Point-pushing actions and configuration-mapping spaces

Martin Palmer and Ulrike Tillmann

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Abstract

Given a manifold M and a point in its interior, the point-pushing map describes a diffeomorphism that pushes the point along a closed path. This defines a homomorphism from the fundamental group of M to the group of isotopy classes of diffeomorphisms of M that fix the basepoint. This map is well-studied in dimension $d = 2$ and is part of the Birman exact sequence. Here we describe for any $d \geqslant 3$ and $k \geqslant 1$ the action, by homotopy classes of homotopy equivalences, of the k-th braid group of M on the k-punctured manifold $M \smallsetminus z$. Equivalently, we describe the monodromy of the universal bundle that associates to a configuration z of size k in M its complement, the space $M \smallsetminus z$. Furthermore, motivated by our work in [PT], we describe the action of the braid group of M on the fibres of configuration-mapping spaces.

Contents

1. Introduction

Let M be a based, connected (smooth) manifold of dimension $d \geqslant 2$ and denote by $C_k(\mathring{M})$ the configuration space of k unordered distinct points in its interior. We may think of it as the *moduli space* of k distinct points in M. Its *universal bundle* is the fiber bundle $U_k(M)$ that associates to each k-tuple $z \in C_k(\mathring{M})$ the k-punctured manifold $M \smallsetminus z$:

$$\begin{array}{ccc} M \smallsetminus z & \longrightarrow & U_k(M) \\ & & \downarrow u \\ & & C_k(\mathring{M}). \end{array}$$

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The primary goal of this paper is to describe the monodromy action (up to homotopy) of the above fibre bundle

$$\mathrm{push}_{(M,z)}\colon \pi_1(C_k(\mathring{M}),z) \longrightarrow \pi_0(\mathrm{hAut}(M \smallsetminus z))$$

where $\mathrm{hAut}(M \smallsetminus z)$ denotes the homotopy equivalences of the complement of z in M; when M has boundary we will consider the relative homotopy equivalences.

Let $(X,*)$ be a fixed connected based space. Applying the functor $\mathrm{Map}(\quad;(X,*))$ (or another continuous functor) to u defines a new fibre bundle:

$$\begin{tikzcd} \mathrm{Map}((M \smallsetminus z,*),(X,*)) \arrow[r] & \mathrm{CMap}_k^*(M;X) \arrow[d, "p"'] \\ & C_k(\mathring{M}). \end{tikzcd}$$

Our second goal is to give explicit formulas for the monodromy action for p (up to homotopy). The total space is an example of the configuration-mapping spaces studied in [EVW; PT]. Indeed, our interest in the monodromy actions was motivated by our study of the homology of configuration-mapping and -section spaces. In [PT] we use the results from this paper to analyse the E^2-page of the Serre spectral sequence associated to p; see also Remark 8.4.

When z is just a single point the monodromy map can be defined in terms of the point-pushing map: it sends an element $[\alpha] \in \pi_1(M,z)$ to the pointed isotopy class of the diffeomorphism that pushes the point z along the curve α and is the identity outside a tubular neighbourhood. It is not difficult to see that the point pushing map and more generally $\mathrm{push}_{(M,z)}$ factors through the (smooth) mapping class group:

$$\mathrm{push}^{\mathsf{sm}}_{(M,z)}\colon \pi_1(C_k(\mathring{M}),z) \longrightarrow \pi_0(\mathrm{Diff}(M;z));$$

here $\mathrm{Diff}(M;z)$ denotes the group of (smooth) diffeomorphisms of M that permute the points in z. If the boundary of M is non-empty we will consider those diffeomorphisms that fix the boundary.

There is a possibly more familiar alternative description of $\mathrm{push}^{\mathsf{sm}}_{(M,z)}$. For z a single point in M, consider the fibration

$$\mathrm{Diff}(M;z) \longrightarrow \mathrm{Diff}(M) \xrightarrow{eval} M$$

where *eval* denotes the map that evaluates a diffeomorphism at z. As M is path-connected, this gives rise to the exact sequence

$$0 \longrightarrow K \longrightarrow \pi_1(M,z) \longrightarrow \pi_0(\mathrm{Diff}(M;z)) \longrightarrow \pi_0(\mathrm{Diff}(M)) \longrightarrow 0.$$

By definition the kernel K of the smooth point-pushing map is a quotient of $\pi_1(\mathrm{Diff}(M,z))$ and hence is abelian. Indeed it is always in the centre of $\pi_1(M,z)$; see [Hat02, page 40, ex. 20].

For $M = S$ a surface of negative Euler characteristic, the connected components of $\mathrm{Diff}(M)$ are contractible [EE69] [ES70] and hence the fibration gives rise to the Birman exact sequence [Bir69]

$$0 \longrightarrow \pi_1(S,z) \longrightarrow \pi_0(\mathrm{Diff}(S;z)) \longrightarrow \pi_0(\mathrm{Diff}(S)) \longrightarrow 0.$$

When α has a two-sided neighbourhood in S, its image is a product of the two Dehn twists around the two curves (oriented oppositely) that form the boundary of a tubular neighbourhood of α. On the other hand, when $S = T$ is the torus, $\mathrm{Diff}(T) \simeq T \rtimes \mathrm{SL}_2(\mathbb{Z})$ [Gra73] and *eval* induces an isomorphism on fundamental groups:

$$\pi_1(\mathrm{Diff}(T);\mathrm{id}_T) \cong K = \pi_1(T,z) \cong \mathbb{Z}^2.$$

Thus the smooth point-pushing map (and hence also the non-smooth version) is well-understood when $d = 2$. Recently, Banks [Ban17] completely determined the kernel K also when $d = 3$. In particular she shows that K is trivial unless the manifold M is prime and Seifert fibered via an S^1 action. In a different direction, Tshishiku [Tsh15] studies the Nielsen realisation problem for the point-pushing map, i.e. asks when the point-pushing map can be factored through $\mathrm{Diff}(M,z)$. However, little seems to be known about the image of the point-pushing map in higher dimensions.

Here we give a complete description, up to homotopy, of the induced self-map of $M \smallsetminus z$ for any element of the fundamental group when M has non-empty boundary. As an example, in section 7, we study the manifolds $M^d_{g,1} = \sharp^g(S^1 \times S^{d-1}) \smallsetminus \mathring{D}^d$ for $d \geqslant 3$ and $g \geqslant 0$ and show that the point-pushing map is injective for these examples (see Proposition 7.4). Inspired by our calculations in these examples, we speculate about a general criterion for injectivity (see Discussion 7.5). We note that for these examples $M^d_{g,1}$, the Nielsen realisation problem is solvable as the fundamental group is free.

Outline and results. The paper is organised as follows. Section 2 contains basic recollections about (relative) monodromy actions associated to fibrations and Section 3 discusses equivalent definitions of the point-pushing map (see Figure 3.1), and considers the induced actions for associated fibre bundles obtained from the universal bundle u by applying a continuous functor. Restricting from now on to manifolds with boundary and dimension $d \geqslant 3$, in Section 4 we note that for a k-tuple z, up to homotopy, $M \smallsetminus z$ decomposes as a wedge of M with a k-fold wedge product of spheres S^{d-1},

$$M \smallsetminus z \simeq M \vee W_k \qquad \text{where} \qquad W_k := \bigvee_k S^{d-1}$$

and $\pi_1(C_k(\mathring{M}), z)$ is the wreath product

$$\pi_1(M)^k \rtimes \Sigma_k.$$

Thus the task of understanding the monodromy action is divided into understanding (on each of the terms M and W_k) the action of the symmetric group elements, which is done in Section 5, and the more complicated action of the loop elements, considered in Section 6. The elements of the symmetric group act, up to homotopy, by the identity on M and by permuting the k summands in the wedge product W_k; compare Proposition 5.1. The precise action of a loop $\alpha \in \pi_1 M$ is the content of Propositions 6.2 and 6.3. Roughly, when α is in the i-th factor of the wreath product, it acts on the summand W_k by taking the i-th sphere S^{d-1} and mapping a neighbourhood of its base point around α before covering itself by a degree ± 1 map depending on whether α lifts to a loop in the orientation double cover of M. The other factors of W_k are mapped by the inclusion. This completely describes the monodromy action of α on $W_k \to M \vee W_k$. The action of α on M depends only on the sequence of intersections of α with the $(d-1)$-cells of M, compare formula (6.8) and Figure 6.2. So, if there are no such intersections, for example when M has no $(d-1)$-cells, then the action on M is simply given by the inclusion. However, if α intersects a $(d-1)$-cell τ of M with intersection number $\sharp(\tau, \alpha)$ then in addition to the inclusion of M, the monodromy action of α takes the cell τ to the i-th factor of W_k by a degree $\sharp(\tau, \alpha)$ map. These assemble to give a map:

$$M \simeq K \longrightarrow\!\!\!\!\!\rightarrow K/K^{(d-2)} \simeq \bigvee_\tau S^{d-1} \longrightarrow S^{d-1}$$

where K is a CW-complex homotopy equivalent to M and $K^{(d-2)}$ denotes its $(d-2)$-skeleton. This completely describes the monodromy action of α on $M \to M \vee W_k$ after projection to each factor M and W_k. The full description of this action in Definition 6.6 takes into account the precise sequence of intersections of α and the $(d-1)$-cells. We illustrate this latter more complicated action of α with several examples in Section 7, and discuss the general question of injectivity for the point-pushing map. Finally in Section 8 the induced action on the fibres of p for configuration mapping spaces is described. As a further application we compute the number of connected components for configuration mapping spaces in Corollary 8.5.

2. Monodromy actions

We first recall the *monodromy action* associated to a fibration. Let $f\colon E \to B$ be a continuous map and write $F = f^{-1}(b)$ for a point $b \in B$. Assume that f satisfies the *homotopy lifting property* (*covering homotopy property*) (*cf.* [Hat02, §4.2] or [May99, §7]) with respect to the spaces F and $F \times [0,1]$. For example, this holds if f is a Hurewicz fibration, or if f is a Serre fibration and F is a CW-complex. In particular it holds whenever f is a fibre bundle and either F is a CW-complex or B is paracompact.

Definition 2.1 For a space F, write $\mathrm{hAut}(F) \subseteq \mathrm{Map}(F,F)$ for the space of continuous self-maps $F \to F$, with the compact-open topology, that admit a homotopy inverse. This is a topological monoid under composition, and *grouplike*, i.e. the discrete monoid $\pi_0(\mathrm{hAut}(F))$ is a group (it is the automorphism group of F in the homotopy category).

For a pair of spaces (F, F_0), we write $\mathrm{End}(F|F_0)$ for the topological monoid (with the compact-open topology) of self-maps of F that are the identity on F_0 and we write $\mathrm{hAut}(F|F_0) \subseteq \mathrm{End}(F|F_0)$ for the union of those path-components of $\mathrm{End}(F|F_0)$ corresponding to the invertible elements of the discrete monoid $\pi_0(\mathrm{End}(F|F_0))$. Note that $\mathrm{hAut}(F|\varnothing) = \mathrm{hAut}(F)$.

Definition 2.2 (*Monodromy actions.*) Under the above assumptions, the *monodromy action* associated to f is the action-up-to-homotopy

$$\mathrm{mon}_f \colon \pi_1(B,b) \longrightarrow \pi_0(\mathrm{hAut}(F)) \tag{2.1}$$

of $\pi_1(B,b)$ on F defined as follows. For an element $[\gamma] \in \pi_1(B,b)$ represented by a loop $\gamma\colon [0,1] \to B$, let $g\colon F \times [0,1] \to E$ be a choice of lift in the diagram:

$$\begin{tikzcd} F \arrow[rr, "\mathrm{incl}"] \arrow[d, "{(-,0)}"'] & & E \arrow[d, "f"] \\ F \times [0,1] \arrow[r, two heads] \arrow[urr, dashed] & {[0,1]} \arrow[r, "\gamma"'] & B \end{tikzcd} \tag{2.2}$$

and define $\mathrm{mon}_f([\gamma]) = [g(-,1)]$.

There is also a relative version of this construction. Let $F_0 \subseteq F$ be a subspace and assume that f satisfies the *relative homotopy lifting property* with respect to the pairs of spaces (F, F_0) and $(F, F_0) \times [0,1]$. For example, this holds if f is a Hurewicz fibration, or if f is a Serre fibration and (F, F_0) is a relative CW-complex. Also assume that we have a topological embedding $i\colon F_0 \times B \hookrightarrow E$ such that $f \circ i$ is the projection onto the second factor and $i(-,b)$ is the inclusion $F_0 \subseteq F \subseteq E$. (This says, essentially, that f contains the trivial fibration over B with fibre F_0 as a sub-fibration.)

Definition 2.3 (*Relative monodromy actions.*) Under these assumptions, the *relative monodromy action* associated to f and F_0 is the action-up-to-homotopy

$$\mathrm{mon}_f \colon \pi_1(B,b) \longrightarrow \pi_0(\mathrm{hAut}(F|F_0)), \tag{2.3}$$

where $\mathrm{hAut}(F|F_0)$ is as in Definition 2.1, constructed as follows. For an element $[\gamma] \in \pi_1(B,b)$ represented by a loop $\gamma\colon [0,1] \to B$, let $g\colon F \times [0,1] \to E$ be a choice of lift in the diagram:

$$\begin{tikzcd} (F_0 \times [0,1]) \cup (F \times \{0\}) \arrow[rr, "(i \circ (\mathrm{id}_{F_0} \times \gamma)) \cup \mathrm{incl}"] \arrow[d, "\mathrm{incl}"'] & & E \arrow[d, "f"] \\ F \times [0,1] \arrow[r, two heads] \arrow[urr, dashed] & {[0,1]} \arrow[r, "\gamma"'] & B \end{tikzcd} \tag{2.4}$$

and define $\mathrm{mon}_f([\gamma]) = [g(-,1)]$.

Lemma 2.4 *The monodromy action* (2.1) *and relative monodromy action* (2.3) *are well-defined.*

Proof. For the monodromy action (2.1), the proof is given in [PT, Lemma 5.3]. The proof for the relative monodromy action (2.3) is similar. ◻

3. Point-pushing actions

This section defines the *point-pushing action* associated to a manifold M and a finite subset $z \subset \mathring{M}$ of its interior. We give two definitions, one (Definition 3.1) via the monodromy action of the "universal" bundle (3.1), and a smooth version (Definition 3.2) via the long exact sequence of the bundle (3.3), as well as a simple geometric description in Lemma 3.4 for manifolds of dimension at least 3. We then describe point-pushing actions on mapping spaces and other spaces associated functorially to the complement $M \smallsetminus z$ (see Definitions 3.11 and 3.12).

Definition 3.1 (*The point-pushing action.*) For a manifold-with-boundary M and a finite subset $z \subseteq \mathring{M}$ of cardinality k, the *point-pushing action* of $\pi_1(C_k(\mathring{M}), z)$ on $M \smallsetminus z$ is defined as follows.

First, define $\bar{C}_{1,k}(M)$ to be the configuration space of k unordered green points in the interior of M and one red point in M, which may lie on the boundary. There is a fibre bundle

$$u\colon U_k(M) = \bar{C}_{1,k}(M) \longrightarrow C_k(\mathring{M}), \tag{3.1}$$

given by forgetting the red point, whose fibre is $F = u^{-1}(z) \cong M \smallsetminus z$. This is the *universal bundle* referred to in the introduction. Let $F_0 = \partial M \subseteq M \smallsetminus z$ and note that $(M \smallsetminus z, \partial M)$ is a relative CW-complex, since it is a (smooth) manifold with boundary. There is an obvious embedding

$$i\colon \partial M \times C_k(\mathring{M}) \longhookrightarrow \bar{C}_{1,k}(M)$$

satisfying the conditions of Definition 2.3. By Definition 2.3 and Lemma 2.4, there is therefore a well-defined relative monodromy action

$$\mathrm{push}_{(M,z)}\colon \pi_1(C_k(\mathring{M}), z) \longrightarrow \pi_0(\mathrm{hAut}(M \smallsetminus z | \partial M)). \tag{3.2}$$

This is, by definition, the *point-pushing action* of $\pi_1(C_k(\mathring{M}), z)$ on $M \smallsetminus z$. For $[\gamma] \in \pi_1(C_k(\mathring{M}), z)$, the homotopy class of maps

$$\mathrm{push}_\gamma = \mathrm{push}_{(M,z)}([\gamma])\colon M \smallsetminus z \longrightarrow M \smallsetminus z$$

(fixing ∂M pointwise) is called the *point-pushing map* of $[\gamma]$ on $M \smallsetminus z$.

Definition 3.2 (*A smooth version.*) The monodromy action (3.2) may be refined to an action by isotopy classes of *diffeomorphisms* of M fixing ∂M. Let $\mathrm{Diff}_\partial(M)$ denote the topological group of diffeomorphisms of M fixing ∂M pointwise, in the smooth Whitney topology. There is a fibre bundle (*cf.* [Pal60; Cer61; Lim63]):

$$\mathrm{Diff}_\partial(M) \longrightarrow C_k(\mathring{M}), \tag{3.3}$$

defined by $\varphi \mapsto \varphi(z)$, whose fibre over z is the subgroup $\mathrm{Diff}_\partial(M, z)$ of diffeomorphisms fixing z as a subset. Denote by

$$\mathrm{push}^{\mathsf{sm}}_{(M,z)}\colon \pi_1(C_k(\mathring{M})) \longrightarrow \pi_0(\mathrm{Diff}_\partial(M, z))$$

the connecting homomorphism in the long exact sequence of homotopy groups of (3.3). This is, by definition, the *smooth point-pushing action* of $\pi_1(C_k(\mathring{M}))$ on $M \smallsetminus z$. For $[\gamma] \in \pi_1(C_k(\mathring{M}), z)$, the isotopy class of diffeomorphisms

$$\mathrm{push}^{\mathsf{sm}}_\gamma = \mathrm{push}^{\mathsf{sm}}_{(M,z)}([\gamma])\colon (M, z) \longrightarrow (M, z)$$

(fixing ∂M pointwise and z setwise) is the *smooth point-pushing map* of $[\gamma]$ on (M, z).

One may check that these constructions are related as follows.

Lemma 3.3 *The point-pushing actions of Definitions 3.1 and 3.2 are related by the commutative diagram*

$$\begin{tikzcd} \pi_1(C_k(\mathring{M}), z) \arrow[r, "\mathrm{push}^{\mathsf{sm}}_{(M,z)}"] \arrow[d, equal] & \pi_0(\mathrm{Diff}_\partial(M, z)) \arrow[d, "i"] \\ \pi_1(C_k(\mathring{M}), z) \arrow[r, "\mathrm{push}_{(M,z)}"'] & \pi_0(\mathrm{hAut}(M \smallsetminus z | \partial M)), \end{tikzcd} \tag{3.4}$$

where i is induced by the inclusion $\mathrm{Diff}_\partial(M, z) \hookrightarrow \mathrm{hAut}(M \smallsetminus z | \partial M)$ given by $\varphi \mapsto \varphi|_{M \smallsetminus z}$.

If $d = \dim(M) \geqslant 3$, there is a useful geometric description of the smooth point-pushing action, which we will use later. An element $\gamma \in \pi_1(C_k(\mathring{M}), z)$ determines a certain number of oriented loops $\gamma_1, \ldots, \gamma_j$ in M, each passing through at least one point of z, such that exactly one of the loops passes through each point of z. (The number $j \leqslant k$ of such loops is the number of cycles in the cycle decomposition of the permutation of z induced by γ.) Choose representatives of the loops $\gamma_1, \ldots, \gamma_j$ that are smoothly embedded and have pairwise disjoint images. Also choose pairwise disjoint closed tubular neighbourhoods $T_1, \ldots, T_j$ of these loops, which we assume to be contained in the interior of M. Define a diffeomorphism

$$\varphi_{(T_1,\ldots,T_j)} \colon (M, z) \longrightarrow (M, z)$$

fixing ∂M pointwise and z setwise as follows. On the complement of the tubular neighbourhoods, $\varphi_{(T_1,\ldots,T_j)}$ is the identity. Suppose that the tubular neighbourhood T_i contains k_i of the points of z (so $k_1 + \cdots + k_j = k$) and identify $T \smallsetminus (z \cap T)$ with

$$((D^{d-1} \times \mathbb{R}) \smallsetminus (\{0\} \times \mathbb{Z}))/{\sim},$$

where $\sim$ is either the equivalence relation given by $(x, t) \sim (x, t + k_i)$ or the equivalence relation given by $(x, t) \sim (r(x), t + k_i)$, where $r \colon D^{d-1} \to D^{d-1}$ is a fixed reflection in a hyperplane passing through 0, depending on whether or not the loop γ_i lifts to a loop in the orientation double cover of M. Choose a smooth function $\lambda \colon [0, 1] \to [0, 1]$ that takes the value 1 on $[0, \epsilon]$ and the value 0 on $[1 - \epsilon, 1]$ for some $\epsilon > 0$. Then the restriction of $\varphi_{(T_1,\ldots,T_j)}$ to T_i, under this identification, is defined by

$$\varphi_{(T_1,\ldots,T_j)}(x, t) = (x, t + \lambda(|x|)).$$

See Figure 3.1 for an illustration. We record this geometric description in the following lemma.

Lemma 3.4 (*Geometric point-pushing.*) *Let M be a smooth manifold-with-boundary of dimension $d \geqslant 3$ and let $[\gamma] \in \pi_1(C_k(\mathring{M}), z)$. Choose a collection of smoothly embedded loops $\gamma_1, \ldots, \gamma_j$ and tubular neighbourhoods $T_1, \ldots, T_j$ as described above. Then*

$$[\varphi_{(T_1,\ldots,T_j)}] = \mathrm{push}^{\mathsf{sm}}_{(M,z)}([\gamma]) \in \pi_0(\mathrm{Diff}_\partial(M, z)).$$

Associated point-pushing actions. We have so far described the "universal" point-pushing action of $\pi_1(C_k(\mathring{M}), z)$ on the complement $M \smallsetminus z$, for a subset $z \subset \mathring{M}$ with $|z| = k$. We now discuss induced point-pushing actions associated to continuous endofunctors $T \colon \mathrm{Top} \to \mathrm{Top}$ or $T \colon \mathrm{Top}_* \to \mathrm{Top}_*$ (or, more generally, to a continuous functor of the form (3.8)).

Definition 3.5 (*Associated fibre bundles.*) We first recall that, if $f \colon E \to B$ is a fibre bundle with fibre F (and structure group $\mathrm{Homeo}(F)$ in the compact-open topology), and if $T \colon \mathrm{Top} \to \mathrm{Top}$ is a continuous endofunctor (covariant or contravariant) of the topologically-enriched category of spaces, there is an associated fibre bundle

$$f_T \colon T_{\mathrm{fib}}(E) \longrightarrow B \tag{3.5}$$

with fibre $T(F)$, constructed by "applying T fibrewise" to E. More precisely, the functor T restricts to a continuous group (anti-)homomorphism

$$\mathrm{Homeo}(F) \longrightarrow \mathrm{Homeo}(T(F)), \tag{3.6}$$

and we define (3.5) to be the Borel construction $\mathrm{Prin}(E) \times_{\mathrm{Homeo}(F)} T(F)$, where $\mathrm{Prin}(E) \to B$ is the principal $\mathrm{Homeo}(F)$-bundle associated to f, and where $\mathrm{Homeo}(F)$ acts on $T(F)$ via (3.6). (See [Ste51, §§8–9] for more details.)

There is an exactly analogous construction if f is equipped with a section and $T \colon \mathrm{Top}_* \to \mathrm{Top}_*$ is a continuous endofunctor of the topologically-enriched category of *based* spaces. In this case the structure group of f reduces to the based homeomorphism group $\mathrm{Homeo}_*(F)$ and T restricts to a continuous group homomorphism

$$\mathrm{Homeo}_*(F) \longrightarrow \mathrm{Homeo}_*(T(F)), \tag{3.7}$$

so we may define (3.5) to be the Borel construction $\mathrm{Prin}_*(E) \times_{\mathrm{Homeo}_*(F)} T(F)$, where $\mathrm{Prin}_*(E) \to B$ is the principal $\mathrm{Homeo}_*(F)$-bundle associated to f, and where $\mathrm{Homeo}_*(F)$ acts on $T(F)$ via (3.7).

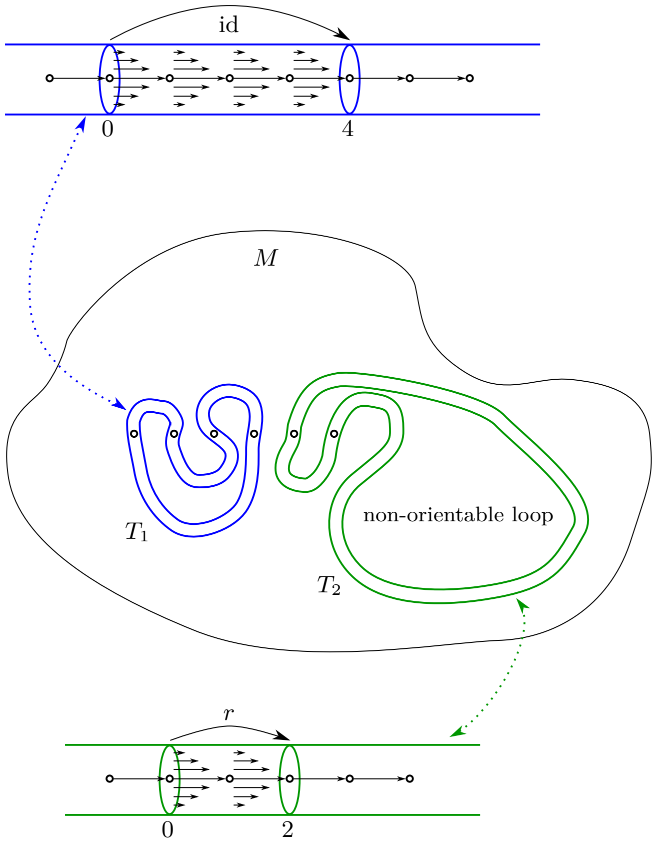

Figure 3.1 An example of the point-pushing action for $|z| = 6$ and where the loop $\gamma \in \pi_1(C_6(\mathring{M}), z)$ induces a permutation of z with one 4-cycle and one 2-cycle.

Definition 3.6 (*Configuration-mapping spaces.*) Let X be any space and consider the (contravariant) continuous functor

$$T = \mathrm{Map}(-, X) \colon \mathrm{Top} \longrightarrow \mathrm{Top}.$$

The fibre bundle associated by T to the bundle (3.1) is denoted by

$$T_{\mathrm{fib}}(\bar{C}_{1,k}(M)) = \mathrm{CMap}_k(M; X) \longrightarrow C_k(\mathring{M}),$$

and its total space is the k-th *configuration-mapping space* of M and X. A point in $\mathrm{CMap}_k(M; X)$ consists of a configuration $z \subset \mathring{M}$ in the interior of M and a continuous map $M \smallsetminus z \to X$.

If $\partial M \neq \varnothing$, the fibre bundle (3.1) admits a canonical section given by $z \mapsto (z, *)$, where $* \in \partial M$ is a choice of basepoint. Thus, choosing a basepoint for X, we may also consider the fibre bundle associated to (3.1) by the continuous functor $T = \mathrm{Map}_*(-, X) \colon \mathrm{Top}_* \to \mathrm{Top}_*$, which is denoted by

$$T_{\mathrm{fib}}(\bar{C}_{1,k}(M)) = \mathrm{CMap}_k^*(M; X) \longrightarrow C_k(\mathring{M}).$$

A point in $\mathrm{CMap}_k^*(M; X)$ consists of a configuration $z \subset \mathring{M}$ in the interior of M together with a *based* continuous map $M \smallsetminus z \to X$.

Definition 3.7 (*Associated fibre bundles, II.*) The structure group of the bundle (3.1) may be reduced further to $\mathrm{Homeo}_{\partial M}(M, z)$, the group of self-homeomorphisms of M that fix z setwise and ∂M pointwise. Hence any continuous functor

$$T \colon \mathrm{Homeo}_{\partial M}(M, z) \longrightarrow \mathrm{Top} \tag{3.8}$$

(i.e., any space with a continuous action of $\mathrm{Homeo}_{\partial M}(M, z)$) associates to (3.1) a new fibre bundle

$$T_{\mathrm{fib}}(\bar{C}_{1,k}(M)) \longrightarrow C_k(\mathring{M}) \tag{3.9}$$

by taking the Borel construction of the associated principal $\mathrm{Homeo}_{\partial M}(M, z)$-bundle.

Remark 3.8 For comparison, the associated fibre bundles of Definition 3.5 above correspond to continuous functors (3.8) that are of the form

$$\mathrm{Homeo}_{\partial M}(M,z) \xrightarrow{-|_{M\smallsetminus z}} \mathrm{Homeo}(M\smallsetminus z) \subset \mathrm{Top} \longrightarrow \mathrm{Top},$$

in other words, that extend to an endofunctor of Top. However, there are interesting (and more subtle) examples that do not extend in this way, as we show in the next example.

Definition 3.9 (*Configuration-mapping spaces, II.*) Fix a basepoint $* \in \partial M$, a based space X and a subset $c \subseteq [S^{d-1}, X]$ of unbased homotopy classes of maps $S^{d-1} \to X$. If M is non-orientable we assume that c consists of fixed points under the involution of $[S^{d-1}, X]$ given by a reflection of S^{d-1}. There is a continuous functor

$$\mathrm{Map}_*^c(-,X)\colon \mathrm{Homeo}_{\partial M}(M,z) \longrightarrow \mathrm{Top} \tag{3.10}$$

defined as follows. The unique object on the left-hand side is sent to the space (with the compact-open topology) of based, continuous maps $f\colon M\smallsetminus z \to X$ with "monodromy" contained in c. The last condition means that, if $e\colon D^d \to M$ is an embedding such that $z \cap e(D^d)$ is a single point in the interior of $e(D^d)$, then the homotopy class of $f \circ e|_{\partial D^d}$ lies in c. (If M is orientable, we fix an orientation and require that e is orientation-preserving in the preceding sentence.) One may then check that the natural action of $\varphi \in \mathrm{Homeo}_{\partial M}(M,z)$ on the mapping space $\mathrm{Map}_*(M\smallsetminus z, X)$ preserves the subspace $\mathrm{Map}_*^c(M\smallsetminus z, X)$. The fibre bundle associated by (3.10) to the bundle (3.1) is denoted by

$$\mathrm{CMap}_k^{c,*}(M;X) \longrightarrow C_k(\mathring{M}), \tag{3.11}$$

and its total space is the k-th *based configuration-mapping space* of M and X with "*monodromy*" or "*charge*" c.

Remark 3.10 Configuration-mapping spaces are discussed in more detail in [PT, §2], and may be generalised to *configuration-section spaces*, which are defined in [PT, §3]. There are of course many other natural continuous functors $T\colon \mathrm{Top} \to \mathrm{Top}$ or $T\colon \mathrm{Homeo}_{\partial M}(M,z) \to \mathrm{Top}$ that may be used to construct interesting fibre bundles associated to the "universal" bundle (3.1).

Definition 3.11 (*Associated point-pushing action.*) For a space T with a continuous action of $\mathrm{Homeo}_{\partial M}(M,z)$, viewed as a continuous functor $T\colon \mathrm{Homeo}_{\partial M}(M,z) \to \mathrm{Top}$, we have from Definition 3.7 a fibre bundle (3.9)

$$T_{\mathrm{fib}}(\bar{C}_{1,k}(M)) \longrightarrow C_k(\mathring{M})$$

with fibre T. The *associated point-pushing action* of $\pi_1(C_k(\mathring{M}), z)$ on T is then the monodromy action of this fibre bundle, denoted by

$$\mathrm{push}_{(M,z,T)}\colon \pi_1(C_k(\mathring{M}), z) \longrightarrow \pi_0(\mathrm{hAut}(T)). \tag{3.12}$$

Definition 3.12 (*Point-pushing action on mapping spaces.*) In particular, if we specialise to the case $T = \mathrm{Map}_*^c(M\smallsetminus z, X)$ for a based space X and a subset $c \subseteq [S^{d-1}, X]$, as in Definition 3.9, we have an associated point-pushing action

$$\mathrm{push}_{(M,z,X,c)}\colon \pi_1(C_k(\mathring{M}), z) \longrightarrow \pi_0(\mathrm{hAut}(\mathrm{Map}_*^c(M\smallsetminus z, X))).$$

which is the monodromy action of the fibre bundle (3.11). This can of course be straightforwardly generalised to a point-pushing action of $\pi_1(C_k(\mathring{M}), z)$ on $\mathrm{Map}^c((M\smallsetminus z, D), (X,*))$ for any subset $D \subseteq \partial M$.

The following elementary lemma relates the *point pushing action* of $\pi_1(C_k(\mathring{M}), z)$ on $M\smallsetminus z$ (Definition 3.1) and its *associated point-pushing action* on the mapping space $\mathrm{Map}^c((M\smallsetminus z, D), (X,*))$ (Definition 3.12).

Lemma 3.13 *The point-pushing action of $\pi_1(C_k(\mathring{M}), z)$ on $\mathrm{Map}^c((M\smallsetminus z, D), (X,*))$ is obtained from its point-pushing action on $M\smallsetminus z$ by pre-composition. In other words, the following diagram* *commutes:*

$$\begin{tikzcd}[column sep=large]
\pi_1(C_k(\mathring{M}), z) \arrow[r, "\mathrm{push}_{(M,z)}"] \arrow[d, equal] & \pi_0(\mathrm{hAut}(M \smallsetminus z|\partial M)) \arrow[d, "\circ"] \\
\pi_1(C_k(\mathring{M}), z) \arrow[r, "\mathrm{push}_{(M,z,X,c)}"'] & \pi_0(\mathrm{hAut}(\mathrm{Map}^c((M \smallsetminus z, D), (X, *)))),
\end{tikzcd} \tag{3.13}$$

where the vertical homomorphism $\circ$ *is defined by composition. In particular, the action up to homotopy of* $\pi_0(\mathrm{hAut}(M \smallsetminus z|\partial M))$ *on the mapping space* $\mathrm{Map}((M \smallsetminus z, D), (X, *))$ *preserves the subspace* $\mathrm{Map}^c((M \smallsetminus z, D), (X, *))$ *for each subset* $c \subseteq [S^{d-1}, X]$*.*

4. Formulas for point-pushing actions

Let M be a connected manifold of dimension $d \geqslant 3$, let $z \subset \mathring{M}$ be a k-point configuration in its interior, $D \subseteq \partial M$ an embedded $(d-1)$-dimensional disc in its boundary, X a based space and $c \subseteq [S^{d-1}, X]$ a non-empty set of unbased homotopy classes of maps $S^{d-1} \to X$. Our goal is to give explicit formulas for the point-pushing action of $\pi_1(C_k(\mathring{M}), z)$ on $M \smallsetminus z$ (Definition 3.1). These will be given in the following two sections; in this section we first fix notation and the identifications that we will use.

Notation 4.1 Let W_k denote a wedge $\bigvee^k S^{d-1}$ of k copies of the $(d-1)$-sphere.

Construction 4.2 Let us choose an explicit homotopy equivalence of pairs

$$(M \smallsetminus z, D) \;\simeq\; (M \vee W_k, *), \tag{4.1}$$

as follows (see Figure 4.1 for an illustration). Choose a d-dimensional closed disc B in M containing the configuration z in its interior and such that $B \cap \partial M$ is a $(d-1)$-dimensional disc in ∂M containing (but not equal to) D. (In Figure 4.1, we may assume that $D = \partial M \cap B'$.) Note that the closure M' of $M \smallsetminus B$ in M is also homeomorphic to M. Choose a basepoint $*$ of M in the intersection $\partial M \cap B \cap M'$. Choose also k embedded $(d-1)$-spheres in B such that each sphere intersects ∂B at the basepoint $*$ and nowhere else, the spheres are pairwise disjoint except for $*$ and each sphere "wraps once around each of the points of z" (this is more formally expressed by the condition that $B \smallsetminus z$ must deformation retract onto the union of the spheres). The union of M' and the spheres is homeomorphic to the wedge sum on the right-hand side of (4.1), and there is a deformation retraction of $M \smallsetminus z$ onto this subspace, supported in $B \smallsetminus z$, fixing the basepoint $*$ and sending D onto $\{*\}$.

Notation 4.3 From now on, we will write $\pi_1(C_k(\mathring{M}), z)$ just as $\pi_1(C_k(M))$, leaving the basepoint z implicit, and using the fact that the inclusion $C_k(\mathring{M}) \hookrightarrow C_k(M)$ is a homotopy equivalence.

Notation 4.4 By the smooth version of the point-pushing action (see Definition 3.2), an element $\gamma \in \pi_1(C_k(M))$ induces (an isotopy class of) a self-diffeomorphism $\mathrm{push}^{\mathsf{sm}}_\gamma \colon M \to M$, fixing ∂M pointwise and z setwise, which has an explicit geometric representative $\varphi_{(T_1,\ldots,T_j)}$ given by Lemma 3.4 if $\dim(M) \geqslant 3$. We denote its restriction to a self-diffeomorphism of $M \smallsetminus z$ by

$$\pi_\gamma \colon M \smallsetminus z \longrightarrow M \smallsetminus z.$$

By abuse of notation, we also denote by π_γ the (homotopy class of a) homotopy self-equivalence of $M \vee W_k$ fixing $*$ induced via the deformation retraction (4.2):

$$\begin{tikzcd}
M \smallsetminus z \arrow[r, "\pi_\gamma"] & M \smallsetminus z \arrow[d, "\simeq"', "(4.2)"] \\
M \vee W_k \arrow[u, "\mathrm{incl}", "\simeq"'] \arrow[r, "\pi_\gamma"] & M \vee W_k.
\end{tikzcd} \tag{4.2}$$

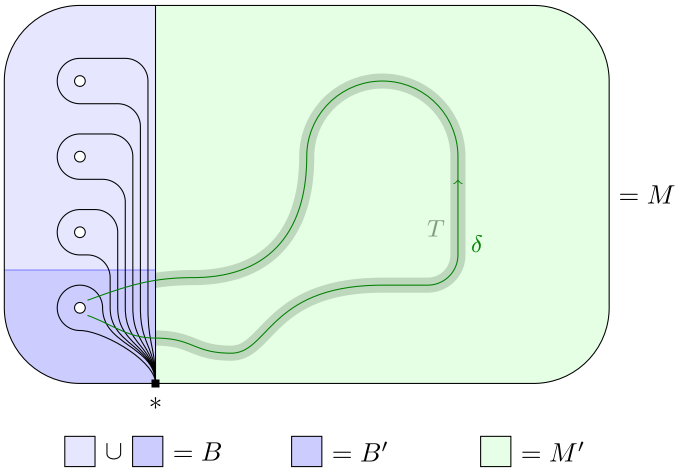

Figure 4.1 An embedding of $M \vee (\bigvee^k S^{d-1})$ into $M \smallsetminus z$ as a deformation retract, together with a loop δ in $B' \cup M'$ based at $z \cap B'$ and a tubular neighbourhood T of its intersection with M'.

Recall [Til16, Lemma 4.1] that, for $\dim(M) \geqslant 3$, the fundamental group $\pi_1(C_k(M))$ decomposes as the semi-direct product $\pi_1(M)^k \rtimes \Sigma_k$. In the next two sections we give explicit formulas for the bottom horizontal map of (4.2) for $\gamma = (\alpha_1, \ldots, \alpha_k; \sigma) \in \pi_1(M)^k \rtimes \Sigma_k$ under this decomposition.

Notation 4.5 We collect here some additional notation that will be used in the following two sections.

- For a wedge $A \vee B$, we write inc_A (resp. inc_B) for the inclusion of the first (resp. second) summand, and similarly we write pr_A (resp. pr_B) for the projection onto the first (resp. second) summand.
- For a map $f\colon A \vee B \to C$ we will sometimes write f as a (1×2)-matrix:

$$f = \left(\begin{array}{c|c} f_A & f_B \end{array} \right),$$

  where $f_A = f \circ \mathrm{inc}_A$ and $f_B = f \circ \mathrm{inc}_B$. Note that f_A and f_B jointly determine f, since $\vee$ is a coproduct.
- For a map $f\colon A \vee B \to C \vee D$ we will also sometimes write f as a (2×2)-matrix:

$$f = \left(\begin{array}{c|c} f_A & f_B \end{array} \right) \rightsquigarrow \left(\begin{array}{c|c} {}_C f_A & {}_C f_B \\ \hline {}_D f_A & {}_D f_B \end{array} \right),$$

  where ${}_C f_A = \mathrm{pr}_C \circ f \circ \mathrm{inc}_A$, etc. Note that the pair of ${}_C f_A$ and ${}_D f_A$ does *not* determine f_A (since $\vee$ is not a product), so the (2×2)-matrix-notation loses information. (This is why we write "$\rightsquigarrow$" instead of "$=$" in this case.)
- As mentioned above, we have for $\dim(M) \geqslant 3$ a splitting $\pi_1(C_k(M)) \cong \pi_1(M)^k \rtimes \Sigma_k$. Thus, for each $\sigma \in \Sigma_k$ and $\alpha \in \pi_1(M)$, we have elements

$$(1, \ldots, 1; \sigma) \text{ and } (\alpha, 1, \ldots, 1; \mathrm{id}) \in \pi_1(C_k(M)),$$

  which we will denote simply by σ and α by abuse of notation. We will always use these letters for elements of these two subgroups of $\pi_1(C_k(M))$, and we will denote a general element of $\pi_1(C_k(M))$ by γ.
- We take the basepoint of S^{d-1} to be the south pole, and write

$$\mathrm{pinch}\colon S^{d-1} \longrightarrow S^{d-1} \vee S^{d-1}$$

  for the map that collapses the equator of S^{d-1} to a point. This is a based map, where we take the convention that the basepoint of $S^{d-1} \vee S^{d-1}$ is contained in the left-hand summand (note that it is *not* the point at which the wedge sum is taken).

- We write
$$\mathrm{coll}\colon S^{d-1} \longrightarrow [0,1]$$
for the "collapse" map that projects $S^{d-1} \subset \mathbb{R}^d$ onto the d-th coordinate (so the south pole goes to -1 and the north pole goes to 1) and then linearly reparametrises by $x \mapsto \frac{1}{2}(x+1)$.

Remark 4.6 Since $\pi_1(C_k(M))$ is generated by elements of the form $(1,\ldots,1;\sigma)$ and $(\alpha,1,\ldots,1;\mathrm{id})$ (which we henceforth denote simply by σ and α) for $\sigma \in \Sigma_k$ and $\alpha \in \pi_1(M)$, it will suffice to give explicit formulas for
$$\pi_\sigma \text{ and } \pi_\alpha \colon M \vee W_k \longrightarrow M \vee W_k$$
up to basepoint-preserving homotopy, for all $\sigma \in \Sigma_k$ and $\alpha \in \pi_1(M)$. This will be done in sections 5 and 6 respectively.

Terminology 4.7 The elements $\sigma = (1,\ldots,1;\sigma)$ will be called *symmetric generators* of $\pi_1(C_k(M))$ and the elements $\alpha = (\alpha,1,\ldots,1;\mathrm{id})$ will be called *loop generators* of $\pi_1(C_k(M))$.

5. Symmetric generators

The action of the *symmetric generators* of $\pi_1(C_k(M))$ on $M \vee W_k$ is fairly easy to describe.

Proposition 5.1 *For any element $\sigma \in \Sigma_k$ we have*
$$\pi_\sigma = \mathrm{id}_M \vee \sigma_\sharp = \left(\ \mathrm{inc}_M \ \middle| \ \mathrm{inc}_{W_k} \circ \sigma_\sharp \ \right) \ \rightsquigarrow \ \left(\begin{array}{c|c} \mathrm{id}_M & * \\ \hline * & \sigma_\sharp \end{array} \right), \tag{5.1}$$
where $\sigma_\sharp$ denotes the obvious self-map of $W_k = \bigvee^k S^{d-1}$ determined by the permutation σ.

Proof. In the geometric model $\varphi_{(T_1,\ldots,T_j)}$ (see Lemma 3.4) for the point-pushing diffeomorphism of (M,z) induced by $\gamma = (1,\ldots,1;\sigma)$, we may assume that the tubular neighbourhoods $T_1,\ldots,T_j$ are all contained in the codimension-zero ball $B \subset M$ (see Figure 4.1). Since $\varphi_{(T_1,\ldots,T_j)}$ is the identity outside of the tubular neighbourhoods, this implies that $\pi_\sigma = \mathrm{id}_M \vee \psi$, for some automorphism ψ of W_k. Moreover, it is clear from this geometric model that (up to homotopy) ψ simply permutes the k embedded $(d-1)$-spheres in Figure 4.1. $\square$

6. Loop generators

For any $\alpha \in \pi_1(M,*)$, the point-pushing map $\pi_\alpha \colon M \smallsetminus z \to M \smallsetminus z$ may be assumed (up to basepoint-preserving homotopy) to be supported in a tubular neighbourhood of a loop α' in M, based at one of the points of the configuration z, in the homotopy class determined by conjugating α with a path in B from $*$ to this point (see Figure 4.1). We may choose α' and its tubular neighbourhood T to be contained in $M' \cup B'$, so the support of $\pi_\alpha \colon M \smallsetminus z \to M \smallsetminus z$ is contained in $M' \cup B'$. Under the identification (4.1), this implies the following.

Lemma 6.1 *For any $\alpha \in \pi_1(M)$, up to based homotopy, $\pi_\alpha \colon M \vee W_k \to M \vee W_k$ is of the form*
$$\pi_\alpha = \bar{\pi}_\alpha \vee \mathrm{id}_{W_{k-1}},$$
where $\bar{\pi}_\alpha$ is a self-map of $M \vee S^{d-1}$, unique up to based homotopy.

We therefore just have to describe the map $\bar{\pi}_\alpha$ for each $\alpha \in \pi_1(M)$. We first do this under an additional assumption on the manifold M. Recall that the *handle-dimension* of a manifold is the smallest i such that M may be constructed using handles of degree at most i. Using the cores of such a handle decomposition, this implies that M deformation retracts onto an embedded CW-complex of dimension equal to the handle dimension of M. Since M, in our situation, is connected and has non-empty boundary, its handle-dimension is necessarily at most $\dim(M)-1$.

Proposition 6.2 *Suppose that the handle dimension of M is at most $\dim(M)-2$. Then, for any element $\alpha \in \pi_1(M)$ we have*

$$\bar{\pi}_\alpha = \big(\ \mathrm{inc}_M \ \big|\ ((\alpha \circ \mathrm{coll}) \vee \mathrm{sgn}(\alpha)) \circ \mathrm{pinch}\ \big) \ \rightsquigarrow\ \left(\begin{array}{c|c} \mathrm{id}_M & \alpha \circ \mathrm{coll} \\ \hline * & \mathrm{sgn}(\alpha) \end{array}\right), \tag{6.1}$$

where $\mathrm{sgn}(\alpha)\colon S^{d-1} \to S^{d-1}$ has degree $+1$ if α lifts to a loop in the orientation double cover of M and degree -1 otherwise. The other notation is explained in Notation 4.5.

If the handle dimension of M is equal to $\dim(M)-1$ (the maximum possible), the formula for $\bar{\pi}_\alpha$ is more complicated. The following proposition gives the general formula.

Proposition 6.3 *For any element $\alpha \in \pi_1(M)$ we have*

$$\bar{\pi}_\alpha = \big(\ \overline{\pitchfork}_\alpha \ \big|\ ((\alpha \circ \mathrm{coll}) \vee \mathrm{sgn}(\alpha)) \circ \mathrm{pinch}\ \big) \ \rightsquigarrow\ \left(\begin{array}{c|c} \mathrm{id}_M & \alpha \circ \mathrm{coll} \\ \hline \pitchfork_\alpha & \mathrm{sgn}(\alpha) \end{array}\right), \tag{6.2}$$

where $\mathrm{sgn}(\alpha)$ is as in Proposition 6.2 and the maps $\overline{\pitchfork}_\alpha$ and $\pitchfork_\alpha$ are described in §6.2 below.

In §6.1 we prove Proposition 6.2. In §6.2 we first define the maps $\overline{\pitchfork}_\alpha$ and $\pitchfork_\alpha$ in the statement of Proposition 6.3 (Definitions 6.5 and 6.6) and then prove Proposition 6.3.

6.1. Below the maximal handle dimension. In this subsection we prove Proposition 6.2. Let us write

- $\bar{\pi}_\alpha^M\colon M \to M \vee S^{d-1}$ for the restriction of $\bar{\pi}_\alpha$ to the M summand of $M \vee S^{d-1}$;
- $\bar{\pi}_\alpha^S\colon S^{d-1} \to M \vee S^{d-1}$ for the restriction of $\bar{\pi}_\alpha$ to the S^{d-1} summand of $M \vee S^{d-1}$.

In this notation, to prove Proposition 6.2, we need to show that

$$\bar{\pi}_\alpha^M \simeq \mathrm{inc}_M \qquad \text{and} \qquad \bar{\pi}_\alpha^S \simeq ((\alpha \circ \mathrm{coll}) \vee \mathrm{sgn}(\alpha)) \circ \mathrm{pinch}. \tag{6.3}$$

We first prove the right-hand side of (6.3). This may in fact be seen purely geometrically from Figure 4.1. We need to describe the effect of π_α on the loop (representing a $(d-1)$-sphere) pictured in the bottom-left corner of that figure. As mentioned at the beginning of this section, π_α may be assumed to be supported in a tubular neighbourhood T of a loop based at the puncture $z \cap B'$ and supported in $M' \cup B'$, as pictured in Figure 4.1. To see the effect of point-pushing along the tube T on the $(d-1)$-sphere based at $*$ pictured in the figure, it is easier first to replace it, up to homotopy equivalence, by a $(d-1)$-sphere encircling the puncture $z \cap B'$ together with a "tether" connecting this sphere to the basepoint $*$ (this corresponds to the pinch and collapse maps in the formula (6.3)). Point-pushing along T has the effect on the tether of sending it around a loop homotopic to α. On the $(d-1)$-sphere encircling the puncture, it acts by a map of degree ± 1 depending on whether the tubular neighbourhood T is orientable or not, in other words, whether or not α lifts to a loop in the orientation double cover of M, which is exactly $\mathrm{sgn}(\alpha)$. Putting this all together, we obtain the desired formula on the right-hand side of (6.3).

We prove the left-hand side of (6.3) in two steps:

- $\bar{\pi}_\alpha^M \simeq \mathrm{inc}_M \circ \theta_\alpha$ for some self-map $\theta_\alpha\colon M \to M$;
- $\theta_\alpha \simeq \mathrm{id}_M$.

Since the handle dimension of M is at most $d-2$, there is an embedded CW-complex $K \subset M$ of dimension at most $d-2$, such that M deformation retracts onto K. (Constructed, for example, using the cores of a handle decomposition of M with handles of index at most $d-2$.) The restriction of $\bar{\pi}_\alpha^M$ to K is a map of the form

$$K \longrightarrow M \vee S^{d-1}.$$

We may homotope this to be *cellular*, i.e., so that every r-cell of K is mapped into a cell of dimension at most r. This implies that the image of the map must intersect S^{d-1} only in the basepoint, so we have a factorisation up to homotopy

$$\bar{\pi}_\alpha^M|_K\colon K \longrightarrow M \hookrightarrow M \vee S^{d-1},$$

for some map $K \to M$. Since the inclusion of K into M is a homotopy equivalence, this implies also that $\bar{\pi}_\alpha^M$ itself factorises up to homotopy as a self-map θ_α of M followed by the inclusion into $M \vee S^{d-1}$. This establishes the first claim above.

We next have to prove that θ_α is homotopic to the identity. Consider the following diagram.

$$\begin{tikzcd}
M \arrow[r, "\theta_\alpha"] \arrow[d, hook] & M \arrow[d, hook] \\
M \vee S^{d-1} \arrow[r, "\bar{\pi}_\alpha"] \arrow[d, hook] & M \vee S^{d-1} \arrow[d, hook] \\
M \arrow[r, "\mathrm{id}"] & M
\end{tikzcd} \tag{6.4}$$

The upper vertical inclusions are both the inclusion of the M summand into $M \vee S^{d-1}$. The lower vertical inclusions are both the embedding of $M \vee S^{d-1}$ into M illustrated in Figure 4.1. The bottom square commutes up to homotopy since any point pushing map becomes homotopic to the identity once the puncture(s) have been filled in. The top square commutes up to homotopy by what we have just proven: that $\bar{\pi}_\alpha^M$ factors through θ_α up to homotopy. The composition of the left-hand vertical maps is homotopic to the identity $M \to M$, and similarly for the right-hand side. Hence three out of the four sides of the outer square of (6.4) are homotopic to the identity, so the fourth side θ_α must also be homotopic to the identity.

This completes the proof of Proposition 6.2.

Remark 6.4 This also proves half of Proposition 6.3, since that proposition is equivalent to the two statements

$$\bar{\pi}_\alpha^M \simeq \overline{\pitchfork}_\alpha \qquad \text{and} \qquad \bar{\pi}_\alpha^S \simeq ((\alpha \circ \mathrm{coll}) \vee \mathrm{sgn}(\alpha)) \circ \mathrm{pinch}, \tag{6.5}$$

and in the proof above we did not use the hypothesis on the handle-dimension of M when proving the right-hand side of (6.3), which is the same as the right-hand side of (6.5).

6.2. In the maximal handle dimension. In this subsection, we first define the maps $\pitchfork_\alpha$ and $\overline{\pitchfork}_\alpha$ appearing in the statement of Proposition 6.3. These depend, a priori, on some additional choices, including a CW-complex $K \subset M$ onto which M deformation retracts. However, Proposition 6.3 implies that they do not depend on these additional choices up to homotopy (see Remark 6.7).

Definition 6.5 Let $K \subset M$ be a CW-complex of dimension at most $d-1$ embedded into M such that M deformation retracts onto K. Assume also that K has exactly one 0-cell and that, for any i-cell τ of K, if $\Phi_\tau \colon D^i \to K$ denotes its characteristic map, then the restriction

$$\Phi_\tau|_{\mathrm{int}(D^i)} \colon \mathrm{int}(D^i) \longrightarrow K \subset M$$

is a smooth embedding. This exists since M is connected and has non-empty boundary, so its handle-dimension is at most $d-1$: such a CW-complex K may be constructed from the cores of a handle decomposition of M with one 0-handle. Let $\alpha \in \pi_1(M)$ and choose a representative loop of α that is a smooth embedding, transverse to the interior of every cell of K and also transverse to ∂M. (For the assumption that the representative of α may be chosen to be an *embedding*, we are using the fact that M has dimension at least 3.)

Given these choices, we define the map $\pitchfork_\alpha \colon M \to S^{d-1}$ as follows:

$$\pitchfork_\alpha \colon M \longrightarrow K \longrightarrow\!\!\!\!\rightarrow K/K^{(d-2)} \cong \bigvee_\tau S^{d-1} \longrightarrow S^{d-1}, \tag{6.6}$$

where the map $M \to K$ is a homotopy inverse of the inclusion, the index τ runs over all $(d-1)$-cells of K and the τ-th component of the last map is a map $S^{d-1} \to S^{d-1}$ of degree $\sharp(\tau, \alpha)$, which is the algebraic intersection number of (the interior of) τ with α.

There are two subtleties in this definition: we need to choose the identification of $K/K^{(d-2)}$ with a wedge of $(d-1)$-spheres unambiguously and we need to ensure that the algebraic intersection number $\sharp(\tau,\alpha)$ is well-defined.

For the first point, we simply choose, arbitrarily and once and for all, an orientation of S^{d-1} and an orientation of each open $(d-1)$-cell $\Phi_\tau(\mathrm{int}(D^{d-1}))$ of K. The identification of $K/K^{(d-2)}$ with a wedge of copies of S^{d-1} is then well-defined, up to based homotopy, by taking it to be *orientation-preserving* on each open $(d-1)$-cell.

For the second point, to ensure that the algebraic intersection number $\sharp(\tau,\alpha)$ is well-defined, we need an orientation of α and of each open $(d-1)$-cell τ, as well as a local orientation of M at each intersection point of α with the interior of τ, i.e., each point of

$$\Phi_\tau(\mathrm{int}(D^{d-1}) \cap \alpha([0,1]). \tag{6.7}$$

We have already chosen orientations of each open $(d-1)$-cell τ, and α is an oriented loop, so it remains to choose local orientations of M at each point of (6.7). We do this in several steps:

- We have already chosen an orientation of S^{d-1}, which is embedded into B' (see Figure 4.1).
- By radial expansion, this determines an orientation of $\partial M \cap B'$.
- In particular, it determines a local orientation of ∂M at the basepoint $*$.
- This, together with α, determines a local orientation of M at $*$ as follows: we take it to be the local orientation of M at $*$ such that the algebraic intersection number of $\alpha|_{[1-\epsilon,1]}$ with ∂M at $*$ is $+1$.
- If M is orientable, this then determines an orientation of M, and in particular local orientations of M at each point of (6.7).
- If M is non-orientable, we have to be more careful. Choose $\epsilon > 0$ such that all intersection points (6.7) are contained in $\alpha([\epsilon,1])$ and choose a closed tubular neighbourhood T of $\alpha|_{[\epsilon,1]}$. Since T is an orientable codimension-zero submanifold of M containing $*$ and each point of (6.7), we may use it to transport the local orientation of M at $*$ to a local orientation of M at each point of (6.7).

We note that this definition does *not* depend on our arbitrary choices of orientations for S^{d-1} and for each open $(d-1)$-cell τ of K:

- Suppose that we reverse the orientation of one $(d-1)$-cell τ_0. This affects the identification of $K/K^{(d-2)}$ with the wedge of $(d-1)$-spheres in a way that corresponds to inserting an automorphism of $\bigvee_\tau S^{d-1}$ that sends each sphere to itself, has degree -1 on the τ_0 component and has degree $+1$ on all other components. However, it also has the effect of reversing the sign of the algebraic intersection number $\sharp(\tau_0,\alpha)$, so these effects cancel each other out after composing all maps in (6.6).
- Suppose that we reverse the orientation of S^{d-1}. This affects the identification of $K/K^{(d-2)}$ with the wedge of $(d-1)$-spheres in a way that corresponds to inserting an automorphism of $\bigvee_\tau S^{d-1}$ that sends each sphere to itself and has degree -1 on each component. However, it also has the effect of reversing the local orientations of M at each intersection point (6.7) for each τ, and so it reverses the sign of each algebraic intersection number $\sharp(\tau,\alpha)$. Again, these effects cancel each other out after composing all maps in (6.6).

This completes the definition of the map $\pitchfork_\alpha \colon M \to S^{d-1}$.

Definition 6.6 Let $K \subset M$ be an embedded CW-complex as in Definition 6.5. We have already assumed that K has a unique 0-cell $*$, and we now assume further that, for each i-cell τ of K, for $i > 0$, the image of its attaching map $\phi_\tau \colon \partial D^i \to K^{(i-1)}$ contains $*$. Choose a representative loop of $\alpha \in \pi_1(M)$ as in Definition 6.5.

We now define a map $\overline{\pitchfork}_\alpha \colon M \to M \vee S^{d-1}$ whose composition with $\mathrm{pr}_{S^{d-1}} \colon M \vee S^{d-1} \to S^{d-1}$ is $\pitchfork_\alpha$. This is the map

$$\overline{\pitchfork}_\alpha \colon M \longrightarrow K \longrightarrow M \vee S^{d-1} \tag{6.8}$$

where the first map is a homotopy inverse of the inclusion and the second map is defined as follows. On the $(d-2)$-skeleton it is defined to be the inclusion $K^{(d-2)} \subset K \subset M \subset M \vee S^{d-1}$. We now extend this to each $(d-1)$-cell of K, in other words, for each $(d-1)$-cell τ of K, we define a map

$$\overline{\pitchfork}_{\alpha,\tau} \colon D^{d-1} \longrightarrow M \vee S^{d-1} \tag{6.9}$$

whose restriction to ∂D^{d-1} is equal to the attaching map $\phi_\tau \colon \partial D^{d-1} \to K^{(d-2)}$ of τ followed by the inclusion $K^{(d-2)} \subset K \subset M \subset M \vee S^{d-1}$. We define the map (6.9) in several steps:

- Choose a point $\widetilde{*} \in \partial D^{d-1}$ such that $\phi_\tau(\widetilde{*}) = *$.
- Denote the intersection points of α with the interior of τ by

$$\Phi_\tau(\mathrm{int}(D^{d-1})) \cap \alpha([0,1]) = \{y_1, \ldots, y_n\}$$

  and write $x_i = \Phi_\tau^{-1}(y_i) \in \mathrm{int}(D^{d-1})$. Let p_i be the straight-line path in D^{d-1} from $\widetilde{*}$ to x_i.
- Fix orientations of D^{d-1} and S^{d-1}. Choose an embedding

$$e_n \colon \bigvee^n S^{d-2} \hookrightarrow D^{d-1}$$

  taking the basepoint to $\widetilde{*}$ and every other point to the interior of D^{d-1}, such that the images of the n copies of S^{d-2} are non-nested in D^{d-1} (see Figure 6.1). There is a unique identification $D^{d-1}/\mathrm{im}(e_n) \cong D^{d-1} \vee \bigvee^n S^{d-1}$ that is orientation-preserving away from the basepoint. We therefore have a map

$$c_n \colon D^{d-1} \longrightarrow D^{d-1} \vee \bigvee^n S^{d-1},$$

  from which we obtain the map (see Notation 4.5 and Figure 6.1 for a picture):

$$\bar{c}_n = (\mathrm{id} \vee \bigvee^n ((\mathrm{coll} \vee \mathrm{id}) \circ \mathrm{pinch})) \circ c_n \colon D^{d-1} \longrightarrow D^{d-1} \vee \bigvee^n \left([0,1] \vee S^{d-1}\right). \tag{6.10}$$

- Finally, we define (6.9) by $\overline{\pitchfork}_{\alpha,\tau} = \overline{\pitchfork}^{\diamond}_{\alpha,\tau} \circ \bar{c}_n$, where the map

$$\overline{\pitchfork}^{\diamond}_{\alpha,\tau} \colon D^{d-1} \vee \bigvee^n \left([0,1] \vee S^{d-1}\right) \longrightarrow M \vee S^{d-1}$$

  is defined on each component as follows.
  - On the D^{d-1} component, $\overline{\pitchfork}^{\diamond}_{\alpha,\tau}$ is the characteristic map $\Phi_\tau \colon D^{d-1} \to K^{(d-2)}$ followed by the inclusion $K^{(d-2)} \subset K \subset M \subset M \vee S^{d-1}$.
  - On the i-th $[0,1]$ component, $\overline{\pitchfork}^{\diamond}_{\alpha,\tau}$ is the element of $\pi_1(M)$ given by

$$\alpha|_{[\alpha^{-1}(y_i),1]} \cdot (\Phi_\tau \circ p_i).$$

  - On the i-th S^{d-1} component, $\overline{\pitchfork}^{\diamond}_{\alpha,\tau}$ is a map $S^{d-1} \to S^{d-1}$ of degree $\epsilon_i \in \{\pm 1\}$, where the sign ϵ_i is determined as follows.
    - As in Definition 6.5, the chosen orientation of S^{d-1} determines a local orientation of M at $*$.
    - We have also chosen an orientation of D^{d-1}, and Φ_τ is a smooth embedding on the interior of D^{d-1}, so we also have an orientation of $\Phi_\tau(\mathrm{int}(D^{d-1}))$. This determines a local orientation of M at the intersection point y_i: namely the one with respect to which the intersection number of $\Phi_\tau(\mathrm{int}(D^{d-1}))$ with $\alpha([0,1])$ at y_i is $+1$.
    - If M is orientable, these two local orientations each determine an orientation of M, and we set ϵ_i to be $+1$ if they agree and -1 if they disagree.
    - If M is non-orientable, we have to be more careful, just as in Definition 6.5. Choose $\delta > 0$ such that all intersection points $y_1, \ldots, y_n$ are contained in $\alpha([\delta,1])$ and choose a tubular neighbourhood T of $\alpha|_{[\delta,1]}$. Since T is an orientable codimension-zero submanifold of M containing $*$ and y_i, the two local orientations of M (at $*$ and at y_i) each determine an orientation of T. We set $\epsilon_i = +1$ if they agree and $\epsilon_i = -1$ if they disagree.

One may see, as in Definition 6.5, that this construction of $\overline{\pitchfork}_\alpha$ is independent of the choices of orientation of S^{d-1} and D^{d-1}. It is also independent of the choice of pre-image $\widetilde{*}$ of the basepoint $* \in K$ under the attaching map of τ: modifying this choice affects the map $\bar{c}_n$ and the map $\overline{\pitchfork}^{\diamond}_{\alpha,\tau}$ on each $[0,1]$ component, and these effects cancel out when we compose them to form $\overline{\pitchfork}_{\alpha,\tau} = \overline{\pitchfork}^{\diamond}_{\alpha,\tau} \circ \bar{c}_n$.

Remark 6.7 A priori, the maps $\pitchfork_\alpha \colon M \to S^{d-1}$ and $\overline{\pitchfork}_\alpha \colon M \to M \vee S^{d-1}$ described in Definitions 6.5 and 6.6 depend on the choice of embedded CW-complex K and the choice of representative of $\alpha \in \pi_1(M)$ that is a smooth embedding and transverse to ∂M and each open cell of K. However,

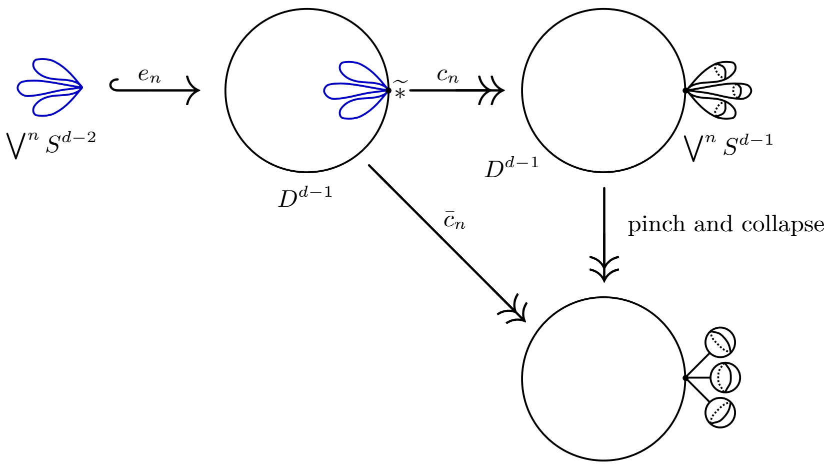

Figure 6.1 The quotient map $\bar{c}_n \colon D^{d-1} \twoheadrightarrow D^{d-1} \vee \bigvee^n ([0,1] \vee S^{d-1})$ from Definition 6.6.

a consequence of Proposition 6.3 is that these maps, up to basepoint-preserving homotopy, do *not* depend on these choices; they depend only on the element $\alpha \in \pi_1(M)$. This is because Proposition 6.3 identifies these two maps with certain maps derived from the point-pushing map π_α, which depends up to homotopy only on $\alpha \in \pi_1(M)$.

Proof of Proposition 6.3. As pointed out in Remark 6.4, we have already proven one half of Proposition 6.3 while proving Proposition 6.2. The remaining statement to prove is

$$\bar{\pi}_\alpha^M \simeq \overline{\pitchfork}_\alpha \colon M \longrightarrow M \vee S^{d-1}. \tag{6.11}$$

We will first prove the two (jointly weaker) statements:

$$\mathrm{pr}_M \circ \bar{\pi}_\alpha^M \simeq \mathrm{id}_M \qquad \text{and} \qquad \mathrm{pr}_{S^{d-1}} \circ \bar{\pi}_\alpha^M \simeq \pitchfork_\alpha, \tag{6.12}$$

which correspond to the (2×2)-matrix description of $\bar{\pi}_\alpha$ on the right-hand side of (6.2). Consider the following homotopy-commutative diagram.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr, "\mathrm{id}"'] & M \vee S^{d-1} \arrow[r, "\bar{\pi}_\alpha"] \arrow[d, hook] & M \vee S^{d-1} \arrow[d, hook] \\ & M \arrow[r, "\mathrm{id}"] & M \end{tikzcd} \tag{6.13}$$

(The square is the same as the bottom square of (6.4).) The two vertical inclusions are both the embedding of $M \vee S^{d-1}$ into M illustrated in Figure 4.1. But this is homotopic to the projection pr_M of $M \vee S^{d-1}$ onto its first summand, so $\mathrm{pr}_M \circ \bar{\pi}_\alpha^M$ is the composition from the top-left to the bottom-right of the diagram, and hence homotopic to the identity. This proves the left-hand side of (6.12).

Next, we prove the right-hand side of (6.12). We start by giving another description of the map

$$w_\alpha = \mathrm{pr}_{S^{d-1}} \circ \bar{\pi}_\alpha^M \colon M \longrightarrow S^{d-1}$$

using Figure 4.1. Choose a path p in B' from $*$ to the point $z \cap B'$ and choose a loop δ in $B' \cup M'$, intersecting $\partial M'$ transversely in two points, in the homotopy class of $p \cdot \gamma_1 \cdot \bar{p}$. Also choose a tubular neighbourhood T of $\delta \cap M'$ in M'. Geometrically, the map $w_\alpha \colon M \to S^{d-1}$ is then given by starting in M', including into M, applying the point pushing map along the loop δ and then collapsing onto the copy of S^{d-1} contained in B'. Clearly the complement $M' \smallsetminus T$ of the tubular neighbourhood T is sent to the basepoint under this map. To describe how w_α acts on T, we use the following identifications. The intersection $T \cap \partial B'$ consists of two disjoint $(d-1)$-discs T_0 and

T_1, where we assume that T_0 contains the intersection point of $\delta \cap \partial B'$ where δ is pointing into M' and T_1 contains the intersection point of $\delta \cap \partial B'$ where δ is pointing into B'. We may then identify T with $T_1 \times [0,1]$ and describe the map w_α on T by

$$T \cong T_1 \times [0,1] \longrightarrow T_1 \longrightarrow T_1/\partial T_1 \simeq S^{d-1}, \tag{6.14}$$

where the two maps are the obvious projections and $T_1/\partial T_1 \simeq S^{d-1}$ is the composition of the canonical identifications

$$T_1/\partial T_1 \simeq \partial B' \simeq S^{d-1},$$

given respectively by the fact that T_1 is a closed disc in the sphere $\partial B'$ and the fact that $\partial B'$ deformation retracts onto the copy of S^{d-1} embedded in B'.

We now use this geometric description of w_α to show that it is homotopic to the map $\pitchfork_\alpha$ defined in Definition 6.5. Let K be a CW-complex of dimension at most $d-1$ embedded into M', such that M' deformation retracts onto K. We need to show that the restriction of w_α to K factors as

$$K \longrightarrow\!\!\!\!\rightarrow K/K^{(d-2)} \cong \bigvee\nolimits_\tau S^{d-1} \longrightarrow S^{d-1}, \tag{6.15}$$

where the τ-th component of the right-hand map is a map $f_\tau \colon S^{d-1} \to S^{d-1}$ of degree $\sharp(\tau,\delta)$. By smooth approximation and transversality, we may assume that each $(d-1)$-cell τ of K is smoothly embedded into M' and that δ and T have been chosen so that (a) each r-cell of K, for $r \leqslant d-2$, is disjoint from T and (b) each $\tau \cap T$, for τ a $(d-1)$-cell of K, consists of finitely many $(d-1)$-discs each intersecting δ transversely in one point.

By property (a), and since $M' \smallsetminus T$ is sent to the basepoint by w_α, we see that its restriction to K must factor through the projection $K \twoheadrightarrow K/K^{(d-2)}$. So we just have to show that f_τ has degree $\sharp(\tau,\delta)$. By property (b) and the description (6.14) of $w_\alpha|_T$, each component of the disjoint union of $(d-1)$-discs $\tau \cap T$ contributes either $+1$ or -1 to $\deg(f_\tau)$. Being careful about (local) orientations as explained in Definition 6.5, we see that the sum of these $+1$'s and -1's is precisely the algebraic intersection number $\sharp(\tau,\delta)$ of τ and δ.

This completes the proof that $w_\alpha|_K$ factors as in (6.15), and hence that $w_\alpha \simeq \pitchfork_\alpha$, in other words, the right-hand side of (6.12).

The proof of (6.11) is similar to the proof above of the right-hand side of (6.12): looking at Figure 4.1 and using a geometric model for the point-pushing map supported in a tubular neighbourhood of an embedded loop representing α, one must check carefully that the definition of $\overline{\pitchfork}_\alpha$ given in Definition 6.6 is a correct description of $\bar{\pi}^M_\alpha$ up to homotopy. Rather than go through this in symbols, we refer the reader instead to Figure 6.2, which depicts the map $\bar{\pi}^M_\alpha$ induced by point-pushing along α, and which one may compare to the definition of $\overline{\pitchfork}_\alpha$ in Definition 6.6. □

7. Two examples

To illustrate the more complicated setting where M is non-simply-connected and has maximal handle dimension, we discuss some explicit examples, namely

$$M = (S^1 \times S^2) \smallsetminus \mathrm{int}(D^3)$$

and more generally

$$M = \underbrace{(S^1 \times S^2)\sharp(S^1 \times S^2)\sharp\cdots\sharp(S^1 \times S^2)}_{g \text{ copies}} \smallsetminus \mathrm{int}(D^3)$$

which all have maximal handle-dimension $\dim(M) - 1 = 2$ and which have fundamental groups $\mathbb{Z}$ and F_g, the free group on g generators, respectively.

Example 7.1 First, consider $M = (S^1 \times S^2) \smallsetminus \mathrm{int}(D^3)$ and let α be a generator of $\pi_1(M) \cong \mathbb{Z}$. By Proposition 6.3, the point-pushing map

$$\bar{\pi}_\alpha \colon M \vee S^2 \longrightarrow M \vee S^2$$

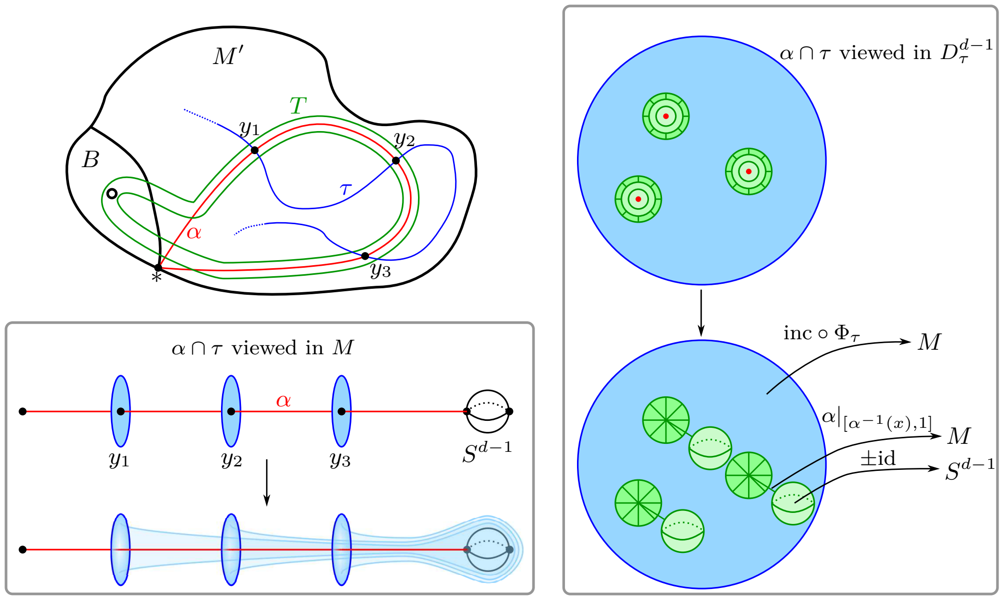

Figure 6.2 Two views of the effect of point-pushing along an embedded arc α on a $(d-1)$-cell τ: (1) Embedded in M. — (2) Intrinsically in the disc parametrising the cell τ.

has a simple explicit description when restricted to the S^2 summand, and is homotopic to the (in general complicated) map $\overline{\pitchfork}_\alpha \colon M \to M \vee S^2$ of Definition 6.6 when restricted to the M summand.

In this example, M is homotopy equivalent to $S^1 \vee S^2$ (see Figure 7.1 for a picture of an embedded $S^1 \vee S^2$ onto which it deformation retracts). So, under this identification, the point pushing map $\bar{\pi}_\alpha$ is an endomorphism of $S^1 \vee S^2 \vee S^2$. We will label these 1- and 2-spheres with subscripts to indicate which of the (light or dark) red spheres in Figure 7.1 they correspond to. Thus our aim is to describe (up to based homotopy) the map

$$\bar{\pi}_\alpha \colon S^1_\alpha \vee S^2_\tau \vee S^2_p = X \longrightarrow X = S^1_\alpha \vee S^2_\tau \vee S^2_p.$$

This is an element of the homotopy set $\langle X, X\rangle = \pi_0(\mathrm{Map}_*(X,X))$, which becomes a monoid under composition. In fact, we know of course that $\bar{\pi}_\alpha$ must be an *invertible* element of this monoid, i.e. an element of $\pi_0(\mathrm{hAut}_*(X))$, but we will describe it as an element of the larger monoid $\langle X, X\rangle$. In order to do this, we first describe the monoid $\langle X, X\rangle$ explicitly.

First, note that there is an obvious bijection

$$\langle X, X\rangle \;\cong\; \pi_1(X) \times \pi_2(X) \times \pi_2(X),$$

and that $\pi_1(X) \cong \mathbb{Z}\{\alpha\}$, the free (abelian) group generated by α. The second homotopy group of X is the same as that of its universal cover, and using Hilton's theorem [Hil55] to compute homotopy groups of wedges of spheres, we see that

$$\pi_2(X) \cong \mathbb{Z}\{\alpha^n\tau, \alpha^n p \mid n \in \mathbb{Z}\},$$

the free abelian group generated by the symbols $\alpha^n\tau$ and $\alpha^n p$ for each $n \in \mathbb{Z}$. Moreover, the action of $\pi_1(X) = \mathbb{Z}\{\alpha\}$ is given by $\alpha.\alpha^n\tau = \alpha^{n+1}\tau$ and $\alpha.\alpha^n p = \alpha^{n+1}p$. This means that we may write $\pi_2(X) \cong \mathbb{Z}[\alpha^{\pm 1}]\{\tau, p\} = \mathbb{Z}[\pi_1(X)]\{\tau, p\}$ as a (free) module over the group-ring of $\pi_1(X)$. Putting these identifications together, we have

$$\langle X, X\rangle \;\cong\; \mathbb{Z}\{\alpha\} \times \mathbb{Z}[\alpha^{\pm 1}]\{\tau, p\} \times \mathbb{Z}[\alpha^{\pm 1}]\{\tau, p\} \tag{7.1}$$

as a set. With a little work, one may check that the operation of composition on $\langle X, X\rangle$ may be

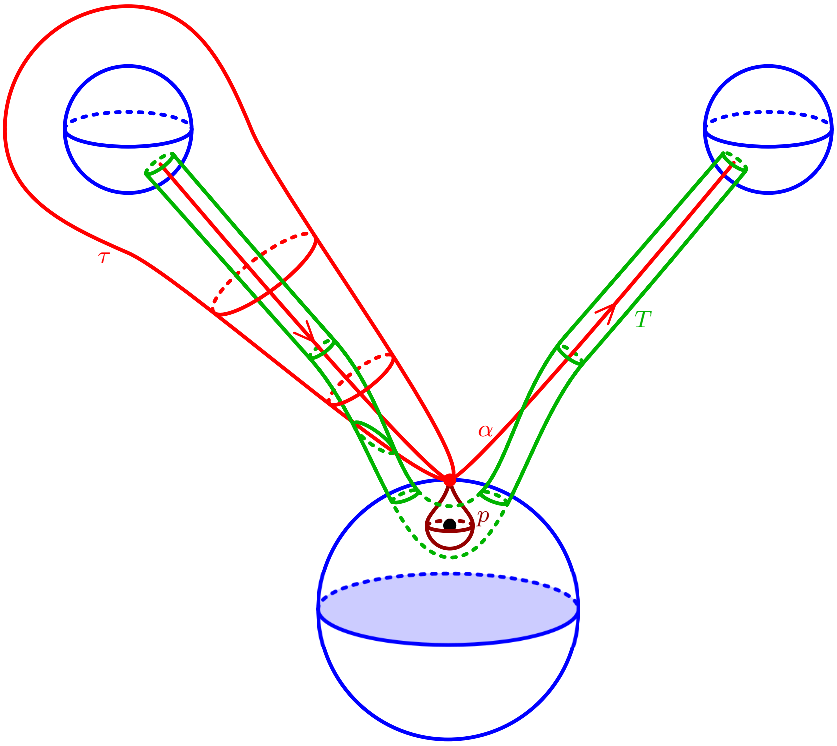

Figure 7.1 The picture is to be thought of as S^3 with three open balls (in blue) cut out, and the boundaries of two of them (the top two) identified by a reflection. This is a model for the manifold $M = (S^1 \times S^2) \smallsetminus \mathrm{int}(D^3)$. The embedded copy of $S^1 \vee S^2$ is drawn in red, consisting of a 1-sphere called α and a 2-sphere called τ. The manifold M deformation retracts onto this subspace. As a model for M with a puncture removed, we glue back in the top half of the lower 3-ball (so the boundary now consists of the light blue shaded 2-disc together with the southern hemisphere of the lower blue 2-sphere) and then remove the black point. This manifold (let us call it M') deformation retracts onto the embedded wedge sum $S^1 \vee S^2 \vee S^2$ consisting of α, τ and the dark red 2-sphere called p. The green solid cylinder called T is a tubular neighbourhood of α, isotoped slightly so that it contains the puncture in its interior. Thus, the effect of the point-pushing map on α may be realised explicitly by a diffeomorphism of the manifold M' supported in the interior of T, as described in Lemma 3.4.

described, under this identification, as follows:

$$\Big(k\alpha, \sum_i \alpha^i(m_i\tau + n_ip), \sum_i \alpha^i(r_i\tau + s_ip)\Big) \circ \Big(k'\alpha, \sum_i \alpha^i(m'_i\tau + n'_ip), \sum_i \alpha^i(r'_i\tau + s'_ip)\Big)$$
$$= \Big(kk'\alpha, \sum_{i,j} \alpha^{jk+i}((m_im'_j + r_in'_j)\tau + (s_in'_j + n_im'_j)p), \sum_{i,j} \alpha^{jk+i}((r_is'_j + m_ir'_j)\tau + (s_is'_j + n_ir'_j)p)\Big). \tag{7.2}$$

Its neutral element is (α, τ, p). With the concrete description (7.1) and (7.2) of the monoid $\langle X, X\rangle$ in hand, we can now write explicitly the element $\bar{\pi}_\alpha$ in terms of this description. Namely, we have

$$\bar{\pi}_\alpha = (\alpha, \tau + p, \alpha p).$$

As a sanity check, one may also calculate that

$$\bar{\pi}_{\alpha^{-1}} = (\alpha, \tau - \alpha^{-1}p, \alpha^{-1}p)$$

and verify using (7.2) that this is indeed an inverse for $\bar{\pi}_\alpha$ in the monoid $\langle X, X\rangle$. We may also read off from this description that $\bar{\pi}_\alpha$ acts on $\pi_1(X) \cong \mathbb{Z}\{\alpha\}$ by the identity and on $H_2(X;\mathbb{Z}) \cong \mathbb{Z}\{\tau, p\}$ by the (2×2)-matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

The element $\bar{\pi}_\alpha = (\alpha, \tau + p, \alpha p) \in \langle X, X \rangle$ has infinite order: this is detected for example by its action on $H_2(-;\mathbb{Z})$, but one may also directly calculate from (7.2) that

$$(\bar{\pi}_\alpha)^n = \underbrace{(\alpha, \tau + p, \alpha p) \circ \cdots \circ (\alpha, \tau + p, \alpha p)}_{n} = (\alpha, \tau + (1 + \alpha + \cdots + \alpha^{n-1})p, \alpha^n p).$$

Hence the point-pushing homomorphism

$$\pi_1(M) \cong \mathbb{Z}\{\alpha\} \longrightarrow \pi_0(\mathrm{hAut}_*(M \vee S^2)) \subset \langle X, X \rangle$$

is injective. This factors through the point-pushing homomorphism

$$\pi_1(M) \longrightarrow \pi_0(\mathrm{Homeo}_*(M \smallsetminus *)),$$

which is therefore also injective.

Example 7.2 Now consider the more general example of

$$M = \underbrace{(S^1 \times S^2) \sharp (S^1 \times S^2) \sharp \cdots \sharp (S^1 \times S^2)}_{g \text{ copies}} \smallsetminus \mathrm{int}(D^3).$$

Now M is homotopy equivalent to a wedge of g circles (labelled by $\alpha_1, \ldots, \alpha_g$) and g two-spheres (labelled by $\tau_1, \ldots, \tau_g$), so the point-pushing homomorphism is of the form

$$\pi_1(M) \cong F_g = \langle \alpha_1, \ldots, \alpha_g \rangle \longrightarrow \pi_0(\mathrm{hAut}_*(X)) \subset \langle X, X \rangle, \tag{7.3}$$

where $X = S^1_{\alpha_1} \vee \ldots \vee S^1_{\alpha_g} \vee S^2_{\tau_1} \vee \ldots \vee S^2_{\tau_g} \vee S^2_p$. Here $\langle \alpha_1, \ldots, \alpha_g \rangle$ denotes the free group generated by $\alpha_1, \ldots, \alpha_g$ and $\langle X, X \rangle$ denotes the monoid $\pi_0(\mathrm{Map}_*(X, X))$, as before. We would like to describe the point-pushing maps $\bar{\pi}_{\alpha_1}, \ldots, \bar{\pi}_{\alpha_g}$ (the images of $\alpha_1, \ldots, \alpha_g$) as elements of this monoid.

Generalising the discussion in the previous example, suppose that X is a wedge of a number of circles indexed by a set A and a number of two-spheres indexed by a set B. We then have

$$\pi_1(X) \cong F_A \qquad \text{and} \qquad \pi_2(X) \cong \mathbb{Z}[F_A]B,$$

where F_A is the free group on the set A, $\mathbb{Z}[F_A]$ is its integral group-ring and $\mathbb{Z}[F_A]B$ is the free $\mathbb{Z}[F_A]$-module on the set B. The underlying set of the monoid $\langle X, X \rangle$ is therefore

$$\langle X, X \rangle \cong \prod_A F_A \times \prod_B \mathbb{Z}[F_A]B,$$

and one may describe the operation of composition, under this identification, by a formula analogous to (7.2). Taking $A = \{\alpha_1, \ldots, \alpha_g\}$ and $B = \{\tau_1, \ldots, \tau_g, p\}$, we may rewrite this as

$$\langle X, X \rangle \cong \prod_{i=1}^{g} \langle \alpha_1, \ldots, \alpha_g \rangle \times \prod_{i=1}^{g+1} \mathbb{Z}\langle \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1} \rangle \{\tau_1, \ldots, \tau_g, p\}, \tag{7.4}$$

where $\mathbb{Z}\langle \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1} \rangle$ denotes the ring of non-commutative Laurent polynomials with coefficients in $\mathbb{Z}$ in the variables $\alpha_1, \ldots, \alpha_g$. We will describe the point-pushing maps $\bar{\pi}_{\alpha_1}, \ldots, \bar{\pi}_{\alpha_g}$ as elements of the right-hand side of (7.4). Namely, we have:

$$\bar{\pi}_{\alpha_i} = (\alpha_1, \ldots, \alpha_g, \tau_1, \ldots, \tau_{i-1}, \tau_i + p, \tau_{i+1}, \ldots, \tau_g, \alpha_i p).$$

Using this description, and a purely algebraic formula for composition in $\langle X, X \rangle$ under the identification (7.4), which is a straightforward generalisation of the formula (7.2), we may compute that, for any word $w = w_1^{\epsilon_1} \cdots w_j^{\epsilon_j}$ in the generators $\alpha_1, \ldots, \alpha_g$, we have

$$\bar{\pi}_w = \bar{\pi}_{w_1}^{\epsilon_1} \circ \cdots \circ \bar{\pi}_{w_j}^{\epsilon_j} = (\alpha_1, \ldots, \alpha_g, \tau'_1, \ldots, \tau'_g, wp),$$

where $\tau'_i = \tau_i + [n_i]_{\alpha_i} p$ and the vector of integers $(n_1, \ldots, n_g)$ is the abelianisation of $w \in F_g$. Here, the *quantum integer* $[n]_{\alpha_i}$ is defined to be the polynomial $1 + \alpha_i + \cdots + \alpha_i^{n-1}$ if $n \geqslant 1$ and the polynomial $-\alpha_i - \cdots - \alpha_i^n$ if $n \leqslant -1$ (and the zero polynomial if $n = 0$).

In particular, we note that the coefficient of the generator p in the last component of $\bar{\pi}_w$ is exactly $w \in F_g \subseteq \mathbb{Z}[F_g] = \mathbb{Z}\langle \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1} \rangle$. This implies that the point-pushing homomorphism (7.3) is injective.

Remark 7.3 The two examples above go through identically if $S^1 \times S^2$ is replaced with $S^1 \times S^{d-1}$ for any $d \geqslant 3$; we obtain the same formulas for the point-pushing maps $\bar{\pi}_\alpha$ and the point-pushing homomorphism $\alpha \mapsto \bar{\pi}_\alpha$ is injective. We summarise this as:

Proposition 7.4 *For the manifold*

$$M = M^d_{g,1} = \underbrace{(S^1 \times S^{d-1})\sharp(S^1 \times S^{d-1})\sharp \cdots \sharp(S^1 \times S^{d-1})}_{g \text{ copies}} \smallsetminus \mathrm{int}(D^d)$$

for any $d \geqslant 3$ and $g \geqslant 0$, the point-pushing homomorphism

$$\mathrm{push}_M \colon \pi_1(M) \longrightarrow \pi_0(\mathrm{Homeo}_*(M \smallsetminus *)) \longrightarrow \pi_0(\mathrm{hAut}_*(M \vee S^{d-1})) \tag{7.5}$$

is injective.

For comparison, we note that in dimension $d = 2$ the point-pushing homomorphism is part of the Birman exact sequence [Bir69]:

$$1 \to \pi_1(M^2_{g,1}) = F_{2g} \longrightarrow \Gamma^1_{g,1} \longrightarrow \Gamma_{g,1} \to 1,$$

in particular it is injective. Note that in this case the fundamental group is larger, freely generated by the loops α_i *and* the $\tau_i = \beta_i$.

Of course, for $g \geqslant 2$ one knows that (7.5) is injective for general reasons: its kernel always lies in the centre of $\pi_1(M)$, as mentioned in the introduction, but in this case $\pi_1(M)$ is a non-abelian free group, so its centre is trivial. Nevertheless, the advantage of our methods is, firstly, that we obtain a *complete description* of (7.5) rather than just a computation that its kernel is trivial. Secondly, our calculations in these examples suggest a general geometric criterion for injectivity:

Discussion 7.5 The above examples suggest that the point-pushing homomorphism (7.5) should be injective as long as each generator of (some generating set of) $\pi_1(M)$ has an associated $(d-1)$-cell (of some skeleton for M given by the cores of a handle decomposition) with which it has non-trivial algebraic intersection number, and with which all of the other generators have trivial algebraic intersection number, which then "detects" its action. Much more generally, it should suffice if we can find independent linear combinations of $(d-1)$-cells paired with each generator. More explicitly we expect:

Let A be a set of generators for $\pi_1(M)$ and B be the set of $(d-1)$-cells in some CW-decomposition of M. Then, for oriented manifolds M with non-empty boundary, the point pushing map is injective if the intersection matrix $(\sharp(\tau, \alpha))_{\tau \in B, \alpha \in A}$ has rank $|A|$.

8. Formulas for associated point-pushing actions on mapping spaces

As an immediate corollary of Proposition 5.1, Lemma 6.1 Proposition 6.2 and Lemma 3.13, we obtain (under certain assumptions on M) a formula for the associated point-pushing action (Definition 3.12) of $\pi_1(C_k(M))$ on the mapping space $\mathrm{Map}^c_*(M \smallsetminus z, X)$, under the identification

$$\mathrm{Map}^c_*(M \smallsetminus z, X) \simeq \mathrm{Map}_*(M, X) \times (\Omega^{d-1}_c X)^k \tag{8.1}$$

induced by the identification (4.2) of $M \smallsetminus z$ with $M \vee \bigvee^k S^{d-1}$. On the right-hand side of (8.1), $\Omega^{d-1}_c X$ denotes the union of path-components of $\Omega^{d-1} X$ corresponding to the subset $c \subseteq [S^{d-1}, X]$.

Remark 8.1 There are two natural actions on the space $\Omega^{d-1}_c X$. First, there is an up-to-homotopy action of $\pi_1(X)$ on $\Omega^{d-1} X$, which restricts to an action-up-to-homotopy on the subspace $\Omega^{d-1}_c X$ (this is because the subset $c \subseteq [S^{d-1}, X]$ corresponds to a union of $\pi_1(X)$-orbits of $\pi_{d-1}(X)$).

Second, there is an involution of $\Omega^{d-1} X$ given by precomposition with a reflection of S^{d-1} in a hyperplane containing the basepoint; this involution commutes with the up-to-homotopy action of $\pi_1(X)$. If $c \subseteq [S^{d-1}, X]$ is invariant under the corresponding involution of $[S^{d-1}, X]$, then this involution restricts to the subspace $\Omega^{d-1}_c X$. In our situation, the involution will only be relevant if M is non-orientable, in which case we have assumed (see Definition 3.9) that $c \subseteq [S^{d-1}, X]$ is a subset of the fixed points under the involution, so in particular it is invariant under the involution.

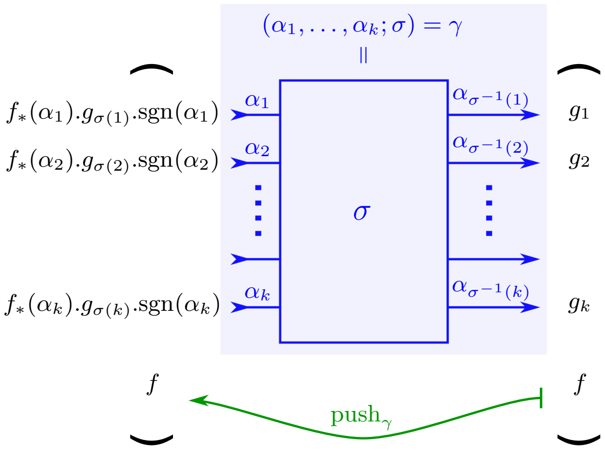

Figure 8.1 The action of the point-pushing map associated to $\gamma = (\alpha_1, \ldots, \alpha_k; \sigma) \in \pi_1(C_k(M))$ on the mapping space $\mathrm{Map}_*(M, X) \times (\Omega_c^{d-1} X)^k$. The loop γ is represented in blue, the elements of the mapping space in black and the point-pushing map is represented in green.

Corollary 8.2 *If $d = \dim(M) \geqslant 3$ and M satisfies at least one of the following conditions:*

- *M is simply-connected, or*
- *the handle-dimension of M is at most $d-2$;*

then the point-pushing action of $\gamma = (\alpha_1, \ldots, \alpha_k; \sigma) \in \pi_1(C_k(M)) \cong \pi_1(M)^k \rtimes \Sigma_k$ on the mapping space $\mathrm{Map}_*^c(M \smallsetminus z, X)$, under the identification (8.1), is given as follows (see also Figure 8.1)

$$(\alpha_1, \ldots, \alpha_k; \sigma) \cdot (f, g_1, \ldots, g_k) = (f, \bar{g}_1, \ldots, \bar{g}_k), \tag{8.2}$$

where $\bar{g}_i = f_*(\alpha_i).g_{\sigma(i)}.\mathrm{sgn}(\alpha_i)$, and

- *for an element $\alpha \in \pi_1(M)$ we write $\mathrm{sgn}(\alpha) = +1$ if α lifts to a loop in the orientation double cover of M and $\mathrm{sgn}(\alpha) = -1$ otherwise,*
- *the actions of $\pi_1(X)$ and of $\{\pm 1\}$ on $\Omega_c^{d-1} X$ are as described in Remark 8.1 above.*

Proof. It suffices to check this for elements of the form $(1, \ldots, 1; \sigma)$ and $(\alpha, 1, \ldots, 1; \mathrm{id})$ (symmetric and loop generators), which we denote simply by σ and α by abuse of notation.

By Proposition 5.1, the action of σ on $M \smallsetminus z \simeq M \vee W_k$ is the identity on the M summand and permutes the k copies of S^{d-1} in $W_k = \bigvee^k S^{d-1}$. Lemma 3.13 tells us that the associated point-pushing action of σ on $\mathrm{Map}_*(M, X) \times (\Omega_c^{d-1} X)^k$ is induced from its point-pushing action on $M \vee W_k$ by precomposition, so we deduce that it acts by the identity on the $\mathrm{Map}_*(M, X)$ component and the $\Omega_c^{d-1} X$ components are permuted by σ^{-1} (the inverse occurs since precomposition is contravariant).

Similarly, Lemma 3.13 implies that the point-pushing action of α on $\mathrm{Map}_*(M, X) \times (\Omega_c^{d-1} X)^k$ is induced from the point-pushing action of α on $M \vee W_k$, which is described by Lemma 6.1 and Proposition 6.2, by precomposition. Putting this together, we see that α sends the tuple $(f, g_1, \ldots, g_k)$ to the tuple $(f, f_*(\alpha).g_1.\mathrm{sgn}(\alpha), g_2, \ldots, g_k)$, as desired. Specifically, the f entry in this tuple follows from the left-hand side of (6.3), the $f_*(\alpha).g_1.\mathrm{sgn}(\alpha)$ entry follows from the right-hand side of (6.3) and the remaining entries follow from Lemma 6.1. □

Remark 8.3 Part of the formula (8.2) remains valid without the additional hypothesis on M. More precisely, assuming still that $\dim(M) \geqslant 3$ but removing the second hypothesis (so M is now allowed to be non-simply-connected and to have maximal handle-dimension), the formula for the action of $\gamma = (\alpha_1, \ldots, \alpha_k; \sigma)$ becomes

$$(\alpha_1, \ldots, \alpha_k; \sigma) \cdot (f, g_1, \ldots, g_k) = (?, \bar{g}_1, \ldots, \bar{g}_k), \tag{8.3}$$

where the entry ? is not in general f, but rather a based map $M \to X$ that depends in a subtle way on f, the loop γ and the elements g_i. For example, when $\gamma = (\alpha, 1, \ldots, 1; \mathrm{id})$, the map $?\colon M \to X$

is given by the composition

$$\mathrm{fold} \circ (f \vee g_1) \circ \overline{\pitchfork}_\alpha \colon M \longrightarrow M \vee S^{d-1} \longrightarrow X \vee X \longrightarrow X,$$

where $\overline{\pitchfork}_\alpha$ is the map defined in Definition 6.6. To see this, recall that the equations (6.3) describe the point-pushing action of a loop generator α under the additional assumptions on M, and the equations (6.5) describe the point-pushing action of α without these assumptions. The right-hand equation of (6.3) agrees with the right-hand equation of (6.5), which is why the tuple $(\bar{g}_1, \ldots, \bar{g}_k)$ occurs in (8.3), just as in (8.2). However, the left-hand equation of (6.3) is simply $\bar{\pi}_\alpha^M \simeq \mathrm{inc}_M$, whereas the left-hand equation of (6.5) is $\bar{\pi}_\alpha^M \simeq \overline{\pitchfork}_\alpha$.

Remark 8.4 Corollary 8.2 is used in [PT, §9] to prove a certain split-injectivity result for maps between configuration-mapping spaces. More precisely, there is a natural map of spectral sequences converging to the map on homology induced by the *stabilisation map*

$$\mathrm{CMap}_k^{c,*}(M;X) \longrightarrow \mathrm{CMap}_{k+1}^{c,*}(M;X).$$

Under the hypotheses on M assumed in Corollary 8.2, this map of spectral sequences is split-injective on E^2 pages. For the precise statement, see [PT, Theorem 9.1].

Corollary 8.2 may also be used to understand the path-components of configuration-mapping spaces of manifolds of dimension at least 3. As an example, we have the following.

Corollary 8.5 *Suppose that $d = \dim(M) \geqslant 3$, M is orientable and either*

- *M is simply-connected, or*
- *the handle-dimension of M is at most $d-2$.*

Then there is a natural bijection

$$\pi_0(\mathrm{CMap}_k^{c,*}(M;X)) \;\cong\; \bigsqcup_{f \in \langle M,X \rangle} SP^k(c_f), \tag{8.4}$$

where $\langle M,X \rangle = \pi_0(\mathrm{Map}_*(M,X))$, the notation $SP^k(\ \ )$ means $(\ \ )^k/\Sigma_k$ and c_f is the pre-image of $c \subseteq [S^{d-1},X]$ under the quotient map

$$\pi_{d-1}(X)/f_*(\pi_1(M)) \longrightarrow \pi_{d-1}(X)/\pi_1(X) = [S^{d-1},X].$$

Proof. By the long exact sequence associated to the bundle (3.11), the left-hand side of (8.4) is naturally in bijection with the set of orbits of

$$\pi_0(\mathrm{Map}_*^c(M \smallsetminus z, X)) \cong \langle M,X \rangle \times \tilde{c}^k$$

under the monodromy (i.e., point-pushing) action of $\pi_1(C_k(M))$, where $\tilde{c}$ denotes the pre-image of $c \subseteq [S^{d-1},X]$ under the quotient map $\pi_{d-1}(X) \to \pi_{d-1}(X)/\pi_1(X) = [S^{d-1},X]$. Corollary 8.2 implies that the elements of $\pi_1(C_k(M))$ act on a tuple $([f],[g_1],\ldots,[g_k])$ by (i) permuting the $[g_i]$'s and (ii) acting on each $[g_i]$ (individually) by $f_*(\pi_1(M)) \leqslant \pi_1(X)$. The formula (8.4) follows. ∎

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Institutul de Matematică Simion Stoilow al Academiei Române, 21 Calea Griviței, 010702 București, România

Mathematical Institute, University of Oxford, Andrew Wiles Building, Woodstock Road, Oxford, OX2 6GG, UK

mpanghel@imar.ro

tillmann@maths.ox.ac.uk