to configurations with labels in a bundle.** Theorem D generalises directly to configuration spaces $C_k(M,\theta)$ with labels in a bundle $\theta\colon E \to M$ with path-connected fibres.

Theorem D′ *Let M be a closed, connected, smooth manifold with Euler characteristic χ and let $\theta\colon E \to M$ be a fibre bundle with path-connected fibres. Choose a field $\mathbb{F}$ of positive characteristic p and let $r \geqslant 2$ be an integer coprime to p such that p divides $(\chi-1)(r-1)$. Alternatively choose a field $\mathbb{F}$ of characteristic $p=0$, let $r \geqslant 2$ be any positive integer and assume that $\chi = 1$. Then there are isomorphisms*

$$H_*(C_k(M,\theta);\mathbb{F}) \;\cong\; H_*(C_{rk}(M,\theta);\mathbb{F})$$

in the range $* \leqslant \lambda_r(k)$.

In the remainder of this section we sketch how to generalise the proof of Theorem D to a proof of Theorem D′. The proof follows the same steps. In §5.2 one also has to choose a trivialisation of this bundle over the embedded disc $D \subseteq M$, and there is a homology cofibre sequence

$$C_k(M \smallsetminus \{0\}) \longrightarrow C_k(M) \longrightarrow \Sigma^n((F \times C_{k-1}(M \smallsetminus \{0\}))_+),$$

where F is the typical fibre of θ. The description of the connecting homomorphism for the long exact sequence on homology is exactly analogous, using the trivialisation of θ on D to determine the label of the new point which is added to the configuration near $0 \in D$.

In the diagram (5.8) the top three spaces are replaced by their cartesian products with F. The maps c_* and α are isomorphisms in the range $* \leqslant \lambda_r(k)$ for the same reasons as before, using Theorem B′ instead of Theorem B. The map b is a composition of maps of fibre bundles over $F \times S^{n-1}$ and the maps of fibres are classical stabilisation maps for configuration spaces with labels in a bundle, and so are isomorphisms on homology in the stable range for the stabilisation map (c.f. Proposition A.9 and the appendix of [KM14a]). The rest of the argument that β is an isomorphism in the range $* \leqslant \lambda_r(k)$ goes through as before.

For commutativity: the map ζ can be defined similarly, using the chosen trivialisation of θ over D. The input is now a configuration of $k-1$ points in $M \smallsetminus \{0\}$ with labels in θ and a configuration of r points in $\mathbb{R}^n \smallsetminus \{0\}$ with labels in the trivial bundle with fibre F, and the output is a configuration of rk points in $M \smallsetminus \{0\}$ with labels in θ. The map $f\colon S^{n-1} \to S^{n-1}$, corresponding to the restriction of the vector field to ∂D, induces a map $\sigma_f\colon F \times S^{n-1} \to C_r(\mathbb{R}^n \smallsetminus \{0\}, F)$. The two ways around the square (5.8) are homotopic to $\zeta \circ (\sigma_f \times \mathrm{id})$ and $\zeta \circ (\sigma_{\mathrm{id}} \times \mathrm{id})$. Hence we just need to show that σ_f and $\sigma_{\mathrm{id}}\colon F \times S^{n-1} \to C_r(\mathbb{R}^n \smallsetminus \{0\}, F)$ induce the same map on $\mathbb{F}$-homology.

Now as in §5.3 we need to find formulas, in terms of more basic classes, for $(\sigma_f)_*(x)$, for any class $x \in H_*(F \times S^{n-1})$. Previously we showed that when $F = *$ and $x = [S^{n-1}]$ we have

$$(\sigma_f)_*([S^{n-1}]) = r\Delta_0 + \deg(f)r(r-1)\pi \in H_{n-1}(C_r(\mathbb{R}^n \smallsetminus \{0\});\mathbb{Z}).$$

By the Künneth decomposition $H_*(F \times S^{n-1}) = H_*(F) \oplus H_{*-n+1}(F)$ it suffices to show that $(\sigma_f)_*(x \times [*]) - (\sigma_{\mathrm{id}})_*(x \times [*])$ and $(\sigma_f)_*(x \times [S^{n-1}]) - (\sigma_{\mathrm{id}})_*(x \times [S^{n-1}])$ are zero on $\mathbb{F}$-homology for any class $x \in H_*(F)$. It is easy to see that in fact

$$(\sigma_f)_*(x \times [*]) = (\sigma_{\mathrm{id}})_*(x \times [*]) \in H_*(C_r(\mathbb{R}^n \smallsetminus \{0\});\mathbb{Z})$$

and therefore also on $\mathbb{F}$-homology. One can define classes $\pi(x), \Delta_i(x), \tau_i(x)$ etc. in $H_{*+n-1}(C_r(\mathbb{R}^n \smallsetminus \{0\});\mathbb{Z})$ just as in §5.3 and by the same arguments as before we have

$$(\sigma_f)_*(x \times [S^{n-1}]) = r\Delta_0(x) + \deg(f)r(r-1)\pi(x) \in H_{*+n-1}(C_r(\mathbb{R}^n \smallsetminus \{0\});\mathbb{Z}).$$

Hence $(\sigma_f)_*(x \times [S^{n-1}]) - (\sigma_{\mathrm{id}})_*(x \times [S^{n-1}])$ is equal to $(\deg(f)-1)r(r-1)\pi(x) = (\chi-1)r(r-1)\pi(x)$ and is therefore zero on $\mathbb{F}$-homology since p divides $(\chi-1)(r-1)$. This completes the sketch of the proof of Theorem D$'$.

5.7. **Proof of Corollary E.**

Proof of Corollary E. The first two lines follow directly from the odd part of Theorem A, noting that a map of spaces which induces an isomorphism with $\mathbb{Z}[\frac{1}{2}]$ coefficients also induces isomorphisms with coefficients in any field of characteristic different from 2. The third and fourth lines follow directly from the even part of Theorem A, since a map of spaces inducing isomorphisms on homology with $\mathbb{Z}_{(p)}$ coefficients also induces isomorphisms with $\mathbb{Q}$ or $\mathbb{F}_p$ coefficients, and therefore with coefficients in any field of characteristic 0 or p. (For the third line: given j, k in the stable range and not equal to $\frac{\chi}{2}$, choose any prime p which divides neither $2j-\chi$ nor $2k-\chi$ and apply Theorem A to get an isomorphism with $\mathbb{Z}_{(p)}$ coefficients.)

Fifth line: First, if p divides neither $2j-\chi$ nor $2k-\chi$ then this follows from the fourth line. On the other hand if p divides both $2j-\chi$ and $2k-\chi$ then $j \equiv k \not\equiv 0$ mod p, using the assumption that $\chi \not\equiv 0$ mod p and that p is odd. We can therefore choose a number l such that $jl \equiv kl \equiv 1$ mod p. We then apply Theorem D twice, with $r = jl$ and $r = kl$, to obtain the required isomorphisms in the range $* \leqslant \lambda_r(j), \lambda_r(k)$.

Sixth line: By Theorems A and D we have isomorphisms for j, k in the range $* \leqslant \lambda_r(j), \lambda_r(k)$ as long as either $(2j-\chi)_p = (2k-\chi)_p$ or $(j)_p = (k)_p$. Let R be the relation on $\mathbb{N}$ given by jRk if and only if one of these conditions holds. It suffices to show that for any $j, k \in \mathbb{N}$ we have either jRk or $jRlRk$ for some $l \geqslant j, k$. If j, k are both in $p\mathbb{Z}$ then $(2j-1)_p = (2k-1)_p$ so jRk. If j, k are both not in $p\mathbb{Z}$ then $(j)_p = (k)_p$ so jRk. Finally suppose that $j \in p\mathbb{Z}$ and $k \notin p\mathbb{Z}$. Then $(2j-1)_p = (2(jk+1)-1)_p$ and $(jk+1)_p = (k)_p$ so $jR(jk+1)Rk$.

The seventh line follows directly from Theorem D: when $p = 2$ we may take r to be any odd integer, so there are isomorphisms in the range $* \leqslant \lambda_r(j), \lambda_r(k)$ between any j, k with the same 2-adic valuation. To deduce the eighth line from the seventh we need to show that there are isomorphisms in this range whenever $(j)_2$ and $(k)_2$ are both at least $(\chi)_2$. In this case we have $(2k-\chi)_2 = (\chi)_2 = (2j-\chi)_2$, and so we can apply Theorem A (since we assumed that χ is even). Finally, the ninth line is a special case of the eighth line. □

Appendix A. The stability range for the torsion in configuration spaces

In this appendix we show that the stable range for homological stability of unordered configuration spaces may be improved to have slope 1 when taking $\mathbb{Z}[\frac{1}{2}]$ coefficients. We note that this has also recently been proved by a different method by [KM14b]. We begin by proving this in a larger range when $M = \mathbb{R}^n$ using Salvatore's description [Sal04] of Cohen's calculations [CLM76], and then use this to deduce the slope 1 statement for general open, connected manifolds M. Our method for this second step is a slight variation of an argument due to Oscar Randal-Williams in [RW13, §8].

From Salvatore's description [Sal04, page 537] of the homology of $C(\mathbb{R}^n)$ (based on [CLM76, page 227]), we obtain the following. A non-empty sequence of positive integers $I = (i_1, \ldots, i_{\ell(I)})$ with $\ell(I) \geqslant 0$ is said to be (n,p)-*admissible* if it is weakly monotone and strictly bounded above by n. If p is odd, an admissible function for I is a function $\epsilon\colon \{1,\dots,\ell(I)\} \to \{0,1\}$ satisfying

$$\epsilon(j) \equiv i_j + i_{j-1} \bmod 2 \qquad \text{for } 2 \leqslant j \leqslant \ell(I).$$

Observe that ϵ is determined by I and $\epsilon(1)$. If $p=2$, define f to be constant with value 0.

If $p=2$, then $H_*(C(\mathbb{R}^n);\mathbb{F}_2)$ is isomorphic to the free commutative graded algebra generated by the symbols $Q_{\epsilon,I}(\iota)$ where I is an (n,p)-admissible sequence and ϵ is an admissible function.

If p is odd, then $H_*(C(\mathbb{R}^n);\mathbb{F}_p)$ is isomorphic to the free commutative graded algebra generated by the symbols $Q_{\epsilon,I}(\iota)$ where I is an (n,p)-admissible sequence with $i_{\ell(I)}$ even and ϵ is an admissible function for I, and also (if n is even) the symbols $Q_{\epsilon,I}([\iota,\iota])$ where I is an (n,p)-admissible sequence with $i_{\ell(I)}$ odd and ϵ is an admissible function for I.

The homological degrees of $\iota := Q_{\varnothing,\varnothing}(\iota)$ and $[\iota,\iota] := Q_{\varnothing,\varnothing}([\iota,\iota])$ are 0 and $n-1$, and the configuration degrees are 1 and 2. The homological and configuration degrees of the other generators are

$$\begin{aligned} h(Q_{\epsilon,(i_1,\dots,i_k)}(\alpha)) &= ph(Q_{\epsilon(2),(i_2,\dots,i_k)}(\alpha)) + i_1(p-1) - \epsilon(1) \\ \nu(Q_{\epsilon,(i_1,\dots,i_k)}(\alpha)) &= p\nu(Q_{\epsilon(2),(i_2,\dots,i_k)}(\alpha)), \end{aligned}$$

where $\alpha = \iota$ or $[\iota,\iota]$ and where by an abuse of notation, for $\delta \in \{0,1\}$, $Q_{\delta,I}(x)$ means $Q_{\epsilon,I}(x)$ for the unique admissible function ϵ for I with $\epsilon(1)=\delta$. The degrees of a product of generators are:

$$h(xy) = h(x) + h(y), \qquad\qquad \nu(xy) = \nu(x) + \nu(y).$$

Multiplication by the class ι raises the configuration degree by 1 and hence defines a homomorphism

$$H_*(C_{k-1}(\mathbb{R}^n)) \longrightarrow H_*(C_k(\mathbb{R}^n)),$$

which is the same as that induced by the stabilisation map.

We say that a class in $H_*(C_k(\mathbb{R}^n);\mathbb{F}_p)$ is *p-inceptive* if it is not in the image of the stabilisation map $C_{k-1}(\mathbb{R}^n) \to C_k(\mathbb{R}^n)$ on mod-p homology. By the above a class is p-inceptive if and only if it is not in the principal ideal generated by ι.

Lemma A.1 *In $H_*(C_k(\mathbb{R}^n);\mathbb{F}_p)$ the first p-inceptive class in a fixed configuration degree k is given in Table 2, where $a = \lfloor k/p \rfloor$ and $m_p(k)$ is the remainder after dividing k by p. Any case not covered in the table has no p-inceptive classes. Hence by the above discussion the stabilisation map $C_{k-1}(\mathbb{R}^n) \to C_k(\mathbb{R}^n)$ induces an isomorphism on $H_*(-;\mathbb{F}_p)$ for smaller homological degrees.*

Proof. First observe that

$$\begin{aligned} h(Q_{\epsilon,(i_1,\dots,i_k)}(\iota)) &\geqslant h(Q_{\epsilon(i_2),(i_2,\dots,i_k)}(\iota)^p) & \nu(Q_{\epsilon,(i_1,\dots,i_k)}(\iota)) &= \nu(Q_{\epsilon(i_2),(i_2,\dots,i_k)}(\iota)^p) \\ h(Q_{\epsilon,(i_1,\dots,i_k)}([\iota,\iota])) &\geqslant h(Q_{\epsilon(i_1),(i_2,\dots,i_k)}([\iota,\iota])^p) & \nu(Q_{\epsilon,(i_1,\dots,i_k)}([\iota,\iota])) &= \nu(Q_{\epsilon(i_2),(i_2,\dots,i_k)}([\iota,\iota])^p) \\ h(Q_{1,(i)}(\iota)) &\leqslant h(Q_{\epsilon,(j)}(\iota)) & \nu(Q_{1,(i)}(\iota)) &= \nu(Q_{\epsilon,(j)}(\iota)) \end{aligned}$$

where $i \leqslant j$ in the bottom row. Hence the lowest p-inceptive class in a fixed configuration degree is a product whose factors are

$$\begin{cases} Q_1(\iota) & \text{if } p=2 \\ Q_{1,(2)}(\iota) & \text{if } p \text{ odd and } n \text{ odd} \\ Q_{1,(2)}(\iota), [\iota,\iota] & \text{if } p \text{ odd and } n \text{ even} \\ [\iota,\iota] & \text{if } p \text{ is odd and } n=2. \end{cases}$$

<table>
<tr><th>p</th><th>n</th><th>k</th><th>class</th><th>homological degree</th></tr>
<tr><td>even</td><td>all</td><td>even</td><td>$Q_{0,(1)}(\iota)^a$</td><td>a</td></tr>
<tr><td>odd</td><td>odd</td><td>$\in p\mathbb{Z}$</td><td>$Q_{1,(2)}(\iota)^a$</td><td>$a(2(p-1)-1)$</td></tr>
<tr><td>odd</td><td>$\geqslant 6$, even</td><td rowspan="2">$\geqslant p$, odd</td><td rowspan="2">$Q_{1,(2)}(\iota)^a[\iota,\iota]^{m_p(k)/2}$</td><td rowspan="2">$a(2(p-1)-1)+(n-1)m_p(k)/2$</td></tr>
<tr><td>$3,5$</td><td>4</td></tr>
<tr><td>odd</td><td>$\geqslant 6$, even</td><td rowspan="2">even</td><td rowspan="2">$Q_{1,(2)}(\iota)^{a-1}[\iota,\iota]^{(p+m_p(k))/2}$</td><td rowspan="2">$(a-1)(2(p-1)-1)+(n-1)(p+m_p(k))/2$</td></tr>
<tr><td>$3,5$</td><td>4</td></tr>
<tr><td rowspan="2">$\neq 2,3,5$</td><td rowspan="2">4</td><td>even</td><td rowspan="2">$Q_{1,(2)}(\iota)^{m_2(k)}[\iota,\iota]^{\lfloor k/2\rfloor}$</td><td rowspan="2">$m_2(k)(2(p-1)-1)+(n-1)\lfloor k/2\rfloor$</td></tr>
<tr><td>$\geqslant p$, odd</td></tr>
<tr><td>odd</td><td>2</td><td>even</td><td>$[\iota,\iota]^{k/2}$</td><td>$k/2$</td></tr>
</table>

TABLE 2. The first p-inceptive class in degree k.

This is enough to deduce the first two rows of the table, as well as the sixth. Second observe that, if p is odd and n is even, the first configuration degree in which a power of $Q_{1,(2)}(\iota)$ and a power of $[\iota,\iota]$ both live is $2p$, where $\nu(Q_{1,(2)}(\iota)^2) = \nu([\iota,\iota]^p)$, and

$$h(Q_{1,(2)}(\iota)^2) = 4p-6 < p(n-1) = h([\iota,\iota]^p) \Leftrightarrow \; n \geqslant 6 \text{ or } n=4, p=3,5,$$

from which the third, fourth and fifth rows of the table follow. □

Lemma A.1 tells us in particular that for odd primes p the stabilisation map $C_k(\mathbb{R}^n) \to C_{k+1}(\mathbb{R}^n)$ induces an isomorphism on homology with $\mathbb{F}_p$ coefficients in the range $* \leqslant k$. We now show that this implies that the same is true for the stabilisation map $C_k(M) \to C_{k+1}(M)$ for any smooth, connected, open manifold M of dimension at least 3. Our method for this is a slight variation of an argument due to Oscar Randal-Williams in [RW13, §8].

Proposition A.2 *Let M be a smooth, connected, open n-manifold with $n \geqslant 3$ and let A be an abelian group. If the stabilisation map on A-homology*

$$H_*(C_k(\mathbb{R}^n);A) \longrightarrow H_*(C_{k+1}(\mathbb{R}^n);A)$$

is an isomorphism in the range $* \leqslant k$ then so is the stabilisation map on A-homology

$$H_*(C_k(M);A) \longrightarrow H_*(C_{k+1}(M);A).$$

So by Lemma A.1 the stabilisation map $C_k(M) \to C_{k+1}(M)$ induces isomorphisms on homology with $\mathbb{F}_p$ coefficients in the range $* \leqslant k$ for any odd prime p.

This result has also been recently proved by [KM14b] using a different method along the lines of [Seg79].

Proof. We will just write $H_*(-)$ for $H_*(-;A)$. Define $R_k(M)$ to be the homotopy cofibre of the stabilisation map $C_k(M) \to C_{k+1}(M)$. Now the stabilisation map $C_k(M) \to C_{k+1}(M)$ is split-injective on homology (see [McD75, page 103]) so it induces an isomorphism on homology in degree $*$ if and only if $\widetilde{H}_*(R_k(M)) = 0$. So the hypothesis of the proposition says that $\widetilde{H}_*(R_k(\mathbb{R}^n)) = 0$ for $* \leqslant k$ and we would like to show that $\widetilde{H}_*(R_k(M)) = 0$ for $* \leqslant k$. We refer to [RW13] for background and any details which we omit in this proof – the line of argument is very similar. The proof is by induction on k. The base case $k=0$ is obvious so we now fix $k \geqslant 1$ for the inductive step.

For $i \geqslant 0$ let $C_l(M)^i$ be the space of l-point subsets c of M together with an injection $\{0, \dots, i\} \to c$. These fit together to form an semi-simplicial space $C_l(M)^\bullet$ augmented by $C_l(M)$. The stabilisation map lifts to a map $C_l(M)^\bullet \to C_{l+1}(M)^\bullet$ of augmented semi-simplicial spaces. There is a fibre bundle $\pi\colon C_l(M)^i \to \widetilde{C}_{i+1}(M)$, where $\widetilde{C}$ denotes the *ordered* configuration space, given by sending an injection $\{0, \dots, i\} \to c$ to its image and remembering the induced ordering. Its fibre over a point is homeomorphic to $C_{l-i-1}(M_{i+1})$, where M_{i+1} denotes the manifold M with $i+1$ points removed. Moreover the projection π commutes with the stabilisation map $C_l(M)^i \to C_{l+1}(M)^i$ and the map of fibres over a point is the stabilisation map $C_{l-i-1}(M_{i+1}) \to C_{l-i}(M_{i+1})$. Any map of Serre fibrations over a fixed base space has an associated relative Serre spectral sequence; in this case it has second page

$$ {}^i\widetilde{E}^2_{s,t} \cong H_s(\widetilde{C}_{i+1}(M); \widetilde{H}_t(R_{l-i-1}(M_{i+1}))) $$

and converges to $\widetilde{H}_*(R_l(M)^i)$, where $R_l(M)^i$ denotes the homotopy cofibre of the lift $C_l(M)^i \to C_{l+1}(M)^i$ of the stabilisation map.

For $1 \leqslant j \leqslant k$ there are maps of augmented semi-simplicial spaces $C_{k-j}(M) \times C_j(\mathbb{R}^n)^\bullet \to C_k(M)^\bullet$ defined similarly to the stabilisation map, except one stabilises by adding the given configuration in $\mathbb{R}^n$ instead of just a single point. In [RW13, §8] it is explained how these induce maps of semi-simplicial spaces $R_{k-j-1}(M) \wedge R_j(\mathbb{R}^n)^\bullet \to R_k(M)^\bullet$ for $1 \leqslant j \leqslant k$. Note that when $j = k$ we have $R_{-1}(M) = S^0$ and this is just the map $R_k(\mathbb{R}^n)^\bullet \to R_k(M)^\bullet$ induced by an embedding $\mathbb{R}^n \hookrightarrow M$. Each semi-simplicial space has an associated spectral sequence so we obtain a map ${}^j\bar{E} \to E$ of spectral sequences whose first pages are

$$ \begin{aligned} {}^j\bar{E}^1_{s,t} &\cong \widetilde{H}_t(R_{k-j-1}(M) \wedge R_j(\mathbb{R}^n)^s) \\ E^1_{s,t} &\cong \widetilde{H}_t(R_k(M)^s). \end{aligned} $$

Note that these are first quadrant plus an extra column $\{s = -1, t \geqslant 0\}$.

The spectral sequence E converges to $\widetilde{H}_{*+1}$ of the homotopy cofibre of the map $\|R_k(M)^\bullet\| \to R_k(M)$ induced by the augmentation map. Since taking homotopy cofibres commutes with taking geometric realisation of semi-simplicial spaces this space can also be obtained as follows: first take the homotopy cofibres of the maps $\|C_k(M)^\bullet\| \to C_k(M)$ and $\|C_{k+1}(M)^\bullet\| \to C_{k+1}(M)$; these are related by a map induced by stabilisation; then take the homotopy cofibre of this map. Now the augmented semi-simplicial space $C_k(M)^\bullet$ is a $(k-1)$-resolution [RW13, Proposition 6.1], i.e. the map $\|C_k(M)^\bullet\| \to C_k(M)$ is $(k-1)$-connected. Hence the spectral sequence E converges to zero in total degree $* \leqslant k-1$.

The inductive hypothesis says that

$$ \widetilde{H}_*(R_l(M)) = 0 \text{ for } * \leqslant l < k \tag{IH} $$

and the hypothesis of the proposition says that

$$ \widetilde{H}_*(R_l(\mathbb{R}^n)) = 0 \text{ for } * \leqslant l. \tag{Hyp} $$

From (IH) we deduce that ${}^i\widetilde{E}^2_{s,t} = 0$ for $t \leqslant l-i-1$ so the spectral sequence ${}^i\widetilde{E}$ converges to zero in total degree $* \leqslant l-i-1$, so

$$ \widetilde{H}_*(R_l(M)^i) = 0 \text{ for } * \leqslant l-i-1 \text{ and } i \geqslant 0. \tag{A.1} $$

In other words:

$$ E^1_{s,t} = 0 \text{ for } t \leqslant k-s-1 \text{ and } s \geqslant 0. \tag{A.2} $$

Also, using the Künneth theorem, (A.1) and (IH) we deduce that

$$ {}^j\bar{E}^1_{s,t} = 0 \text{ for } t \leqslant k-s-1, \tag{A.3} $$

where for the case $\{s = -1 \text{ and } j = k\}$ we also need to use (Hyp). We now make the following:

Claim *For $1 \leqslant j \leqslant k$ the map ${}^j\bar{E}^1_{j,k-j} \to E^1_{j,k-j}$ is surjective.*

The verification of this claim is delayed until the end of the proof. Now a diagram chase in the following:

$$\begin{tikzcd}
{}^j\bar{E}^1_{j,k-j} \arrow[r, two heads] \arrow[d, two heads] & {}^j\bar{E}^{j+1}_{j,k-j} \arrow[r, "\bar{d}_{j+1}"] \arrow[d] & {}^j\bar{E}^{j+1}_{-1,k} \arrow[d] & {}^j\bar{E}^1_{-1,k} = 0 \arrow[l, two heads] \arrow[d] \\
E^1_{j,k-j} \arrow[r, two heads] & E^{j+1}_{j,k-j} \arrow[r, "d_{j+1}"'] & E^{j+1}_{-1,k} & E^1_{-1,k} \arrow[l, two heads]
\end{tikzcd}$$

shows that the differential $d_{j+1}\colon E^{j+1}_{j,k-j} \to E^{j+1}_{-1,k}$ is zero for $1 \leqslant j \leqslant k$.

Now we can deduce that the first differential $d_1\colon E^1_{0,t} \to E^1_{-1,t}$ is surjective in a range. First, for $t \leqslant k-1$ note that the differentials hitting $E^{\square}_{-1,t}$ have source $E^j_{j-1,t-j+1}$ for $1 \leqslant j \leqslant t+1$. By (A.2) these groups are all zero, so $E^1_{-1,t} = E^{\infty}_{-1,t}$. The spectral sequence E converges to zero in total degree $t-1$ so $E^1_{-1,t} = E^{\infty}_{-1,t} = 0$ and so the first differential $d_1\colon E^1_{0,t} \to E^1_{-1,t}$ is vacuously surjective. For $t = k$ we use the result of the diagram chase above, which tells us that the only possible non-zero differential hitting $E^{\square}_{-1,k}$ is the first differential. We know that $E^{\infty}_{-1,k} = 0$ since E converges to zero in total degree $k-1$ so the first differential $d_1\colon E^1_{0,k} \to E^1_{-1,k}$ must be surjective. This can be identified as the map on homology induced by the augmentation map $R_k(M)^0 \to R_k(M)$. Hence we have established:

Fact A.3 *The augmentation map $a\colon R_k(M)^0 \to R_k(M)$ induces surjections on A-homology up to degree k.*

Now consider the maps $p\colon C_k(M) \to C_k(M_1)$ and $u\colon C_k(M_1) \to C_k(M)$, defined as follows. The map p is defined similarly to the stabilisation map. Write $M = \mathrm{int}(\bar{M})$ for a manifold $\bar{M}$ with non-empty boundary and choose a self-embedding $e'\colon \bar{M} \hookrightarrow \bar{M}$ which is isotopic to the identity and whose image does not contain the missing point of M_1. Then p is defined by applying e' to each point of the configuration. The map u is simply the map induced by the inclusion $M_1 \hookrightarrow M$. Since e' is isotopic to the identity the composition $u \circ p$ is homotopic to the identity, and so the induced maps u_* and p_* on homology are semi-inverses: $u_* \circ p_* = \mathrm{id}$. If we are careful to define p using a self-embedding $e'\colon \bar{M} \hookrightarrow \bar{M}$ whose support is disjoint from the self-embedding $e\colon \bar{M} \hookrightarrow \bar{M}$ used to define the stabilisation map s, then p commutes on the nose with s and there are induced maps $p\colon R_k(M) \to R_k(M_1)$ and $u\colon R_k(M_1) \to R_k(M)$ on mapping cones. Again we have $u \circ p \simeq \mathrm{id}$ so $u_* \circ p_* = \mathrm{id}$.

The methods of the proof of Proposition 6.3 in [RW13] show that

$$h\mathrm{conn}_A(u\colon R_{k-1}(M_1) \to R_{k-1}(M)) \geqslant h\mathrm{conn}(s\colon C_{k-2}(M) \to C_{k-1}(M)) + \dim(M)$$

where $h\mathrm{conn}_A(f)$ is the A-homology-connectivity of f, i.e. the largest $*$ such that $\widetilde{H}_*(mc(f); A) = 0$, where $mc(f)$ is the mapping cone of f. By inductive hypothesis the right-hand side is at least $k-2+\dim(M) \geqslant k+1$ since we have assumed that M is at least 3-dimensional. Therefore the A-homology-connectivity of $p\colon R_{k-1}(M) \to R_{k-1}(M_1)$ is at least k. In particular we have:

Fact A.4 *The map $p\colon R_{k-1}(M) \to R_{k-1}(M_1)$ induces surjections on A-homology up to degree k.*

For our third and final fact, consider the spectral sequence ${}^0\widetilde{E}$ with $l = k$ and recall from just before (A.1) that ${}^0\widetilde{E}^2_{s,t} = 0$ for $t \leqslant k-1$. This is the relative Serre spectral sequence for the map of fibre bundles $C_k(M)^0 \to C_{k+1}(M)^0$ over $\widetilde{C}_1(M) = M$. The inclusion of the fibre over a point $* \in M$ is the map $C_{k-1}(M_1) = C_{k-1}(M \smallsetminus \{*\}) \to C_k(M)^0$ which adds the point $*$ to a configuration and labels it by 0. This induces a map $f\colon R_{k-1}(M_1) \to R_k(M)^0$ on mapping cones. The map on $\widetilde{H}_*$ induced by f can be identified with the composition of the edge homomorphism

$$\widetilde{H}_*(R_{k-1}(M_1)) = {}^0\widetilde{E}^2_{0,*} \twoheadrightarrow {}^0\widetilde{E}^\infty_{0,*}$$

and the inclusion

$${}^0\widetilde{E}^\infty_{0,*} \hookrightarrow \widetilde{H}_*(R_k(M)^0)$$

given by all the extension problems in total degree $*$. But since the second page is trivial for $t \leqslant k-1$ there are no extension problems in total degree $* \leqslant k$, and so this inclusion is an isomorphism. Hence we have:

Fact A.5 *The map $f\colon R_{k-1}(M_1) \to R_k(M)^0$ induces surjections on A-homology up to degree k.*

The composition $s' \coloneqq a \circ f \circ p\colon R_{k-1}(M) \to R_k(M)$ is defined exactly like the stabilisation map $s\colon R_{k-1}(M) \to R_k(M)$ except that it uses the self-embedding e' of $\bar{M}$ instead of e. Since we chose e and e' to have disjoint support, the maps s and s' commute. If we now ensure that we picked e and e' to be isotopic, we have that s and s' are homotopic. The square $s \circ s' = s' \circ s$ induces a map of long exact sequences:

$$\begin{array}{ccccccccc}
\xrightarrow{s_*} & \widetilde{H}_t(C_k(M)) & \xrightarrow{c} & \widetilde{H}_t(R_{k-1}(M)) & \longrightarrow & \widetilde{H}_{t-1}(C_{k-1}(M)) & \xrightarrow{b = s_*} & \widetilde{H}_{t-1}(C_k(M)) & \to \\
& \Big\downarrow {\scriptstyle s'_*} & & \Big\downarrow {\scriptstyle a = s'_*} & & & & & \\
\xrightarrow[s_*]{} & \widetilde{H}_t(C_{k+1}(M)) & \xrightarrow[d]{} & \widetilde{H}_t(R_k(M)) & \longrightarrow & & & &
\end{array}$$

Let $t \leqslant k$ – our aim is to show that $\widetilde{H}_t(R_k(M)) = 0$. By Facts A.3, A.4 and A.5 above, the map a in this diagram is surjective. As mentioned at the beginning of the proof, the stabilisation map is split-injective on homology in all degrees [McD75, page 103], so the map b is injective, and so by exactness the map c is surjective. Hence the composite $a \circ c$ is surjective. But

$$a \circ c = d \circ s'_* = d \circ s_* = 0,$$

so its codomain $\widetilde{H}_t(R_k(M))$ must be trivial.

It now remains to prove the claim we made earlier in the proof, namely that the map

$${}^j\bar{E}^1_{j,k-j} = \widetilde{H}_{k-j}(R_{k-j-1}(M) \wedge R_j(\mathbb{R}^n)^j) \longrightarrow \widetilde{H}_{k-j}(R_k(M)^j) = E^1_{j,k-j}$$

is surjective. In fact we will show that the map $R_{k-j-1}(M) \wedge R_j(\mathbb{R}^n)^j \to R_k(M)^j$ induces surjections on homology in degrees $t \leqslant k-j$. First note that

$$\begin{aligned}
R_j(\mathbb{R}^n)^j &= mc(C_j(\mathbb{R}^n)^j \to C_{j+1}(\mathbb{R}^n)^j) \\
&= mc(\varnothing \to \widetilde{C}_{j+1}(\mathbb{R}^n)) \\
&= \widetilde{C}_{j+1}(\mathbb{R}^n)_+
\end{aligned}$$

and the map $R_{k-j-1}(M) \wedge \widetilde{C}_{j+1}(\mathbb{R}^n)_+ \to R_k(M)^j$ is given by taking mapping cones of the horizontal arrows in the commutative square:

$$\begin{tikzcd} C_{k-j-1}(M) \times \widetilde{C}_{j+1}(\mathbb{R}^n) \arrow[r, "s \times \mathrm{id}"] \arrow[d] & C_{k-j}(M) \times \widetilde{C}_{j+1}(\mathbb{R}^n) \arrow[d] \\ C_k(M)^j \arrow[r, "s"'] & C_{k+1}(M)^j \end{tikzcd}$$

To do this we begin by defining some more explicit models for various maps. As before, write $M = \mathrm{int}(\bar{M})$ for a manifold $\bar{M}$ with non-empty boundary and choose two isotopic self-embeddings $e, e' \colon \bar{M} \hookrightarrow \bar{M}$ which are both non-surjective and have disjoint support. Choose an embedding $\phi \colon \mathbb{R}^n \hookrightarrow M \smallsetminus e'(\bar{M})$ and pairwise disjoint points $p_0, \ldots, p_j \in \mathbb{R}^n$. Write $M_{j+1} = M \smallsetminus \{\phi(p_0), \ldots, \phi(p_j)\}$. We have a square of maps

$$\begin{tikzcd} C_{k-j-1}(M) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & C_{k-j-1}(M) \times \widetilde{C}_{j+1}(\mathbb{R}^n) \arrow[d, "\delta"] \\ C_{k-j-1}(M_{j+1}) \arrow[r, "\beta"'] & C_k(M)^j \end{tikzcd} \tag{A.4}$$

defined by

$$\begin{aligned} \alpha(c) &= (c, (p_0, \ldots, p_j)) \\ \gamma(c) &= e'(c) \\ \beta(c) &= c \cup \{\phi(p_0), \ldots, \phi(p_j)\}; i \mapsto \phi(p_i) \\ \delta(c, (q_0, \ldots, q_j)) &= e'(c) \cup \{\phi(q_0), \ldots, \phi(q_j)\}; i \mapsto \phi(q_i). \end{aligned}$$

Choose a point $* \in M \smallsetminus e(\bar{M})$ and take an explicit model for the stabilisation map to be defined by $c \mapsto e(c) \cup \{*\}$. Since e and e' have disjoint support this induces a map of squares from (A.4) to (A.4)$[k \mapsto k+1]$. Taking mapping cones along this map of squares gives us the following:

$$\begin{tikzcd} R_{k-j-1}(M) \arrow[r, "\bar{\alpha}"] \arrow[d, "\bar{\gamma}"'] & R_{k-j-1}(M) \wedge \widetilde{C}_{j+1}(\mathbb{R}^n)_+ \arrow[d, "\bar{\delta}"] \\ R_{k-j-1}(M_{j+1}) \arrow[r, "\bar{\beta}"'] & R_k(M)^j \end{tikzcd}$$

We need to show that $\bar{\delta}$ induces surjections on homology in degrees $t \leqslant k-j$. This will follow if we can prove this for $\bar{\gamma}$ and $\bar{\beta}$. But $\bar{\gamma}$ is the composition of $j+1$ instances of the map p from Fact A.4, and so this does induce surjections on homology up to degree $k-j$ by Fact A.4.[2] When $j=0$ the map $\bar{\beta}$ is surjective on homology up to degree k by Fact A.5. Moreover, the argument proving Fact A.5 generalises (using the spectral sequence ${}^j\widetilde{E}$ instead of ${}^0\widetilde{E}$) to prove precisely that the map $\bar{\beta}$ is surjective on homology up to degree $k-j$ in general. $\square$

[2]The proofs of Facts A.4 and A.5 earlier did not depend on the claim which we are currently proving, so this is not circular.

Remark A.6 When $\dim(M) \geqslant 3$ we have homological stability in the range $* \leqslant k$ for $\mathbb{Q}$ coefficients (by [RW13, Theorem B]) and for $\mathbb{Z}/p$ coefficients with p odd (by Proposition A.2 above). Using the short exact sequences of coefficients $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^{l+1} \to \mathbb{Z}/p^l \to 0$ and

$$0 \to \mathbb{Z}[\tfrac{1}{2}] \to \mathbb{Q} \to \textstyle\bigoplus_{p\neq 2} \operatorname{colim}_{l\to\infty} \mathbb{Z}/p^l \to 0$$

this implies homological stability in the range $* \leqslant k-1$ for $\mathbb{Z}[\frac{1}{2}]$ coefficients. This recovers Theorem 1.4 of [KM14b], except without surjectivity in degree k.

Remark A.7 Configuration spaces also satisfy homological stability with respect to finite-degree twisted coefficient systems: for the case of symmetric groups this was proved by [Bet02], and the general case was proved in [Pal13]. A twisted coefficient system for M is a functor from the partial braid category $\mathcal{B}(M)$ to $\mathbb{Z}$-mod. The partial braid category $\mathcal{B}(M)$ has objects $\{0, 1, 2, \dots\}$ and a morphism from m to n is a path in $C_k(M)$ from a subset of $\{p_1, \dots, p_m\}$ to a subset of $\{p_1, \dots, p_n\}$, up to endpoint-preserving homotopy, where $\{p_1, p_2, p_3, \dots\}$ is a fixed injective sequence in M.

If the twisted coefficient system has degree d the stable range obtained is $* \leqslant \frac{k-d}{2}$, which arises since homological stability with untwisted $\mathbb{Z}$ coefficients in the range $* \leqslant \frac{k}{2}$ is an input for the proof. However, if the twisted coefficient system takes values in the subcategory $\mathbb{Z}[\frac{1}{2}]$-mod of $\mathbb{Z}$-mod and $\dim(M) \geqslant 3$, then we may instead input [KM14b] or Proposition A.2 to obtain a stable range of $* \leqslant k-d$ for $C_k(M)$ with coefficients in a functor $\mathcal{B}(M) \to \mathbb{Z}[\frac{1}{2}]$-mod of degree d.

In particular we may adapt this last remark to generalise Proposition A.2 to configuration spaces with labels in a bundle over M.

Definition A.8 Let $\theta\colon E \to M$ be a fibre bundle with path-connected fibres F and define

$$C_k(M, \theta) \coloneqq \{\{p_1, \dots, p_k\} \subset E \mid \theta(p_i) \neq \theta(p_j) \text{ for } i \neq j\}.$$

Choose a self-embedding $e\colon \bar{M} \hookrightarrow \bar{M}$ which is non-surjective and isotopic to the identity. Choose an open neighbourhood $U \subseteq \bar{M}$ containing the support of e, write $V = U \smallsetminus \partial\bar{M}$ and choose a trivialisation $\phi\colon \theta^{-1}(V) \to V \times F$ of E over V. Define a self-embedding $\tilde{e}\colon E \hookrightarrow E$ by

$$p \mapsto \begin{cases} p & p \notin \theta^{-1}(V) \\ \phi^{-1} \circ (e \times \mathrm{id}) \circ \phi(p) & p \in \theta^{-1}(V) \end{cases}$$

and note that $\theta \circ \tilde{e} = e \circ \theta$. Also choose points $* \in M \smallsetminus e(\bar{M}) \subseteq V$ and $x \in F$. We can then define the stabilisation map $C_k(M,\theta) \to C_{k+1}(M,\theta)$ by

$$\{p_1, \dots, p_k\} \mapsto \{\tilde{e}(p_1), \dots, \tilde{e}(p_k), \phi^{-1}(*, x)\}.$$

Proposition A.9 *Let M be a smooth, connected, open n-manifold with $n \geqslant 2$ and $\theta\colon E \to M$ a fibre bundle with path-connected fibres. Then the stabilisation map $C_k(M,\theta) \to C_{k+1}(M,\theta)$ induces isomorphisms on $H_*(-;\mathbb{Z})$ in the range $* \leqslant \frac{k}{2} - 1$. It induces isomorphisms in the range $* \leqslant k$ on $H_*(-;\mathbb{Q})$, unless M is an orientable surface in which case the range is only $* \leqslant k-1$. If $n \geqslant 3$ it induces isomorphisms on $H_*(-;\mathbb{Z}[\frac{1}{2}])$ in the range $* \leqslant k-1$.*

The worse range $* \leqslant k-1$ for rational homology of configurations on orientable surfaces is necessary: for example $H_1(C_1(\mathbb{R}^2);\mathbb{Q}) = 0 \ncong \mathbb{Q} \cong H_1(C_2(\mathbb{R}^2);\mathbb{Q})$.

Proof. This will follow by the same considerations as in Remark A.6 if we show that it induces isomorphisms on $H_*(-;A)$ in the range $* \leqslant \frac{k}{2}$ when $A = \mathbb{F}_p$, in the range $* \leqslant k$ if either (a) $A = \mathbb{Q}$ and M is not an orientable surface or (b) $A = \mathbb{F}_p$ for p odd and $n \geqslant 3$, and in the range $* \leqslant k-1$ when $A = \mathbb{Q}$ and M is an orientable surface. The loss of one degree from the range occurs when going from $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ coefficients to $\mathbb{Z}$ coefficients (resp. $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}[\frac{1}{2}]$ coefficients to $\mathbb{Z}[\frac{1}{2}]$ coefficients).

Let $A = \mathbb{Q}$ or $\mathbb{F}_p$ for a prime p. There are fibre bundles $C_k(M,\theta) \to C_k(M)$, given by forgetting labels, with fibre F^k. The stabilisation maps $C_k(M) \to C_{k+1}(M)$ and $C_k(M,\theta) \to C_{k+1}(M,\theta)$ commute with these fibre bundles and the map of fibres is the inclusion $F^k \hookrightarrow F^{k+1}$. There is then a map of Serre spectral sequences

$$\begin{array}{ccc} E^2_{s,t} \cong H_s(C_k(M); H_t(F^k; A)) & \Rightarrow & H_*(C_k(M,\theta); A) \\ \downarrow & & \downarrow \\ E^2_{s,t} \cong H_s(C_{k+1}(M); H_t(F^{k+1}; A)) & \Rightarrow & H_*(C_{k+1}(M,\theta); A) \end{array}$$

and our aim is to prove that the map in the limit is an isomorphism in a certain range depending on A and M.

Now by Lemma 4.2 of [Pal13] and since A is a field the assignment $k \mapsto H_t(F^k; A)$ extends to form a twisted coefficient system of degree at most $\frac{t}{h+1} \leqslant t$ where $h = h\mathrm{conn}_A(F)$. Hence the map on the second page is an isomorphism in the range $s \leqslant \frac{k-t}{2}$ by Theorem 1.3 of [Pal13]. In particular it is an isomorphism in total degree at most $\frac{k}{2}$ and therefore the same holds for the map in the limit.

To obtain the improved range in certain cases note that, by Remark 6.5 of [Pal13], if homological stability with (untwisted) A coefficients holds for (unlabelled) configuration spaces on M in the range $* \leqslant f(k)$, then twisted homological stability will hold in the range $* \leqslant f(k-d)$ for any twisted coefficient system of degree d which factors through the forgetful functor A-mod $\to \mathbb{Z}$-mod. C.f. Remark A.7 above.

If $A = \mathbb{Q}$ then the above twisted coefficient system factors through the inclusion $\mathbb{Q}$-mod $\to \mathbb{Z}$-mod. By Theorem C of [RW13] we may take $f(k) = k$ if $n = \dim(M) \geqslant 3$. For orientable surfaces we may take $f(k) = k-1$ by Corollary 3 of [Chu12] or Theorem 1.3 of [Knu14], and for non-orientable surfaces we may take $f(k) = k$ by Theorem 1.3 of [Knu14]. By the above paragraph the map of spectral sequences is an isomorphism on the second page in the range $s \leqslant f(k-t)$, and therefore in total degree at most k (resp. total degree at most $k-1$ for orientable surfaces). Hence so is the map in the limit.

If $A = \mathbb{F}_p$ for p odd and $n \geqslant 3$ then the twisted coefficient system factors through the inclusion $\mathbb{Z}[\frac{1}{2}]$-mod $\to \mathbb{Z}$-mod. By Proposition A.2 (or Theorem 1.4 of [KM14b]) we may take $f(k) = k$. So as above the map of spectral sequences is an isomorphism in total degree at most k, and therefore so is the map in the limit. □

Remark A.10 A version of Proposition A.9 is also proved in the appendix of [KM14a]. One of the proofs given there is essentially the same as the proof above, and a sketch proof using semi-simplicial resolutions by collections of disjoint arcs in M is also given. This latter method has the advantage that it gives a range of $* \leqslant \frac{k}{2}$ for $\mathbb{Z}$ coefficients (at least when M is orientable), rather than the smaller range $* \leqslant \frac{k}{2} - 1$ for $\mathbb{Z}$ coefficients obtained in Proposition A.9.

Appendix B. Homology stability for configuration spaces with labels in a fibre bundle

The theorem in this appendix can be obtained following step by step the proof in [McD75], as pointed out in the introduction to that paper. We give here a sketch of this proof with some shortcuts, taking advantage of knowing the homology stability theorem with labels (Proposition A.9) in the spirit of [GMTW09].

Definition B.1 Let $\theta\colon X \to M$ be a fibre bundle, let $c\colon \partial M \times [0,1] \to M$ be a collar, let $N \subset \partial M$, write $N_1 = N \times [0,1]$, $M \cup N_{-1} = M \cup_{N\times\{0\}} N \times [-1,0]$ and take $\delta > 0$ smaller than the injectivity radius of M.

(1) $C_k(M;\theta)$ is the space of pairs $(\mathbf{q}, f)$, where $\mathbf{q} \in C_k(M)$ is a configuration and $f\colon \mathbf{q} \to X_{|\mathbf{q}}$ is a section.
(2) $C(M;\theta) := \coprod_{k=0}^{\infty} C_k(M;\theta)$
(3) $\Psi(M,N;\theta)$ is the underlying set of $C(M;\theta)$ with the following topology: Consider the quotient Y of $C(M \cup N_{-1};\theta)$ under the relation $\mathbf{q} \sim \mathbf{q}'$ if and only if $\mathbf{q} \cap (\mathring{M}) = \mathbf{q}' \cap (\mathring{M})$. The natural inclusion $C(M;\theta) \to C(M \cup N_{-1};\theta)$ induces a bijection $C(M;\theta) \cong Y$, which we use to endow $C(M;\theta)$ with a new topology.
(4) $\Psi(TM;\theta)$ is the result of applying Ψ fibrewise to TM and the fibrewise fibre bundle $\theta^* TM \to TM$.

Lemma B.2 ([Hes92, §2.3]) *If the inclusion $N \subset M$ induces a epimorphism on π_0, then the scanning map*

$$\mathfrak{s}^{\gamma,\theta}\colon \Psi(M,N;\theta) \longrightarrow \Gamma(\Psi(TM;\theta),)$$

defined as

$$\mathfrak{s}^{\gamma,\theta}(q,f)(x) = (\tfrac{1}{\gamma} \exp_x^{-1}(\mathbf{q}), f \circ \exp_{x|\gamma\mathbf{q}})$$

is a homotopy equivalence.

In the paper, Hesselholt considers X to be a fibre bundle of based spaces. Our case is a particular case of his theorem taking a disjoint basepoint in each fibre.

Let $\pi_1\colon M \dashrightarrow I$ be the partially defined function that sends a point in the image of the collar to the second coordinate. Define $\Psi(M,N;\theta)_\bullet$ to be the semi-simplicial space whose space of i-simplices is the space of tuples $(\mathbf{q}, f, a_0, \dots, a_i)$, where $(\mathbf{q}, f) \in \Psi(M,N;\theta)$ and $(a_0,\dots,a_i) \in I^{i+1}$ and $\pi_1(\mathbf{q}) \cap \{a_0,\dots,a_i\} = \emptyset$. The jth face map forgets a_j, and there is an augmentation to $\Psi(M,N;\theta)$ that forgets all the a_j's.

Lemma B.3 *The realization of the augmentation*

$$\|\Psi(M,N;\theta)_\bullet\| \to \Psi(M,N;\theta)$$

is a weak homotoy equivalence.

Proof. This is an augmented topological flag complex [GRW14] satisfying the conditions of Theorem 6.2 in that paper, hence a weak homotopy equivalence. □

Proposition B.4 *If M is a manifold with non-empty boundary and $\theta\colon X \to M$ has path-connected fibres, then the restriction of the scanning map*

$$\mathfrak{s}^{\gamma,\theta}\colon C(M;\theta) \longrightarrow \Gamma_c(\Psi(TM;\theta))$$

is a homology isomorphism in the range in which the stabilisation map of Proposition A.9 is a homology isomorphism.

Proof. We have constructed the following commutative diagram

$$\begin{tikzcd} \|\Psi(M,N_1;\theta)_\bullet\| \arrow[r] \arrow[d] & \Psi(M;\theta) \arrow[r] \arrow[d] & \Gamma(\Psi(TM;\theta)) \arrow[d] \\ \|\Psi(N_1;\theta)_\bullet\| \arrow[r] & \Psi(N_1;\theta) \arrow[r] & \Gamma(\Psi(TM;\theta)) \end{tikzcd} \tag{B.1}$$

All the horizontal maps are homotopy equivalences, by the previous two lemmas. The rightmost vertical map is a fibration. We now choose a ball $L \subset \partial M \smallsetminus N$, and take the colimit

$$P(M,N_1;\theta)_\bullet := \operatorname{colim}\Big(\Psi(M,N_1;\theta)_\bullet \xrightarrow{s} \Psi(M,N_1;\theta)_\bullet \longrightarrow \dots\Big)$$

with respect to the stabilisation maps s that push the configurations outside L_1 and add a point in L_1. The scanning of this operation $\mathfrak{s}^{\gamma,\theta}(s)$ gives also a sequence of maps between spaces of sections, whose colimit we denote by

$$G(M,N_1;\theta) := \operatorname{colim}\Big(\Gamma(\Psi(TM;\theta)) \xrightarrow{\mathfrak{s}(s)} \Gamma(\Psi(TM;\theta)) \longrightarrow \dots\Big)$$

Observe that the maps $\mathfrak{s}^{\gamma,\theta}(s)$ increase the degree by 1. We can consider instead the maps that push the source of the scanning map away from L_1 and glue there the *reflection* of the scanning of some point in L_1. This latter map is a homotopy inverse of $\mathfrak{s}^{\gamma,\theta}(s)$, hence $\mathfrak{s}^{\gamma,\theta}(s)$ homotopy equivalences.

By Proposition A.9, it follows that the semi-simplicial map

$$P(M,N_1;\theta)_\bullet \longrightarrow \Psi(N_1;\theta)_\bullet$$

satisfies the hypotheses of [MS76, Proposition 4], so the realization

$$\|P(M,N_1;\theta)_\bullet\| \longrightarrow \|\Psi(N_1;\theta)_\bullet\|$$

is a homology fibration. Its fibre over any point is the colimit of the space $C(M;\theta)$ with respect to the stabilisation map s. The map

$$G(M,N_1;\theta) \longrightarrow \Gamma(\Psi(TM;\theta))$$

is a Serre fibration (it is a union of Serre fibrations). Its fibre over any point is the colimit of $\Gamma_c(\Psi(TM);\theta)$ with respect to the map obtained by scanning s:

$$\begin{tikzcd} \operatorname{colim} C(M;\theta) \arrow[r] \arrow[d] & \operatorname{colim}\Gamma_c(\Psi(TM);\theta) \arrow[d] \\ \|P(M,N_1;\theta)_\bullet\| \arrow[r, "\simeq"] \arrow[d, "a"] & G(\Psi(TM;\theta)) \arrow[d, "b"] \\ \|\Psi(N_1;\theta)_\bullet\| \arrow[r, "\simeq"] & \Gamma(\Psi(TM;\theta)) \end{tikzcd} \tag{B.2}$$

The fibres of (B.2) together with the maps to the fibres of (B.2) give the following commutative diagram

$$\begin{tikzcd} C(M;\theta) \arrow[r, "\mathfrak{s}^{\gamma,\theta}"] \arrow[d] & \Gamma_c(\Psi(TM);\theta) \arrow[d] \\ \operatorname{colim} C(M;\theta) \arrow[r] & \operatorname{colim}\Gamma_c(\Psi(TM);\theta). \end{tikzcd}$$

The bottom map is a homology equivalence because the horizontal maps in diagram (B.1) are homotopy equivalences. The left vertical map is a homology equivalence in the stable range of Proposition A.9. The right vertical map is a homotopy equivalence. As a consequence, the upper horizontal map is a homology equivalence in the range provided by Proposition A.9 □

The following is proved in the same way as Theorem 1.1 at the bottom of page 34 in [McD75].

Corollary B.5 *If M is a manifold with empty boundary, then the restriction of the scanning map*

$$C(M;\theta) \longrightarrow \Gamma_c(\Psi(TM;\theta))$$

is a homology isomorphism in the range in which the stabilisation map is a homology isomorphism.

Definition B.6 Let $\theta\colon X \to M$ be a fibre bundle, and let $\delta > 0$ be bounded by the injectivity radius of M.

(1) Let θ^*TM be the pullback of TM along θ. This is a vector bundle over X, and the composite $\pi_\theta\colon \theta^*TM \to X \to M$ is a fibre bundle whose fibres are vector bundles. We let $\dot{T}^\theta M$ be the fibrewise Thom construction on the fibres of π_θ.
(2) $\Psi^\delta(M,N;\theta)$ is the space of triples $(\mathbf{q},\epsilon,f)$, where $(\mathbf{q},f) \in \Psi(M;\theta)$ and $0<\epsilon<\delta$ is such that $d(q,q')>2\epsilon$ for all $q,q' \in \mathbf{q}$.

There are fibrewise maps

$$i\colon \dot{T}^\theta M \hookrightarrow \Psi(TM;\theta)$$
$$h\colon \Psi(TM;\theta) \longrightarrow \dot{T}^\theta M$$

defined by

$$i(y,q) = (\theta(y),(q,f)) \qquad \text{with } f(q)=y \qquad i(x,\infty)=(x,\infty)$$
$$h(x,(\mathbf{q},f)) = \begin{cases} \left(f\left(\mathbf{q}_{\text{first}}\right), \frac{\mathbf{q}}{\mathbf{q}_{\text{second}}}\right) & \text{if } \mathbf{q}_{\text{first}} < \mathbf{q}_{\text{second}}, \\ (\theta(y),\infty) & \text{if } \mathbf{q}_{\text{first}} = \mathbf{q}_{\text{second}} \end{cases} \qquad h(x,\infty)=(x,\infty)$$

which are mutually fibrewise homotopy inverses by the same argument as on page 7 (where the definition of $\mathbf{q}_{\text{second}}$ is also given).

Finally, there is a linear scanning map

$$\mathscr{S}^{\delta,\theta}\colon \Psi^\delta(M;\theta) \longrightarrow \Gamma(\dot{T}^\theta M)$$

given by

$$\mathscr{S}^{\delta,\theta}((\mathbf{q},\epsilon),f)(x) = \begin{cases} \infty & \text{if } x \notin B_\epsilon(q) \forall q \in \mathbf{q}, \\ (f(q), \frac{\exp_x^{-1}(q)}{\epsilon}) & \text{if } x \in B_\epsilon(q), q \in \mathbf{q}. \end{cases}$$

As on page 7, there is a commutative square

$$\begin{tikzcd} \Psi^\delta(M;\theta) \arrow[r, "\mathscr{S}^\theta"] \arrow[d] & \Gamma(\dot{T}^\theta M) \arrow[d, "i"] \\ \Psi(M;\theta) \arrow[r, "\mathfrak{s}^{\gamma,\theta}"] & \Gamma(\Psi(TM;\theta)) \end{tikzcd}$$

where the vertical maps are easily seen to be homotopy equivalences, and the lower horizontal map is a homotopy equivalence too by Lemma B.2. This together with Corollary B.5 give:

Theorem B.7 (McDuff's Theorem with labels) *The linear scanning map with labels*

$$\mathscr{S}^{\delta,\theta} \colon C_k(M;\theta) \longrightarrow \Gamma_c(\dot{T}^\theta(M))_k$$

induces an isomorphism in homology groups in the stable range provided by Proposition A.9.

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