Twisted homology fibrations and scanning for oriented configuration spaces

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Abstract

In [Pal13], the second author proved that "oriented" configuration spaces exhibit homological stability. To complement that result we identify the limiting space, up to homology equivalence, as a certain explicit double cover of a section space. Along the way we also prove that the scanning map of [McD75] for unordered configuration spaces is acyclic in the limit.

1 Introduction

There are many interesting examples of collections of spaces $\{Y_k\}$ such that the homology groups $H_i(Y_k)$ are independent of k for k sufficiently large compared to i. Examples include the classifying spaces of general linear groups [Qui73, Cha79], mapping class groups [Har85] and automorphism groups of free groups [Hat95, HV98, HW10], moduli spaces of instatons [BHMM93], and configuration spaces of particles in a manifold [McD75, Seg79]. In many of these cases (c.f. [McD75, Tau89, MW07, Gal11]), it is possible to find an easy-to-understand limiting space Z such that $hocolim_k(Y_k)$ is homology equivalent to Z. In [Pal13], the second author proved homological stability for oriented configuration spaces and in this paper we describe the corresponding limiting space.

Oriented configuration spaces are natural generalizations of the classifying spaces of the alternating groups. One possible motivation for studying oriented configuration spaces was given in [GKY]. They showed that homological stability for oriented configuration spaces implies stability for the homotopy groups of spaces of positive and negative particles.[1] We will also describe an application of these ideas to the study of the homology of the spaces appearing in the generalized Snaith splitting of [Böd87].

Before we state the results of this paper and of [Pal13], we first fix some notation and review some classical theorems regarding configuration spaces of

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[1]The spaces of positive and negative particles they consider are a filtration of those considered by McDuff in [McD75].

unordered particles. Let $F_k(M) \coloneqq M^k \smallsetminus \Delta_f$ where Δ_f is the fat diagonal. Define $C_k(M)$ to be the quotient of $F_k(M)$ by the action of the symmetric group Σ_k, and $C_k^+(M)$ to be the quotient by the action of the alternating group A_k. We call these spaces respectively the configuration spaces of ordered, unordered and oriented collections of points in M.

Throughout, we require that the manifold M be connected and of dimension at least 2. We say that a manifold *admits a boundary* if it is the interior of a (not necessarily compact) manifold with (not necessarily compact) boundary. For such manifolds, Segal proved in [Seg79] the following theorem.

Theorem 1.1 ([Seg79, Proposition A.1]). *Let M be a manifold admitting a boundary. Then there is a map $t \colon C_k(M) \to C_{k+1}(M)$ which induces an isomorphism on homology for $* \leq k/2$.*

We call the map t the "stabilization map". Roughly, it involves moving all the particles away from the boundary and then adding a new particle near the boundary; see §3.1 for precise definitions. No such map exists for closed manifolds and in fact homological stability fails for closed manifolds [FVB62].[2]

Let $\dot{T}M \to M$ denote the fiberwise one-point compactification of the tangent bundle of M and let $\Gamma(M)$ denote the space of compactly supported sections of this bundle. The path-components of $\Gamma(M)$ are indexed by the *degree* of the section; we will denote the path-component consisting of degree-k sections by $\Gamma_k(M)$. For orientable manifolds, the degree of a section σ can be defined as the signed intersection number of σ with the zero section of $\dot{T}M \to M$. For non-orientable manifolds, the orientation double cover $\widetilde{M} \to M$ induces a map $\Gamma(M) \to \Gamma(\widetilde{M})$ and one can define the degree of $\sigma \in \Gamma(M)$ as half of the degree of its image in $\Gamma(\widetilde{M})$ [BM12]. In [McD75], McDuff defined a scanning map $s \colon C_k(M) \to \Gamma_k(M)$ and proved the following two theorems.

Theorem 1.2 ([McD75, Theorem 1.2]). *If M admits a boundary, the scanning map $s \colon C_k(M) \to \Gamma_k(M)$ induces a homology equivalence $s \colon C_\infty(M) \to \Gamma_\infty(M)$.*

Here $C_\infty(M)$ denotes the homotopy colimit of the maps

$$\cdots \to C_k(M) \to C_{k+1}(M) \to \cdots$$

from Theorem 1.1, and $\Gamma_\infty(M)$ denotes the homotopy colimit of analogous "stabilization" maps for the path-components $\Gamma_k(M)$ of $\Gamma(M)$.

Theorem 1.3 ([McD75, Theorem 1.1]). *The scanning map $s \colon C_k(M) \to \Gamma_k(M)$ induces an isomorphism on homology in the same range $(* \leq k/2)$ as the map $t \colon C_k(M \smallsetminus pt) \to C_{k+1}(M \smallsetminus pt)$.*

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[2]It does, however, hold rationally [Chu12] or for mod-2 coefficients [ML88, BCT89]. See also [RW13] and [BM12].

In [Pal13], the second author proved an analogue of Theorem 1.1 for oriented configuration spaces.

Theorem 1.4 ([Pal13]). *Let M be a manifold admitting a boundary. There is a map $t\colon C_k^+(M) \to C_{k+1}^+(M)$ which induces an isomorphism on homology for $* \le (k-5)/3$ and a surjection for $* \le (k-2)/3$.*

The goal of this paper is to provide analogues of Theorem 1.2 and Theorem 1.3 for oriented configuration spaces. For $k \ge 2$, the scanning map $s\colon C_k(M) \to \Gamma_k(M)$ induces an isomorphism on $H^1(-;\mathbb{Z}/2)$, by Theorem 1.3 above and the universal coefficient theorem. Cohomology with mod-2 coefficients classifies path-connected double covers of a path-connected space, so this fact says that any double cover of $C_k(M)$ is the pullback of some double cover of $\Gamma_k(M)$. In particular $C_k^+(M)$ fits into a pullback square:

$$
\begin{array}{ccc}
C_k^+(M) & \xrightarrow{\ \widetilde{s}\ } & \Gamma_k^+(M) \\
\downarrow \quad {}^{\lrcorner} & & \downarrow \\
C_k(M) & \xrightarrow{\ s\ } & \Gamma_k(M).
\end{array}
\tag{1.1}
$$

There is an alternative, more concrete description of the associated double cover $\Gamma_k^+(M) \to \Gamma_k(M)$ which will be given in §3.1. Our analogue of Theorem 1.3 is:

Theorem 1.5. *The lifted scanning map $\widetilde{s}\colon C_k^+(M) \to \Gamma_k^+(M)$ induces an isomorphism on homology groups in the range $* \le (k-5)/3$ and a surjection for $* \le (k-2)/3$.*

Unlike configuration spaces of unordered particles or labeled configuration spaces [Sal01], oriented configuration spaces are not local: to determine a point in $C_k^+(M)$, one needs more information than information attached to each point in the configuration. Nevertheless, these oriented configuration spaces still exhibit homological stability and we can still describe a limiting space.

To prove this result we will first prove the following strengthening of Theorem 1.2:

Theorem 1.6. *If M is a manifold which admits a boundary, the scanning map in the limit $s\colon C_\infty(M) \to \Gamma_\infty(M)$ is acyclic.*

This theorem, combined with the stability result from [Pal13], will give Theorem 1.5. Note that in Theorem 1.5 (unlike in Theorem 1.6) the manifold M is allowed to be closed, so it does not immediately follow; it is proved in §3.4.

To prove Theorem 1.6 we will need the notion of a twisted homology fibration, defined in §2.1. We generalize McDuff's homology fibration criterion from [McD75] to give a criterion for a map to be a twisted homology fibration. We believe that this lemma will be useful for studying other questions involving identifying the stable homology with twisted coefficients of sequences of spaces.

3

Terminology. We now fix some terminology for different notions of homology equivalence, which will be used later. Let $f \colon X \to Y$ be a continuous map of spaces, and let $\mathcal{F}$ denote a local coefficient system on Y – this can be thought of as a $\mathbb{Z}[\pi_1(Y)]$-module, a functor $\pi(Y) \to \mathsf{Ab}$ from the fundamental groupoid of Y to abelian groups, or as a bundle of abelian groups over Y. It is called *abelian* if the action $\pi_1(Y) \to \mathrm{Aut}(\mathcal{F})$ factors through the abelianisation $\pi_1(Y)^{\mathrm{ab}}$ or, in the bundle viewpoint, if the monodromy of any fiber around a commutator loop is trivial.

The map f is called *acyclic*, or a *twisted homology equivalence*, if it induces an isomorphism $H_*(X; f^*\mathcal{F}) \to H_*(Y; \mathcal{F})$ for all (not necessarily abelian) local coefficient systems $\mathcal{F}$. It is called an *abelian homology equivalence* if it induces isomorphisms for all abelian local coefficient systems, and it is called a *trivial homology equivalence*, or just a *homology equivalence*, if it induces isomorphisms for the trivial coefficient system $\mathbb{Z}$ (with trivial $\pi_1(Y)$-action, or in the bundle viewpoint the product bundle $\mathbb{Z} \times Y \to Y$).

An alternative characterization (see [Ber82, Proposition 4.3]) of acyclicity of f is that $\widetilde{H}_*(hofib(f); \mathbb{Z}) = 0$ in all degrees (where $\mathbb{Z}$ is just a trivial coefficient group). From this it follows that the pullback of an acyclic map is acyclic. In particular in diagram (1.1), once $k \to \infty$, acyclicity of s will imply acyclicity of $\widetilde{s}$.

The sign representation. One can rephrase results about oriented configuration spaces in terms of homology of unordered configuration spaces with certain twisted coefficients. Let $\rho \colon \pi_1(C_k(M)) \to \mathbb{Z}/2$ be the composition of the natural map $\pi_1(C_k(M)) \to \Sigma_k$ and the sign homomorphism $\Sigma_k \to \mathbb{Z}/2$. For a ring R, the group-ring $R[\mathbb{Z}/2]$ is given the structure of an $R[\pi_1(C_k(M))]$-module by the homomorphism ρ. By the definition of local homology, or a trivial application of the Serre spectral sequence to the fibration $S^0 \to C_k^+(M) \to C_k(M)$, we have an isomorphism

$$H_*(C_k^+(M); R) \ \cong \ H_*(C_k(M); R[\mathbb{Z}/2]).$$

When 2 is invertible in R, the module $R[\mathbb{Z}/2]$ decomposes as $R \oplus R^{(-1)}$, where $\pi_1(C_k(M))$ acts trivially on R, and on $R^{(-1)}$ it acts by ρ and the action of $\mathbb{Z}/2$ given by $r \mapsto -r$ (the "sign representation"). Hence we have a further decomposition

$$H_*(C_k^+(M); R) \ \cong \ H_*(C_k(M); R) \oplus H_*(C_k(M); R^{(-1)}).$$

Theorem 1.6 allows one to study the homology of the spaces $C_k(M)$ with this twisted coefficient system. The groups $H_*(C_k(M); \mathbb{Z}^{(-1)})$ are interesting for the following reason. Let M be an almost parallelizable d-manifold. For $m > 0$, the space $Map_*(M, S^{d+m})$ of based maps splits stably (in the sense of stable homotopy theory) into summands which are Thom spaces of vector bundles over $C_k(M \setminus pt)$ [Böd87, BCT89]. This construction recovers Snaith splitting ([Sna74])

when $M = S^d$. The Thom isomorphism theorem implies that the homology these Thom spaces are shifts of $H_*(C_k(M); \mathbb{Z})$ or $H_*(C_k(M); \mathbb{Z}^{(-1)})$ depending on whether or not the relevant vector bundles are orientable. Thus, to understand the homology of the spaces appearing in generalized Snaith splitting, one needs to understand the homology of configuration spaces with sign-twisted coefficients.

The fact that the groups $H_i(C_k(M); \mathbb{Q}^{(-1)})$ stabilize (and indeed are eventually zero) is originally due to Church using representation stability [Chu12]. First note that $H_i(C_k(M); \mathbb{Q}^{(-1)})$ has the same dimension as the cohomology $H^i(C_k(M); \mathbb{Q}^{(-1)})$, which is the number of copies of the sign representation in the Σ_k-representation $H^i(F_k(M); \mathbb{Q})$. The main result of [Chu12] implies that the irreducible Σ_k-representation V_λ corresponding to a partition $k = \lambda_1 + \cdots + \lambda_\ell$ may only appear in $H^i(F_k(M); \mathbb{Q})$ if $2\lambda_2 + \cdots + \lambda_\ell \leq 2i$ (see the discussion on p.469 of [Chu12]). In particular the sign representation $(k = 1 + \cdots + 1)$ does *not* appear in $H^i(F_k(M); \mathbb{Q})$ for $k > 2i$, and hence $H_i(C_k(M); \mathbb{Q}^{(-1)}) = 0$ in this range.

The organization of this paper is as follows. In Section 2, we generalize McDuff's homology fibration criterion and discuss how the group completion theorem works with twisted coefficients. In Section 3, we first use the group completion theorem to prove Theorem 1.6 in the case where the manifold is of the form $\mathbb{R}^2 \times N$. Then we use this and the twisted homology fibration criterion to prove Theorem 1.6 for general manifolds admitting a boundary. Finally we use Theorem 1.6 to deduce Theorem 1.5.

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2   Twisted homology fibrations

Two of the most important tools for studying stable homology are the homology fibration criterion of [McD75] (which is the analogue of a criterion for quasi-fibrations from [DT58]) and the group completion theorem of [MS76].[3] The goal of this section is to describe versions of these theorems for homology with twisted coefficients. Firstly, following [MS76], we introduce two notions of twisted homology fibration and prove that they agree under reasonable point-set topological hypotheses. This then allows one to prove a twisted homology fibration criterion.

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[3]Many other versions of the group completion theorem exist. A simplicial version which follows the spirit of [MS76] is proved in [PS04], which also uses a simplicial notion of homology fibration.

In the final subsection of this section we check in detail that the arguments of [MS76] go through with twisted coefficients, so that we also have twisted versions of the group-completion theorem (different versions depending on whether you consider all local coefficient systems, or just certain kinds). Although Remark 2 of [MS76] addresses homology with twisted coefficients, no proofs are given. We also recall work of Randal-Williams [RW] which verifies the hypotheses of a twisted version of the group-completion theorem in the case of a homotopy-commutative monoid acting by left-multiplication on a stabilization of itself by right-multiplication – this will be needed when we apply this twisted group-completion theorem in §3.

2.1 Two definitions of twisted homology fibrations

In this subsection, we introduce two definitions of homology fibration and prove that they are equivalent. Much of this is implicit but not explicit in the work of McDuff and Segal in [MS76] (Proposition 5, Proposition 6, and Remark 2). The equivalence of these two definitions will be used in the next subsection to generalize McDuff's homology fibration criterion. We call one type a Serre homology fibration because such maps naturally have an associated Serre spectral sequence, and we call the other type a Leray homology fibration since those maps naturally have an associated Leray spectral sequence. We will denote the homotopy fiber of a map $r\colon Y \to X$ at a point x by $hofib_x(r)$. For a subset $U \subseteq X$, the symbol $hofib_U(r)$ will denote the homotopy limit of the following diagram:

$$
\begin{array}{ccc}
hofib_U(r) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
U & \longrightarrow & X.
\end{array}
$$

Definition 2.1. Let Z be a subspace of a space X. A map $r\colon Y \to X$ is called a *twisted Serre homology fibration on Z* if for all $z \in Z$, the natural inclusion $r^{-1}(z) \to hofib_z(r|_Z)$ induces an isomorphism on homology with coefficients in any system of local coefficients coming from $hofib_z(r)$. That is, if $\mathcal{F}$ is a system of local coefficients on $hofib_z(r)$ and $i\colon r^{-1}(z) \to hofib_z(r|_Z)$ and $j\colon hofib_z(r|_Z) \to hofib_z(r)$ are the natural inclusions, then i induces an isomorphism:

$$
i_*\colon H_*(r^{-1}(z); i^*j^*\mathcal{F}) \to H_*(hofib_z(r|_Z); j^*\mathcal{F}).
$$

The map $r\colon Y \to X$ is called simply a twisted Serre homology fibration if it is a twisted Serre homology fibration on all of X.

An *abelian Serre homology fibration (on Z)* is defined in exactly the same way, except that only local coefficient systems $\mathcal{F}$ which are abelian (see the end of the introduction) are considered.

Definition 2.2. Let X be a space and let $Z \subseteq X$ be a locally contractible subspace. A map $r\colon Y \to X$ is called a *twisted Leray homology fibration on Z* if there is a basis $\mathcal{U}$ for the topology of Z (consisting of contractible sets) such that for all $z \in U \in \mathcal{U}$ and any system of local coefficients $\mathcal{F}$ on $hofib_U(r)$, the inclusion $i\colon r^{-1}(z) \to r^{-1}(U)$ induces an isomorphism on homology with coefficients in pullbacks of $\mathcal{F}$. That is, if $j\colon r^{-1}(U) \to hofib_U(r)$ is the natural inclusion, then

$$i_*\colon H_*(r^{-1}(z); i^* j^* \mathcal{F}) \to H_*(r^{-1}(U); j^* \mathcal{F})$$

is an isomorphism. The map $r\colon Y \to X$ is called simply a *twisted Leray homology fibration* if it is a twisted Leray homology fibration on all of X. We call such a basis $\mathcal{U}$ an *acceptable* basis for the map r.

An *abelian Leray homology fibration* (*on Z*) is defined in exactly the same way, except that only local coefficient systems $\mathcal{F}$ which are abelian are considered.

Mimicking the proofs of Proposition 5 and Proposition 6 of [MS76] and Lemma 5.2 of [McD75], we prove the following propositions.

Proposition 2.3. *Let $r\colon Y \to X$ be a map of spaces and let $Z \subseteq X$ be a subspace. Assume that each $z \in Z$ has a basis of neighborhoods U in the topology of Z with a deformation retraction onto $\{z\}$ which lifts to a deformation retraction of $r^{-1}(U)$ into $r^{-1}(z)$. If r is a twisted (abelian) Serre homology fibration on Z, then it is also a twisted (abelian) Leray homology fibration on Z.*

Proof. Let $\mathcal{U}$ be the basis of the topology of Z described above. Let $z \in U \in \mathcal{U}$ be arbitrary and let $w \in U$ be a point such that $r^{-1}(U)$ deformation retracts onto $r^{-1}(w)$. Consider the following commuting diagram:

$$
\begin{array}{ccccc}
r^{-1}(z) & \longrightarrow & r^{-1}(U) & \longleftarrow & r^{-1}(w) \\
\downarrow & & \downarrow & & \downarrow \\
hofib_z(r|_Z) & \longrightarrow & hofib_U(r|_Z) & \longleftarrow & hofib_w(r|_Z).
\end{array}
$$

Since $\{z\} \to U$ and $\{w\} \to U$ are homotopy equivalences, so are $hofib_z(r) \to hofib_U(r|_Z)$ and $hofib_w(r|_Z) \to hofib_U(r|_Z)$. By assumption, $r^{-1}(w) \to r^{-1}(U)$ is a homotopy equivalence. Since r is a twisted (abelian) Serre homology fibration on Z, the maps $r^{-1}(z) \to hofib_z(r|_Z)$ and $r^{-1}(w) \to hofib_w(r|_Z)$ induce isomorphisms on homology with twisted (abelian) coefficients coming from $hofib_U(r)$. Thus the same is true for the map $r^{-1}(z) \to r^{-1}(U)$ and hence r is a twisted (abelian) Leray homology fibration on Z. $\square$

Proposition 2.4. *Let $r\colon Y \to X$ be a map of spaces and let $Z \subseteq X$ be a locally contractible subspace. If r is a twisted (abelian) Leray homology fibration on Z then it is also a twisted (abelian) Serre homology fibration on Z.*

Proof. Fix $z_0 \in Z$, let $W = r^{-1}(Z)$ and let $P = \{\alpha \colon [0,1] \to Z \text{ such that } \alpha(0) = z_0\}$. Let $p \colon P \to Z$ be the evaluation at 1 map. Let $g \colon p^*W \to P$ be the pullback along p of r and consider the following commuting diagram:

$$
\begin{array}{ccc}
p^*W & \xrightarrow{\ \ g\ \ } & P \\
\downarrow & & \downarrow{\scriptstyle p} \\
W & \xrightarrow{\ \ r\ \ } & Z.
\end{array}
$$

Note that P is contractible, p^*W is the homotopy fiber of $r|_Z$ at z_0 and the fibers of g and r are equal. Therefore it suffices to prove that the inclusions of the fibers of g into p^*W induce isomorphisms in homology with (abelian) coefficients coming from the homotopy fiber of r.

Let $\mathcal{U}$ be an acceptable basis for the topology of Z, as in Definition 2.2. A sequence of elements of $\mathcal{U}$ is called a *chain* if it satisfies the following pattern of inclusions:

$$U_1 \supseteq V_1 \subseteq U_2 \supseteq V_2 \subseteq \ldots \subseteq U_n \supseteq V_n.$$

If $c = (U_1, V_1, \ldots U_n, V_n)$ is a chain and $\vec{t} = (0 = t_0 < t_1 < \cdots < t_n = 1)$ is a sequence of increasing real numbers, let $W_{c,\vec{t}} \subseteq P$ be the set of paths $\alpha \in P$ with $\alpha(t_i) \in V_i$ and $\alpha([t_{i-1}, t_i]) \subseteq U_i$. Let $\mathcal{U}_P$ be the collection of all such $W_{c,\vec{t}}$. The collection $\mathcal{U}_P$ is a basis of the topology of P consisting of contractible sets. Note that $p|_{W_{c,\vec{t}}} \to V_{n-1}$ is a homotopy equivalence and a fibration. Since pulling back preserves fibers and pulling back along fibrations preserves homotopy fibers, the map g is a twisted (abelian) Leray homology fibration with $\mathcal{U}_P$ an acceptable basis.

Let $\mathcal{F}$ be a system of local coefficients on p^*W which is the pullback of a system of (abelian) local coefficients on the homotopy fiber of r. Let $\mathfrak{B}$ be the category whose objects are elements of $\mathcal{U}_P$ and morphisms are containments. Consider the Mayer-Vietoris spectral sequence associated to the cover of p^*W by sets of the form $g^{-1}(W_{c,\vec{t}})$ for $W_{c,\vec{t}} \in \mathcal{U}_P$ with homology twisted by $\mathcal{F}$. See [Bre68] for a description of this spectral sequence. This spectral sequence has E^2 page equal to $H_p(|\mathfrak{B}|; \mathcal{G}_q)$. Here $\mathcal{G}_q$ is the local system associated to the copresheaf $U \to H_q(p^{-1}(U); i^*\mathcal{F})$ with $i \colon U \to P$ the inclusion of an open subset. Since P is not necessarily paracompact, it does not necessarily follow that the geometric realization $|\mathfrak{B}|$ is homotopy equivalent to P. However, this is proved by other methods in [McD80, page 110]. Thus $|\mathfrak{B}|$ is contractible and we have:

$$
E^2_{pq} = \begin{cases} H_q(g^{-1}(\alpha); i^*\mathcal{F}) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}
$$

with $\alpha \in P$ arbitrary and $i \colon g^{-1}(\alpha) \to p^*W$ the natural inclusion. This spectral sequence collapses immediately since the range or domain of every differential

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is zero, so its edge homomorphisms are isomorphisms. On the other hand, it converges to $H_*(p^*W; \mathcal{F})$ and hence $g^{-1}(\alpha) \to p^*W$ induces an isomorphism on $\mathcal{F}$-twisted homology. Taking α to be a path with $\alpha(1) = z_0$, we see that $r^{-1}(z_0) \to hofib_{z_0}(r|_Z)$ induces an isomorphism on $\mathcal{F}$-twisted homology and hence $r: Y \to X$ is a twisted (abelian) Serre homology fibration on Z. $\square$

2.2 A criterion for a map to be a twisted homology fibration

In this subsection, we generalize Proposition 5.1 of [McD75] and give a criterion for a map $r: Y \to X$ to be a twisted (abelian) homology fibration (when we are in the regime where the two notions agree, we will omit the "Serre" or "Leray" adjective). The criterion is the following theorem.

Theorem 2.5. *Let $X = \bigcup_{k \in \mathbb{N}} X_k$ with $X_{k-1} \subseteq X_k$ and each X_i closed. Then a map $r: Y \to X$ is a twisted (abelian) homology fibration, in either sense, if the following three conditions are satisfied:*

(i) *each $x \in X_k$ has a basis of neighborhoods U in the topology of X_k, each with a deformation retraction onto $\{x\}$ which lifts to a deformation retraction of $r^{-1}(U)$ onto $r^{-1}(x)$;*

(ii) *each restriction $r: r^{-1}(X_k \smallsetminus X_{k-1}) \to X_k \smallsetminus X_{k-1}$ is a twisted (abelian) homology fibration, as is the restriction $r: r^{-1}(X_0) \to X_0$;*

(iii) *for each k there is an open subset U_k of X_k such that $X_{k-1} \subseteq U_k$, and there are homotopies $h_t: U_k \to U_k$ and $H_t: r^{-1}(U_k) \to r^{-1}(U_k)$ satisfying:*

   (a) *$h_0 = id, h_t(X_{k-1}) \subseteq X_{k-1}, h_1(U_k) \subseteq X_{k-1}$;*

   (b) *$H_0 = id, r \circ H_t = h_t \circ r$;*

   (c) *for all $x \in U_k$, $H_1: r^{-1}(x) \to r^{-1}(h_1(x))$ induces an isomorphism on homology with twisted (abelian) coefficients coming from $hofib_{h_1(x)}(r)$.*

Proof. For the abelian version of the proof below, replace "twisted homology fibration" by "abelian homology fibration" and consider only abelian local coefficient systems; we will now just write about the twisted homology fibration version.

By condition (i) and Propositions 2.3 and 2.4, r is a twisted Serre homology fibration on X_k if and only if it is a twisted Leray homology fibration on X_k. The proof will follow by induction on the claim: the map r is a twisted homology fibration on X_n. This is true for $n = 0$ by condition (ii). Now assume that the claim is true for some fixed n, and we shall prove it for $n + 1$.

We assumed in (ii) that r is a twisted homology fibration on $X_{n+1} \smallsetminus X_n$. We will now prove that r is a twisted Serre homology fibration on U_{n+1}. Fix $x \in U_n$ and consider the following commuting diagram:

$$\begin{array}{ccccc}
r^{-1}(x) & \longrightarrow & \mathit{hofib}_x(r|_U) & \longrightarrow & \mathit{hofib}_x(r) \\
\downarrow {\scriptstyle H_1} & & \downarrow {\scriptstyle H_1'} & & \downarrow {\scriptstyle \simeq} \\
r^{-1}(h_1(x)) & \longrightarrow & \mathit{hofib}_{h_1(x)}(r|_{X_n}) & \longrightarrow & \mathit{hofib}_{h_1(x)}(r).
\end{array}$$

Here $H_1' \colon \mathit{hofib}_x(r|_U) \to \mathit{hofib}_{h_1(x)}(r|_{X_n})$ is the natural map induced on homotopy fibers by the maps h_1 and H_1. Conditions (iii)(a) and (iii)(b) imply that H_1' is a homotopy equivalence. The map $H_1 \colon r^{-1}(x) \to r^{-1}(h_1(x))$ induces an isomorphism on homology with twisted coefficients coming from the homotopy fiber of r by condition (iii)(c). The natural inclusion $r^{-1}(h_1(x)) \to \mathit{hofib}_{h_1(x)}(r|_{X_n})$ induces an isomorphism on homology with twisted coefficients coming from the homotopy fiber of r since r is a twisted Serre homology fibration on X_n by our inductive hypothesis. Thus, the inclusion $r^{-1}(x) \to \mathit{hofib}_x(r|_U)$ induces an isomorphism on homology with twisted coefficients coming from the homotopy fiber of r. Therefore r is a twisted Serre homology fibration on U_{n+1}.

By condition (i) and condition (iii)(b), the space U_{n+1} satisfies the hypotheses of Proposition 2.3. Thus r is also a twisted Leray homology fibration on U_{n+1}. It is clear that if a subspace Z is the union of two open (with respect to the subspace topology on Z) sets V_1 and V_2 then r is a twisted Leray homology fibration on Z if it is on V_1 and V_2. Since $X_{n+1} = (X_{n+1} \smallsetminus X_n) \cup U_{n+1}$ and r is a twisted Leray homology fibration on $X_{n+1} \smallsetminus X_n$ and U_{n+1}, the map r is also a twisted Leray homology fibration on X_{n+1}.

By induction, we see that r is a twisted homology fibration on X_n for all n. The homotopy fiber of the map r is the colimit of the homotopy fibers of the maps $r|_{X_n}$. Since homology (with twisted coefficients) commutes with direct limits, we can conclude that r is a twisted Serre homology fibration. Since the space X satisfies the hypothesis of Proposition 2.3, r is also a twisted Leray homology fibration. $\qquad\square$

2.3   The twisted group completion theorem

In this section we will prove the "twisted" version of McDuff-Segal's group-completion theorem [MS76, Proposition 2]. In fact this was asserted as Lemma 3.1 of [McD80], but no details were given there or in [MS76] about generalizing the proof for twisted coefficients; we will go through the details in the present section. We do not claim any originality here; the proofs closely follow the ideas of [MS76]. The statement is as follows:

Theorem 2.6. *Let $\mathcal{M}$ be a topological monoid acting on a space X; associated to this action we have a natural map*

$$E\mathcal{M} \times_{\mathcal{M}} X \longrightarrow B\mathcal{M}. \tag{2.1}$$

Assume that for all $m \in \mathcal{M}$ the action $m \cdot - \colon X \to X$ is a twisted (abelian) homology equivalence. Then the map (2.1) is a twisted (abelian) Leray homology fibration.

The difference between this and [MS76, Proposition 2] is that the hypotheses and conclusion have both been strengthened to all (abelian) local coefficient systems.

The most important application of this for us will be as follows. Let $\mathcal{M}$ be a topological monoid with $\pi_0(\mathcal{M}) = \mathbb{N}$ (we could work in more generality, but this case will be enough for our applications), denote its components by $\mathcal{M}_k$ and choose an element $m \in \mathcal{M}_1$. We then form $\mathcal{M}_\infty$ as the mapping telescope of the sequence $\mathcal{M} \to \mathcal{M} \to \mathcal{M} \to \cdots$ where each map is right-multiplication by m. There is then an obvious left-action of $\mathcal{M}$ on $\mathcal{M}_\infty$.

If we now assume that $\mathcal{M}$ is *homotopy-commutative*, then for each $m' \in \mathcal{M}$ this action $m' \cdot - \colon \mathcal{M}_\infty \to \mathcal{M}_\infty$ is a trivial homology equivalence (i.e. with trivial $\mathbb{Z}$ coefficients). To see this: say $m' \in \mathcal{M}_k$. Then the the map we are interested in is the map on homotopy colimits induced by the vertical maps in the diagram:

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathcal{M} & \xrightarrow{\ -\cdot m^k\ } & \mathcal{M} & \longrightarrow & \cdots \\
 & & {\scriptstyle m'\cdot-}\big\downarrow & \nearrow{\scriptstyle \mathrm{id}} & \big\downarrow{\scriptstyle m'\cdot-} & & \\
\cdots & \longrightarrow & \mathcal{M} & \xrightarrow[\ -\cdot m^k\ ]{} & \mathcal{M} & \longrightarrow & \cdots
\end{array}
\tag{2.2}
$$

in which, by homotopy-commutativity of $\mathcal{M}$, the triangles commute up to homotopy. This induces a factorization on homology which implies that the induced map in the homotopy colimit is a homology equivalence.

Note also that the Borel construction $E\mathcal{M} \times_{\mathcal{M}} \mathcal{M}_\infty$ is the mapping telescope of contractible spaces $E\mathcal{M} \times_{\mathcal{M}} \mathcal{M} = E\mathcal{M}$, and so is itself weakly contractible. Hence the homotopy fiber of the map $\pi \colon E\mathcal{M} \times_{\mathcal{M}} \mathcal{M}_\infty \to B\mathcal{M}$ is weakly equivalent to $\Omega B\mathcal{M}$. Applying the group-completion theorem [MS76, Proposition 2] we obtain that for a homotopy-commutative monoid $\mathcal{M}$ there is a homology equivalence

$$
\mathcal{M}_\infty = \mathrm{fib}(\pi) \to \mathit{hofib}(\pi) \simeq_w \Omega B\mathcal{M}
\tag{2.3}
$$

(this is essentially Proposition 1 of [MS76]).

Similarly to above one can show that, for homotopy-commutative monoids $\mathcal{M}$, the maps $m' \cdot - \colon \mathcal{M}_\infty \to \mathcal{M}_\infty$ are *surjective* on homology with coefficients in any (not necessarily abelian) local coefficient system on $\mathcal{M}_\infty$. The argument fails for injectivity, however, and indeed it is in general *not* true that homotopy-

commutative monoids $\mathcal{M}$ act on their mapping telescopes $\mathcal{M}_\infty$ by acyclic maps.[4] However, it has recently been explicitly proved by Randal-Williams [RW] that the maps $m' \cdot - \colon \mathcal{M}_\infty \to \mathcal{M}_\infty$ *are* injective on homology with coefficients in any *abelian* local coefficient system on $\mathcal{M}_\infty$. Hence applying the "twisted abelian" version of the group-completion theorem (Theorem 2.6) and the equivalence between abelian Serre/Leray homology fibrations of the previous section we obtain:

Corollary 2.7. *For a homotopy-commutative monoid $\mathcal{M}$ the map* (2.3) *is an abelian homology equivalence. Since all local coefficient systems on its codomain $\Omega B\mathcal{M}$ are abelian, this means that it is an acyclic map.*

This is the statement which we will need in §3. Theorem 2.6 will follow easily from the more general fact:

Proposition 2.8. *Let $p \colon E \to B$ be a map of semi-simplicial spaces which is a twisted (abelian) Leray homology fibration on each level $p_n \colon E_n \to B_n$, and such that for each face map $d_j \colon B_n \to B_{n-1}$ and element $b \in B_n$, the map*

$$d_j|_{p_n^{-1}(b)} \colon p_n^{-1}(b) \to p_{n-1}^{-1}(d_j(b))$$

is a twisted (abelian) homology equivalence. Then the map of geometric realizations[5] *$\|p\| \colon \|E\| \to \|B\|$ is a twisted (abelian) Leray homology fibration.*

When we applied Proposition 2.6 to obtain Corollary 2.7, we were actually interested in the fact that (2.1) is an abelian *Serre* homology fibration (using the equivalence between abelian Leray/Serre homology fibrations established in §2.1). Our reason for introducing the two notions of abelian homology fibration is that the next two lemmas, which are needed to prove Proposition 2.8, depend on the 'Leray' notion of an abelian homology fibration.

The rest of this subsection is devoted to carefully proving Theorem 2.6. We will only speak of *twisted* Leray homology fibrations in the proofs and in the statements of lemmas below, but everything goes through in exactly the same way for *abelian* Leray homology fibrations by only considering abelian local coefficient systems everywhere.

---

[4]To apply the argument, one needs to produce a factorization into triangles as in (2.2), but for spaces equipped with local coefficient systems, thought of as bundles of abelian groups (say). For the bottom-right triangle (corrsponding to surjectivity) this can be done since given homotopic maps $f, g \colon X \rightrightarrows Y$ and a bundle $\mathcal{F}$ over Y, it is not hard to factor the pullback along f as a bundle map covering id_X followed by the pullback along g. However, one cannot in general factorize it as the pullback along g followed by a bundle map covering id_Y – this is the problem one needs to solve in the top-left triangle, corresponding to injectivity.

[5]Since we are using *semi*-simplicial spaces here, this necessarily means the thick geometric realization.

Lemma 2.9. *Suppose that we have a diagram*

$$
\begin{array}{ccccc}
E_1 & \xleftarrow{\ \widetilde{f}_1\ } & E_0 & \xrightarrow{\ \widetilde{f}_2\ } & E_2 \\
\downarrow{p_1} & & \downarrow{p_0} & & \downarrow{p_2} \\
B_1 & \xleftarrow{\ f_1\ } & B_0 & \xrightarrow{\ f_2\ } & B_2
\end{array}
\qquad (2.4)
$$

in which p_i is a twisted Leray homology fibration for $i = 0, 1, 2$ and for all $b \in B_0$, the restriction $p_0^{-1}(b) \to p_i^{-1}(f_i(b))$ of $\widetilde{f}_i$ is a twisted homology equivalence for $i = 1, 2$. Then the map $p \colon E \to B$, induced by taking homotopy colimits levelwise (in other words $E = \mathrm{hocolim}(E_1 \leftarrow E_0 \to E_2)$, etc.), is again a twisted Leray homology fibration.

Lemma 2.10. *Suppose that we have a ladder diagram*

$$
\begin{array}{ccccccc}
\cdots \xrightarrow{\ f_{n-1}\ } & X_n & \xrightarrow{\ f_n\ } & X_{n+1} & \xrightarrow{\ f_{n+1}\ } \cdots & \cdots \longrightarrow & X = \mathrm{hocolim}_n(X_n) \\
& \downarrow{p_n} & & \downarrow{p_{n+1}} & & & \downarrow{p} \\
\cdots \xrightarrow{\ g_{n-1}\ } & Y_n & \xrightarrow{\ g_n\ } & Y_{n+1} & \xrightarrow{\ g_{n+1}\ } \cdots & \cdots \longrightarrow & Y = \mathrm{hocolim}_n(Y_n)
\end{array}
\qquad (2.5)
$$

in which p_n is a twisted Leray homology fibration for all n, and the restriction $p_n^{-1}(y) \to p_{n+1}^{-1}(g_n(y))$ of f_n is a twisted homology equivalence for all n and all $y \in Y_n$. Then the map $p \colon X \to Y$, induced by taking homotopy colimits levelwise (in other words taking mapping telescopes), is again a twisted Leray homology fibration.

We will prove Lemmas 2.9 and 2.10 first, then use them to deduce Proposition 2.8, and then finally show that this implies the "twisted group-completion theorem" (Proposition 2.6) as a special case.

Proof of Lemma 2.9. Let $\mathcal{U}_i$ be a basis for B_i. Then the following is a basis $\mathcal{U}$ for the double mapping cylinder $B = \mathrm{hocolim}(B_1 \leftarrow B_0 \to B_2)$:

(a) $\quad V \cup_{f_1} \left(\bigcup_\alpha U_\alpha \times [0, \varepsilon_\alpha) \right) \quad \Big| \quad V \in \mathcal{U}_1, \varepsilon_\alpha > 0$ and $U_\alpha \in \mathcal{U}_0 : \bigcup_\alpha U_\alpha = f_1^{-1}(V)$

(b) $\quad U \times (\beta, \gamma) \quad \Big| \quad U \in \mathcal{U}_0$ and $0 < \beta < \gamma < 1$

(c) $\quad \left(\bigcup_\alpha U_\alpha \times (1 - \varepsilon_\alpha, 1] \right) \cup_{f_2} V \quad \Big| \quad V \in \mathcal{U}_2, \varepsilon_\alpha > 0$ and $U_\alpha \in \mathcal{U}_0 : \bigcup_\alpha U_\alpha = f_2^{-1}(V)$.

Pictorially:

$$V \left\{ \quad \right\} f_1^{-1}(V) \qquad\qquad \left\{ \quad \right\} U \qquad\qquad f_2^{-1}(V) \left\{ \quad \right\} V$$

$$0 \qquad\qquad \beta \quad \gamma \qquad\qquad 1$$

$$\text{(a)} \qquad\qquad\qquad \text{(b)} \qquad\qquad\qquad \text{(c)}$$

It is not enough to simply take (ă): sets of the form $V \cup_{f_1} \left(f_1^{-1}(V) \times [0, \varepsilon) \right)$ (and similarly (č)), by the following counterexample pointed out to the authors by Ilya Grigoriev. Take $B_0 = \mathbb{R}$ and $B_1 = B_2 = pt$. Then the subset $pt \cup \{(s, t) \, | \, s < (1 + t^2)^{-1}\}$ of the double mapping cylinder is open but is not covered by sets of the form (ă), (b) and (č).[6] But if one allows the more general sets of the form (a) and (c) then it is not hard with some point-set topology to check that this is, as claimed, a basis for the double mapping cylinder.

If the bases $\mathcal{U}_i$ consist of contractible sets then so will $\mathcal{U}$. Now assume that $\mathcal{U}_i$ is an acceptable basis (witnessing that $p_i \colon E_i \to B_i$ is a twisted Leray homology fibration) for B_i. We will show that $p \colon E \to B$ is also a twisted Leray homology fibration by showing that $\mathcal{U}$ is an acceptable basis for B w.r.t. this map. To do this we need to show that for any $b \in W \in \mathcal{U}$, the inclusions

$$p^{-1}(b) \xrightarrow{i} p^{-1}(W) \xrightarrow{j} \mathit{hofib}_W(p)$$

induce an isomorphism $H_*(p^{-1}(b); i^* j^* \mathcal{F}) \cong H_*(p^{-1}(W); j^* \mathcal{F})$ for any twisted local coefficient system $\mathcal{F}$ on $\mathit{hofib}_W(p)$. There are three essentially different cases of $b \in W \in \mathcal{U}$ to check:

$$V \left\{ \bullet\, b \right. \qquad\qquad V \left\{ \bullet\, b \right. \qquad\qquad \left. \bullet\, b \right\} U$$

$$0 \qquad\qquad\qquad 0 \quad \delta \qquad\qquad \beta \quad \delta \quad \gamma$$

$$\text{(i)} \qquad\qquad\qquad \text{(ii)} \qquad\qquad\qquad \text{(iii)}$$

Note that in case (i) the point b is an element of $V \subseteq W$, whereas in cases (ii) and (iii) we have $b = (a, \delta)$ for some a in $f_1^{-1}(V)$ or U, and $\delta > 0$.

Suppose first we are in case (i) and fix a local coefficient system $\mathcal{F}$ on $\mathit{hofib}_W(p)$. We will say $\mathcal{F}$-homology to mean homology with coefficients in pullbacks of $\mathcal{F}$. The inclusion $p^{-1}(b) \hookrightarrow p^{-1}(W)$ factors through the inclusion $p^{-1}(V) \hookrightarrow p^{-1}(W)$, which is a homotopy equivalence since there is an evident deformation retraction of $p^{-1}(W)$ onto $p^{-1}(V)$. Hence it suffices to show that $p^{-1}(b) \hookrightarrow p^{-1}(V)$ induces an isomorphism on $\mathcal{F}$-homology. But this is the same as $p_1^{-1}(b) \hookrightarrow p_1^{-1}(V)$, and the coefficients $\mathcal{F}$ are pulled back to this through $\mathit{hofib}_V(p_1)$, so this does induce

---

[6]In [MS76] it appears to be assumed (in the proof of Proposition 3) that the collection (ă), (b) and (č) is a basis for the double mapping cylinder, but this is not a major problem as the proof is not made any more complicated by having to admit sets of the more general form (a) and (c).

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an isomorphism on $\mathcal{F}$-homology since p_1 is a twisted Leray homology fibration (and V is part of an acceptable basis for it).

Case (iii) is very similar to case (i), but case (ii) requires a little more work. Again fix a local coefficient system $\mathcal{F}$ on $hofib_W(p)$. As before, $p^{-1}(W)$ deformation retracts onto $p^{-1}(V)$, so the inclusion $p^{-1}(b) \hookrightarrow p^{-1}(W)$ factors up to homotopy as

$$p^{-1}(b) \to p^{-1}(f_1(a)) \hookrightarrow p^{-1}(V) \hookrightarrow p^{-1}(W),$$

where the first map is a restriction of $\widetilde{f_1}$. The middle map is the same as $p_1^{-1}(f_1(a)) \hookrightarrow p_1^{-1}(V)$, and the coefficients $\mathcal{F}$ are pulled back through $hofib_V(p_1)$, so it induces an isomorphism on $\mathcal{F}$-homology since p_1 is a twisted Leray homology fibration. The third map is a homotopy equivalence, and the first map is the same as $p_0^{-1}(a) \to p_1^{-1}(f_1(a))$, which is a twisted homology equivalence by hypothesis. Hence $p^{-1}(b) \hookrightarrow p^{-1}(W)$ induces an isomorphism on $\mathcal{F}$-homology, as required. $\qquad\square$

Proof of Lemma 2.10. This is similar to the above proof, but we will provide some of the details. Let $\mathcal{U}_n$ be an acceptable basis for Y_n (w.r.t. p_n). Then there is a basis of contractible sets $\mathcal{U}$ for the mapping telescope Y consisting of

$$\left(\bigcup_\alpha U_\alpha \times (1 - \varepsilon_\alpha, 1] \right) \cup_{f_{n-1}} \left(V \times [0, \varepsilon) \right)$$

for $V \in \mathcal{U}_n$, $\varepsilon_\alpha, \varepsilon > 0$ and $U_\alpha \in \mathcal{U}_{n-1}$ such that $\bigcup_\alpha U_\alpha = f_{n-1}^{-1}(V)$, and

$$U \times (\beta, \gamma)$$

for $U \in \mathcal{U}_n$ and $0 < \beta < \gamma < 1$. Pictorially:

There are four cases to check to show that this is an acceptable basis and p is a twisted Leray homology fibration:

The interesting case is case (ii) (it is analogous to case (ii) of the previous proof). Denote the open set (ii) above by W and write $b = (a, \delta)$ for $a \in f_{n-1}^{-1}(V)$ and

$\delta > 0$. Consider the diagram

$$
\begin{array}{ccccccc}
\mathcal{F} & & & & & & \\
\downarrow & & & & & & \\
\mathit{hofib}_W(p) & \longleftarrow & p^{-1}(W) & \xleftarrow{\ \ i\ \ } & p^{-1}(b) & = & p_{n-1}^{-1}(a) \\
\Big\updownarrow \simeq & & \Big\updownarrow \simeq & \circlearrowleft & \downarrow & & \Big\downarrow f_{n-1} \\
\mathit{hofib}_{V\times\{0\}}(p) & \longleftarrow & p^{-1}(V\times\{0\}) & \longleftarrow & p^{-1}((g_{n-1}(a),0)) & & \\
\Big\uparrow & & \| & & \diagdown & & \Big\downarrow \\
\mathit{hofib}_V(p_n) & \longleftarrow & p_n^{-1}(V) & \xleftarrow{\qquad\ \ j\qquad} & & & p_n^{-1}(g_{n-1}(a))
\end{array}
\tag{2.6}
$$

in which $\mathcal{F}$ is a system of local coefficients on $\mathit{hofib}_W(p)$. We wish to show that the inclusion i induces an isomorphism on $\mathcal{F}$-homology.

Note that $p^{-1}(W)$ deformation retracts onto the subspace $p^{-1}(V \times \{0\})$, so the inclusion $p^{-1}(V \times \{0\}) \hookrightarrow p^{-1}(W)$ is a homotopy equivalence and the square marked $\circlearrowleft$ commutes up to homotopy (the rest of the diagram commutes on the nose). Hence it suffices to show that the restriction of f_{n-1} to $p_{n-1}^{-1}(a)$ and the inclusion j indicated in (2.6) induce isomorphisms on $\mathcal{F}$-homology. But for the restriction of f_{n-1} this is precisely what was assumed by hypothesis. For j it is also true since p_n is a twisted Leray homology fibration (with V part of an acceptable basis for it) and the coefficients $\mathcal{F}$ pull back to j via $\mathit{hofib}_V(p_n)$. $\qquad\square$

Remark 2.11. We will not need it, but one can see from the similarity of the preceding two proofs that they generalize to prove the following fact. Let $F, G\colon \mathcal{A} \to \mathsf{Top}$ be two diagrams of spaces, and $\tau\colon F \Rightarrow G$ a natural transformation between them such that, firstly, the map $\tau_A\colon FA \to GA$ is a twisted Leray homology fibration for all objects A in $\mathcal{A}$, and secondly, for each morphism $f\colon A \to B$ in $\mathcal{A}$ and point $a \in GA$, the restriction $\tau_A^{-1}(a) \to \tau_B^{-1}(Gf(a))$ of Ff is a twisted homology equivalence. Then the induced map $\tau_*\colon \mathit{hocolim}_{\mathcal{A}}(F) \to \mathit{hocolim}_{\mathcal{A}}(G)$ is a twisted Leray homology fibration.

Remark 2.12. We have given a direct proof of Lemmas 2.9 and 2.10; one could also, at the cost of introducing extra point-set topological assumptions, use the twisted homology fibration criterion (Theorem 2.5) instead, as follows. To show that a map is a twisted Leray homology fibration, it is enough to show this restricted to each set in an open cover of the target space – so we need only consider maps of mapping cylinders (e.g. the left-hand side of (2.4)). For this we can apply the twisted homology fibration criterion with

$$
X_0 = B_1, \quad U_1 = B_1 \cup_{f_1} (B_0 \times [0, \tfrac{1}{2})), \quad X_1 = B_1 \cup_{f_1} (B_0 \times [0,1]).
$$

The extra point-set topological assumptions are needed to ensure that condition (i) of the twisted homology fibration criterion holds.

Proof of Proposition 2.8.
- *Step 1 ($n = 0$).* The map $\|p\|_{(0)}\colon \|E\|_{(0)} \to \|B\|_{(0)}$ of 0-skeleta is just the map $p_0\colon E_0 \to B_0$, and so is a twisted Leray homology fibration by hypothesis.
- *Step 2 ($n \geq 1$).* The map $\|p\|_{(n)}\colon \|E\|_{(n)} \to \|B\|_{(n)}$ of n-skeleta is the map of double mapping cylinders induced by

$$
\begin{array}{ccccc}
\|E\|_{(n-1)} & \longleftarrow & \partial\Delta^n \times E_n & \lhook\joinrel\longrightarrow & \Delta^n \times E_n \\
\Big\downarrow{\scriptstyle\|p\|_{(n-1)}} & & \Big\downarrow{\scriptstyle 1 \times p_n} & & \Big\downarrow{\scriptstyle 1 \times p_n} \\
\|B\|_{(n-1)} & \longleftarrow & \partial\Delta^n \times B_n & \lhook\joinrel\longrightarrow & \Delta^n \times B_n.
\end{array}
$$

Hence we just need to verify the conditions of Lemma 2.9 in this case. The left vertical map is a twisted Leray homology fibration by induction, and the other two vertical maps are too since p_n is. The right-hand square induces homeomorphisms (and therefore twisted homology equivalences) on set-theoretic fibers, since the horizontal maps are just inclusions. Writing $\partial\Delta^n = \cup_{n+1}\Delta^{n-1}$, the left-hand square factorizes as follows:

$$
\begin{array}{ccccc}
\|E\|_{(n-1)} & \longleftarrow & \Delta^{n-1} \times E_{n-1} & \xleftarrow{\ \cup_j(1 \times d_j)\ } & \bigcup_{n+1}(\Delta^{n-1} \times E_n) \\
\Big\downarrow{\scriptstyle\|p\|_{(n-1)}} & & \Big\downarrow{\scriptstyle 1 \times p_{n-1}} & & \Big\downarrow{\scriptstyle \cup_{n+1}(1 \times p_n)} \\
\|B\|_{(n-1)} & \longleftarrow & \Delta^{n-1} \times B_{n-1} & \xleftarrow[\ \cup_j(1 \times d_j)\ ]{} & \bigcup_{n+1}(\Delta^{n-1} \times B_n)
\end{array} \qquad (2.7)
$$

By inspection (considering that $\|E\|_{(n-1)} = \big(\coprod_{k \leq n-1}(\Delta^k \times E_k)\big)/\sim$), the left-hand square of this decomposition also induces homeomorphisms on set-theoretic fibers. So it just remains to show that the right-hand square of (2.7) induces twisted homology equivalences on set-theoretic fibers. In other words for all $(x, b) \in \Delta^{n-1} \times B_n$, we want

$$
(1 \times d_j)\colon (1 \times p_n)^{-1}(x, b) \longrightarrow (1 \times p_{n-1})^{-1}(x, d_j(b))
$$

to be a twisted homology equivalence. But this is exactly the map $d_j\colon p_n^{-1}(b) \to p_{n-1}^{-1}(d_j(b))$ which was assumed to be a twisted homology equivalence by hypothesis.
- *Step 3 ($n = \infty$).* If $E_\bullet$ and $B_\bullet$ are finite-dimensional we are done by the previous step; in general we need to apply Lemma 2.10 to finish the proof. We take the map $p_n\colon X_n \to Y_n$ in the lemma to be $\|p\|_{(n)}\colon \|E\|_{(n)} \to \|B\|_{(n)}$ and the horizontal maps f_n and g_n to be the inclusions of skeleta. By above the vertical maps are twisted Leray homology fibrations, and since the horizontal maps are just inclusions, the squares in the ladder induce homeomorphisms (and therefore

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twisted homology equivalences) on set-theoretic fibers. Hence by Lemma 2.10 the mapping telescope $\|p\|\colon \|E\| \to \|B\|$ is a twisted Leray homology fibration. $\square$

Proof of Proposition 2.6. An explicit model for the spaces $E\mathcal{M} \times_{\mathcal{M}} X$ and $B\mathcal{M}$, and the map between them, is the (thick) geometric realization of the map $p_\bullet\colon E_\bullet \to B_\bullet$ of semi-simplicial spaces, where $E_n = \mathcal{M}^n \times X$, $B_n = \mathcal{M}^n$, with the usual face maps of the bar construction, and $p_n\colon \mathcal{M}^n \times X \to \mathcal{M}^n$ is the projection.

Hence we just need to check the conditions of Proposition 2.8 in this case. The projection map $p_n\colon \mathcal{M}^n \times X \to \mathcal{M}^n$ is a trivial fiber bundle, so obviously a twisted Leray homology fibration. We also need to check that for all face maps

$$
\begin{array}{ccc}
\mathcal{M}^n \times X & \xrightarrow{\ \ p_n\ \ } & \mathcal{M}^n \\
{\scriptstyle d_j}\downarrow & & \downarrow{\scriptstyle d_j} \\
\mathcal{M}^{n-1} \times X & \xrightarrow[\ \ p_{n-1}\ \ ]{} & \mathcal{M}^{n-1}
\end{array}
$$

and elements $b = (m_1, \ldots, m_n) \in \mathcal{M}^n$, the map $p_n^{-1}(b) \to p_{n-1}^{-1}(d_j(b))$ is a twisted homology equivalence. For $0 \leq j < n$ this map is just the identity $X \to X$. For $j = n$, it is the map $m_n \cdot - \colon X \to X$ which acts on X by m_n. But this is a twisted homology equivalence by hypothesis. $\square$

3 Scanning for oriented configuration spaces

In this section we apply the tools developed in the previous section to obtain our scanning results. In §3.1 we recall a few facts about oriented configuration spaces and give the more geometric description of the covering space $\Gamma^+(M)$ promised in the introduction. Then in §3.2 we prove acyclicity of the scanning map in the limit (Theorem 1.6) in the special case when the manifold M is of the form $\mathbb{R}^2 \times N$; this requires the use of the "twisted group-completion theorem" (Theorem 2.6) to show that a certain map is an abelian homology fibration.

Then in §3.3 we use this special case to prove the general result, this time using the twisted homology fibration criterion (Theorem 2.5) to show that a certain map is an abelian homology fibration. We also deduce the corollary that the *lifted* scanning map is a homology equivalence in a stable range when the manifold admits a boundary. Finally, in §3.4 we show how to extend this corollary to general manifolds (including closed manifolds).

3.1 Oriented configurations and the double cover of $\Gamma(M)$

In the classical theory of configuration spaces, the stabilization and scanning maps

$$t \colon C_k(M) \to C_{k+1}(M) \text{ and } s \colon C_k(M) \to \Gamma_k(M)$$

play key roles. In this section we recall their construction and describe the modifications needed to define similar maps for oriented configuration spaces. In particular we describe a geometric model for the double cover of $\Gamma(M)$ mentioned in the introduction.

The stabilization maps $t \colon C_k(M) \to C_{k+1}(M)$ and $\widetilde{t} \colon C_k^+(M) \to C_{k+1}^+(M)$ only exist for noncompact manifolds, as there needs to be some place "at infinity" from which to add the new point. We will moreover assume that our manifold M *admits boundary*: it is the interior of some manifold $\overline{M}$ which has non-empty boundary (neither $\overline{M}$ nor $\partial \overline{M}$ are required to be compact). This is equivalent to the existence of a proper embedding $\mathbb{R}_+ \hookrightarrow M$.[7]

The ray $\mathbb{R}_+ \hookrightarrow M$ can be extended to a proper embedding $\mathbb{R}_+^d \hookrightarrow M$ of the Euclidean half-space $\mathbb{R}_+^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 \geq 0\}$. (This is called a "flange" in [CKS12].) Let $\phi \colon M \hookrightarrow M$ be the self-embedding which is given by

$$(x_1, \dots, x_d) \mapsto (\tanh(x_1), x_2, \dots, x_d)$$

on $\mathbb{R}_+^d \subseteq M$ and the identity elsewhere,[8] and let p_0 be the point $(2, 0, \dots, 0)$. Properness of the embedding $\mathbb{R}_+^d \hookrightarrow M$ ensures that this is continuous: schematically, it looks like (a) $\mapsto$ (c), rather than (a) $\mapsto$ (b), in Figure 3.1. The stabilization map t is now defined by:

$$\{p_1, \dots, p_k\} \mapsto \{\phi(p_1), \dots, \phi(p_k), p_0\}.$$

The stabilization map $\widetilde{t}$ is defined in exactly the same way, except one now has to choose a convention for the ordering-up-to-even-permutations of the new configuration. Up to homotopy, these maps depend only on the *end* of the manifold M determined by the ray $\mathbb{R}_+ \hookrightarrow M$ (plus an ordering convention, for $\widetilde{t}$).

Many homotopic descriptions of a map $s \colon C_k(M) \to \Gamma_k(M)$ have been given. The first by Segal in [Seg73] involved sending a configuration to the electric field produced if one places unit charges at each point. The name "scanning map" comes from the following description. Choose a metric on M. Given $\vec{p} = \{p_1, \dots p_k\} \in C_k(M)$ and $m \in M$, we need a continuous way of defining a vector $s(\vec{p})(m) \in \dot{T}_m M$. Pick an $\varepsilon > 0$ continuously depending on $\vec{p}$ and m such

---

[7]One direction is obvious; for the other, one can use the embedded ray to do a "connect sum at infinity" [CKS12] to attach some boundary to M.

[8]The point of $\tanh(\cdot)$ here is just that it is an order-preserving homeomorphism from $[0, \infty)$ to $[0, 1)$.
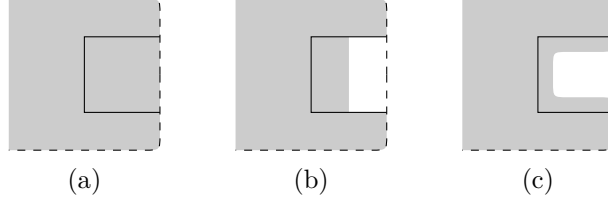
(a)   (b)   (c)

Figure 3.1   In each of (a), (b), (c) the image of $\mathbb{R}^d_+$ in M is represented by the interior square. Since it is embedded *properly*, the self-embedding $M \hookrightarrow M$ defined above looks like (a) $\mapsto$ (c).

that there is at most one $p_i \in B_\varepsilon(m)$ and ε is less than the injectivity radius. Use the exponential map to construct a homeomorphism $e\colon B_\varepsilon(m) \to T_m M$. We now define the value of the section $s(\vec{p})$ at the point $m \in M$ by the formula:

$$s(\vec{p})(m) = \begin{cases} e(p_i) & \text{if } p_i \in B_\varepsilon(m) \\ \infty & \text{otherwise.} \end{cases}$$

Segal describes this as looking at the point m under a microscope and recording the nearby part of the configuration. As you vary $m \in M$, you "scan" the microscope across the manifold.

For non-compact M admitting boundary one can also define stabilization maps $T\colon \Gamma_k(M) \to \Gamma_{k+1}(M)$ as follows. Let $\phi\colon M \hookrightarrow M$ be the self-embedding defined above, choose a degree $+1$ section $\tau \in \Gamma_1\big(M \smallsetminus \overline{\phi(M)}\big)$ and define:

$$T(\sigma)(m) = \begin{cases} \sigma(m) & m \in M \smallsetminus \mathbb{R}^d_+ \\ P^m_{\phi^{-1}(m)}(\sigma(\phi^{-1}(m))) & m \in \mathbb{R}^d_+ \cap \phi(M) \\ \infty & m \in \partial\,\overline{\phi(M)} \\ \tau(m) & m \in M \smallsetminus \overline{\phi(M)}, \end{cases} \tag{3.1}$$

where we fix a trivialization of the bundle $\dot{T}M \to M$ over the contractible subset $\mathbb{R}^d_+ \subseteq M$ to get canonical identifications $P^{m_2}_{m_1}\colon \dot{T}_{m_1}M \to \dot{T}_{m_2}M$ for $m_1, m_2 \in \mathbb{R}^d_+$. See Figure 3.2 for a schematic picture of the four cases.

Note that the same construction replacing τ by a degree -1 section yields a homotopy inverse to T, so $\Gamma_j(M) \simeq \Gamma_k(M)$ for all j, k. Using the scanning map to pick τ allows us to construct a map T making the following diagram commute on the nose:

$$\begin{array}{ccc} C_k(M) & \xrightarrow{\ s\ } & \Gamma_k(M) \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ C_{k+1}(M) & \xrightarrow{\ s\ } & \Gamma_{k+1}(M) \end{array}$$

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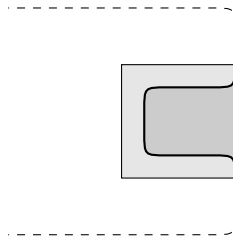
Figure 3.2　A schematic picture of the chosen end of the manifold M, where again the image of $\mathbb{R}^d_+$ is denoted by the interior square. The four cases in (3.1) are colored white, light gray, black and dark gray respectively.

We now turn to the double cover $\Gamma^+(M)$ of $\Gamma(M)$ mentioned in the introduction. To make its definition more geometric, we will describe a geometric construction for the associated map $\pi_1(\Gamma(M)) \to \mathbb{Z}/2$.

First note that one can describe the natural maps $\pi_1(C_k(M)) \to \Sigma_k \to \mathbb{Z}/2$ on the space level as follows. Pick a proper embedding $\iota\colon M \to \mathbb{R}^\infty$ and note that it induces a map of configuration spaces $i\colon C_k(M) \to C_k(\mathbb{R}^\infty)$. Composing with the scanning map gives:

$$C_k(M) \xrightarrow{i} C_k(\mathbb{R}^\infty) \xrightarrow{s} \Omega^\infty_k S^\infty$$

which induces maps $\pi_1(C_k(M)) \to \Sigma_k \to \mathbb{Z}/2$ after taking fundamental groups. The first map forgets everything but the permutation of the basepoint configuration (since a braid in $\mathbb{R}^\infty$ is determined by the permuation of its endpoints), and for $k \geq 2$ the second map is the abelianization of Σ_k, i.e. the sign map, since the scanning map is an isomorphism on H_1.

We now describe a similar map $j\colon \Gamma(M) \to \Omega^\infty S^\infty$. This will give a homomorphism $\pi_1(\Gamma(M)) \to \mathbb{Z}/2$ which can be used to define the covering space $\Gamma^+(M)$. Equivalently, $\Gamma^+(M)$ is defined to be the pullback along j of the universal cover of $\Omega^\infty S^\infty$. To define this map we just need a way of extending a compactly-supported section of $\dot{T}M$ to a compactly-supported section of $\dot{T}\mathbb{R}^\infty$, for example as follows. Let $U \subset \mathbb{R}^\infty$ be a tubular neighborhood of $\iota(M) \subset \mathbb{R}^\infty$ with projection map $p\colon U \to M$. Fix $v \in \mathbb{R}^\infty$ non-zero and $f\colon \mathbb{R}^\infty \to S^\infty = \mathbb{R}^\infty \cup \{\infty\}$ a function which is zero on $\iota(M)$ and ∞ outside of U. For $\sigma \in \Gamma(M)$ and $w \in \mathbb{R}^\infty$, define $j\colon \Gamma(M) \to \Omega^\infty S^\infty$ by the formula:

$$j(\sigma)(w) = \begin{cases} \iota_*(\sigma(p(w))) + f(w)v & \text{if } w \in U \\ \infty & \text{otherwise.} \end{cases}$$

This gives, up to contractible choices, a natural way of extending a section of $\dot{T}M$ to a section of $\dot{T}\mathbb{R}^\infty$.

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Since the following diagram homotopy commutes, the map j and the construction from the introduction give the same double cover of $\Gamma(M)$.

$$
\begin{array}{ccc}
C_k(M) & \xrightarrow{\ i\ } & C_k(\mathbb{R}^\infty) \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\
\Gamma_k(M) & \xrightarrow{\ j\ } & \Omega_k^\infty S^\infty
\end{array}
$$

In particular, commutativity of this diagram shows that j induces a *surjective* map $\pi_1(\Gamma(M)) \to \mathbb{Z}/2$, since $s_* \circ i_*$ is surjective, and so $\Gamma^+(M)$ is not the trivial disconnected double cover.

3.2 Manifolds of the form $\mathbb{R}^2 \times N$

In this subsection, we will prove Theorem 1.6 for manifolds M which are of the form $\mathbb{R}^2 \times N$. Let $C(M)$ denote the disjoint union $\bigsqcup_{k \in \mathbb{N}} C_k(M)$, and write $C_\infty(M)$ and $\Gamma_\infty(M)$ for the homotopy colimits of the sequences of components $C_k(M)$ and $\Gamma_k(M)$ respectively (with maps given by the stabilization maps defined in the previous section).

It will actually be slightly more convenient to work with the homotopy colimits of the direct systems $C(M) \to C(M) \to \cdots$ and $\Gamma(M) \to \Gamma(M) \to \cdots$, which in this notation are $\mathbb{Z} \times C_\infty(M)$ and $\mathbb{Z} \times \Gamma_\infty(M)$ respectively. Note that the stabilization map $T \colon \Gamma_k(M) \to \Gamma_{k+1}(M)$ is a homotopy equivalence, so the components of $\Gamma(M)$ are all homotopy-equivalent and $\mathbb{Z} \times \Gamma_\infty(M) \simeq \Gamma(M)$. The scanning map in the limit

$$
s \colon \mathbb{Z} \times C_\infty(M) \to \Gamma(M) \tag{3.2}
$$

is clearly a bijection on π_0, so Theorem 1.6 is the same as the statement that this map is acyclic, which is what we will prove.

Since our proof will use the group completion theorem, we will first recall a construction of a monoid homotopy equivalent to $C(M)$ in the case that M is of the form $\mathbb{R} \times N$ (see [Seg73] for a similar construction when the manifold is $\mathbb{R}^d$).

For such manifolds, let $C'(M)$ be the subspace of $C((0, \infty) \times N) \times [0, \infty)$ of pairs $\{x_1 \ldots x_k; t\}$ such that the points x_i lie in $(0, t) \times N$. Choose a diffeomorphism $\phi \colon \mathbb{R} \to (0, 1)$, inducing $\hat{\phi} \colon M \to (0, 1) \times N$. Then the map $C(M) \to C'(M)$ given by

$$
\{x_1, \ldots x_k\} \mapsto \{\hat{\phi}(x_1), \ldots \hat{\phi}(x_k); 1\}
$$

is a homotopy equivalence. The space $C'(M)$ can be given the structure of a monoid by sending two configurations to the union of one configuration with a
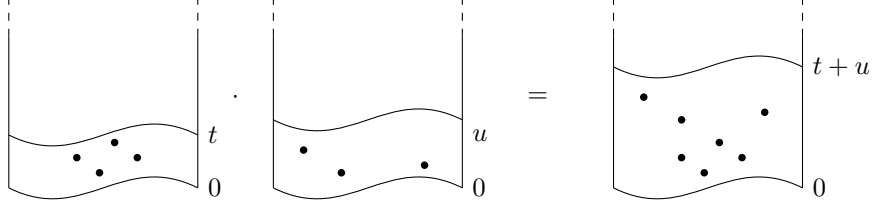
Figure 3.3   The monoid structure on $C'(\mathbb{R} \times N)$. The wavy line represents N.

translation of the other. More precisely, we define $\mu\colon C'(M) \times C'(M) \to C'(M)$ by the formula

$$\mu(\{x_1 \ldots x_k; t\}, \{y_1 \ldots y_j; s\}) = \{x_1, \ldots x_k, y_1 + t, \ldots y_j + t; t + s\}$$

where $y_i + t$ is shorthand for adding the real number t to the first coordinate of $y_i \in \mathbb{R} \times N$. See Figure 3.3. The unit of this monoid is given by the empty configuration and the number zero. This monoid is never commutative, but it is *homotopy*-commutative when M is of the form $\mathbb{R}^2 \times N$. The proof of this is identical to the proof that the higher homotopy groups are abelian. The monoid $C'(M)$ should be compared to the Moore loop space construction which converts the A_∞-space ΩX into a monoid.

Remark 3.1. One can rephrase the algebraic structure of configuration spaces as follows: the space $C(\mathbb{R}^k \times N)$ is an algebra over the framed little k-cubes operad (see [Get94]). This is a special case of the fact that the topological chiral homology of a $(d-k)$-manifold with coefficients in a framed E_d-algebra is a framed E_k-algebra [Lur09, p.91].

Note that, when $M = \mathbb{R} \times N$, the stabilization map $s\colon C_k(M) \to C_{k+1}(M)$ described in §3.1 is homotopic, under the identification above, to the map $C'_k(M) \to C'_{k+1}(M)$ induced by monoid multiplication with any fixed element of $C'_1(M)$. In particular, the mapping telescopes of these two maps are homotopy equivalent: $C'_\infty(M) \simeq C_\infty(M)$.

Proof of Theorem 1.6 when $M = \mathbb{R}^2 \times N$. Note that a similar construction makes the space $\Gamma(M)$ a homotopy commutative monoid, in particular an H-space. Since $C'(M)$ is a homotopy-commutative monoid, there is an acyclic map $\mathbb{Z} \times C'_\infty(M) \to \Omega BC'(M)$ by Corollary 2.7. We therefore have two maps

$$\Omega BC(M) \xleftarrow{\ i\ } \mathbb{Z} \times C_\infty(M) \xrightarrow{\ s\ } \Gamma(M), \qquad (3.3)$$

with i acyclic, and we can deduce acyclicity of s from acyclicity of i as follows.

Firstly, we could assume that M is in fact $\mathbb{R}^d$ for $d \geq 2$, since this is the only case which is needed from this subsection to prove the general case in the next subsection. In this case, there is a map $\Omega BC(\mathbb{R}^d) \to \Gamma(\mathbb{R}^d) = \Omega^d S^d$ commuting with (3.3) which is a homotopy-equivalence, by [Seg73, Theorem 1].

Alternatively, remaining in the more general case of $M = \mathbb{R}^2 \times N$, we may argue as follows. Showing that s is acyclic is equivalent to showing that s^+ (the effect of applying the Quillen plus-construction w.r.t. maximal perfect subgroups to all spaces and maps) is a weak equivalence. In the plus-constructed diagram $(3.3)^+$ we have that i^+ is a weak equivalence, so the middle space is a *simple* space: π_1 acts trivially on π_n for all n. As noted above, $\Gamma(M)$ is an H-space, so it is also simple (and hence also $\Gamma(M)^+ = \Gamma(M)$). Thus s^+ is a homology-equivalence [McD75, Theorem 1.2] between simple spaces, so by the homology Whitehead theorem[9] it is a weak equivalence, as desired. □

Remark 3.2. There are certain general conditions on a discrete group G which ensure that BG^+ is a simple space [Wag72, Proposition 1.2]. These conditions apply to the infinite braid group β_∞ and the infinite symmetric group Σ_∞, so this tells us that the middle space in $(3.3)^+$ is a simple space in the two cases $M = \mathbb{R}^2$ and $M = \mathbb{R}^\infty$ respectively. However, this does not work in general, as $C_\infty(M)$ is not in general the classifying space of a discrete group (and the method of proof in [Wag72] depends critically on properties of group homology).

3.3 Manifolds admitting boundary

In this subsection we let M be any connected manifold of dimension $d \geq 2$ which admits a boundary; suppose it is the interior of some manifold $\overline{M}$ with non-empty boundary. Choose a closed ball $B \subseteq \overline{M}$ of dimension d, intersecting $\partial\overline{M}$ in a closed ball of dimension $d-1$, and assume that the stabilization map is supported inside B: in other words it adds a new configuration point in B without moving the configuration in $M \smallsetminus B$. Let $\mathring{B}$ denote the interior of B.

We start with some notation for relative configuration and section spaces.

Definition 3.3. For a subspace $N \subseteq M$, let $\Gamma(M,N)$ be the subspace of all sections of the restriction of $\dot{T}M \to M$ to $M \smallsetminus N$ (with the subspace topology) such that the support of each section is contained in a compact subset of M. In particular, we have $\Gamma(M,\varnothing) = \Gamma(M)$. Define $C(M,N)$ to be the quotient of $C(M)$ where we identify two configurations if they agree on $M \smallsetminus N$. There are natural maps $\pi \colon C(M) \to C(M,N)$ and $\pi' \colon \Gamma(M) \to \Gamma(M,N)$: the quotient map

---

[9]See for example [Hat02, Proposition 4.74] for a statement of this. The domain and codomain should be path-connected, but in our case s^+ is a bijection on π_0 and we can consider each path-component separately.

and the forgetful map respectively. The scanning map descends to a well-defined map $C(M, N) \to \Gamma(M, N)$.[10]

Before we prove Theorem 1.6, we first prove that the map $\pi_\infty \colon C_\infty(M) \to C(M, \mathring{B})$ in the colimit induced by $\pi \colon C_k(M) \to C(M, \mathring{B})$ is an abelian homology fibration with fiber $C_\infty(\mathring{B})$.

Lemma 3.4. *Let M be a manifold admitting boundary and let B be as above. The projection map $\pi_\infty \colon C_\infty(M) \to C(M, \mathring{B})$ induced by $\pi \colon C_k(M) \to C(M, \mathring{B})$ is an abelian homology fibration with fiber $C_\infty(\mathring{B})$.*

Proof. We will begin by choosing explicit models of $C_\infty(M)$ and $C_\infty(\mathring{B})$. Define $D \coloneqq B \cap \partial \overline{M}$, which is a closed ball of dimension $d-1$. Let $\hat{M} \coloneqq \overline{M} \cup_D D \times [0, \infty)$ and choose a point $p_i \in D \times (i, i+1)$ for each $i \geq 2$. Define the topological space Y as follows. Firstly, as a set: an element of Y is a countably infinite configuration c of distinct points in $\hat{M}$, together with a parameter $t \in [2, \infty)$. These data are required to have the property that the part of the configuration in $D \times [\lfloor t \rfloor, \infty)$ is precisely $\{p_i \mid i \geq \lfloor t \rfloor\}$ and the part of the configuration in $\overline{M} \cup_D D \times [1, \lfloor t \rfloor)$ has cardinality $\lfloor t \rfloor - 2$, where $\lfloor t \rfloor$ denotes the floor of t. We can view the p_i's as "parking spaces" and we only allow a finite number of points, controlled by t, to leave their parking spaces. The parking spaces only start at $i = 2$ since we will use $D \times [1, 2]$ to model B and $D \times (\frac{1}{2}, 1]$ as a collar neighborhood. An example of a configuration $(c, t) \in Y$ with $\lfloor t \rfloor = 5$ is pictured in Figure 3.4.

With this description, it is topologized so that basic open neighborhoods of a point $(c, t) \in Y$ consist of elements $(c', t') \in Y$ so that c' is a configuration 'close' to c in the usual sense and $|t - t'| < \varepsilon$. This means for example that in a continuous path in Y, a configuration point $x \in c$ may not move past the 'barrier' $D \times \{\lfloor t \rfloor\}$ unless the parameter t is first continuously increased to at least $\lfloor t \rfloor + 1$. Equally, t may not be decreased below $\lfloor t \rfloor$ until $c \cap (D \times [\lfloor t \rfloor - 1, \infty))$ has first been made equal to $\{p_i \mid i \geq \lfloor t \rfloor - 1\}$ – in other words there is a point back in the parking space $p_{\lfloor t \rfloor - 1}$ and there are no other points in $D \times [\lfloor t \rfloor - 1, \lfloor t \rfloor]$.

We define F to be the subspace of Y of configurations $(c, t) \in Y$ such that c contains no points in $\overline{M} \cup_D D \times [0, 1)$, and define X to be the relative configuration space $C(\hat{M}, D \times [1, \infty))$. There is an obvious map $\phi \colon Y \to X$, forgetting the parameter t and the points of the configuration c which are in $D \times [1, \infty)$. It is

---

[10]This is strictly false, since in $C(M, N)$ points may disappear when they enter N, whereas in $\Gamma(M, N)$ the "electric field" induced by a point is not permitted to suddenly vanish when its center enters N. This can be rescued by modifying the definition of $\Gamma(M, N)$ slightly: replace it with all sections of the restriction of $\hat{T}M \to M$ to $M \setminus \mathrm{N}_\varepsilon(N)$ whose support is contained in a compact subset of M. Here we have chosen a metric on M, and $M \setminus \mathrm{N}_\varepsilon(N)$ is all points of M whose ε-neighborhood is disjoint from N, where ε is chosen to be larger than the radius of the "magnifying glass" used to define the scanning map.
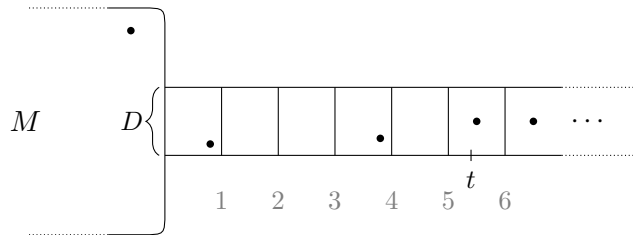
Figure 3.4 An element of Y in which 3 points have left their parking spaces.

then clear that the sequence

$$C_\infty(\mathring{B}) \hookrightarrow C_\infty(M) \xrightarrow{\pi_\infty} C(M, \mathring{B})$$

is homotopy equivalent to the sequence $F \hookrightarrow Y \xrightarrow{\phi} X$. (Included in this claim is the fact that the configuration space on an open ball is homotopy equivalent to the configuration space on a closed ball, and similarly for relative configuration spaces.) Note that the set-theoretic fiber of the map ϕ over the point of X corresponding to the *empty* relative configuration is precisely F. Hence it remains for us to prove that ϕ is an abelian homology fibration. We will do this using the abelian homology fibration criterion (Theorem 2.5), as well as the main result of [RW] to verify one of the conditions in the criterion.

Let X_k be the subspace of X of relative configurations with at most k points in $\overline{M} \cup_D D \times [0, 1)$. This is an increasing filtration of X by closed subsets, so we just need to verify the conditions (i)–(iii) of Theorem 2.5 in this case.

The deformation retractions required for condition (i) can be easily constructed (cf. the proof of Lemma 4.1 on page 106 of [McD75]), but we will not do this here. Since X_0 is just a point, the part of condition (ii) concerned with X_0 is trivially true.

As a brief aside, define the space $F^{(\ell)}$, for a non-negative integer ℓ, similarly to F, to be the space of pairs (c, t) where c is an infinite configuration in $D \times [0, \infty)$ agreeing with $\{p_i\}_{i \geq 2}$ to the right of $\lfloor t \rfloor$ and with exactly $\lfloor t \rfloor - 2 - \ell$ points to the left of $\lfloor t \rfloor$ (so necessarily $t \in [\ell + 2, \infty)$). We have $F^{(0)} = F$ and $F^{(\ell)} \cong F$ for all ℓ. For all $k \geq 1$ the preimage $\phi^{-1}(X_k \smallsetminus X_{k-1})$ is $(X_k \smallsetminus X_{k-1}) \times F^{(k)} \cong (X_k \smallsetminus X_{k-1}) \times F$, with ϕ restricting to the obvious projection map. So over each layer $X_k \smallsetminus X_{k-1}$ of the filtration it is a trivial fiber bundle (therefore certainly an abelian homology fibration) with fiber F, verifying condition (ii).

For condition (iii), define U_k to be the open subset of X_k consisting of those configurations with at most $k-1$ points in $\overline{M} \cup_D D \times [0, \frac{1}{2}]$. Let $f_t \colon [0, \infty) \to [0, \infty)$ be the function which is

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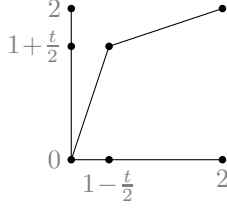
on $[0, 2]$ and the identity on $[2, \infty)$. This induces an automorphism g_t of $\hat{M}$ which is the identity on $\overline{M}$ and $\mathrm{id} \times f_t$ on $D \times [0, \infty)$. We can then define the homotopies h_t and H_t needed for condition (iii) by simply applying g_t to each point of the configuration or relative configuration. Properties (a) and (b) of these homotopies are immediate from their definition.

Finally, to show property (c) we need to show that a certain map is an abelian homology equivalence.[11] Recall the monoid $C'(\mathbb{R}^d)$ constructed in §3.2, and let $C'_\infty(\mathbb{R}^d)$ be the mapping telescope $hocolim(C'(\mathbb{R}^d) \to C'(\mathbb{R}^d) \to \cdots)$ for right-multiplication by a fixed element $m \in C'_1(\mathbb{R}^d)$. Up to homotopy equivalence, the map which we need to be an abelian homology equivalence is the composition of finitely many copies – depending on how many particles we pushed into $D \times [1, \infty)$ during the homotopy H_t; possibly zero – of the map $m \cdot - : C'_\infty(\mathbb{R}^d) \to C'_\infty(\mathbb{R}^d)$ given by left-multiplication by m. This is precisely the map which is shown by [RW] to be an abelian homology equivalence as long as the monoid is homotopy-commutative, which is true in our case since $d \geq 2$. $\square$

Proof of Theorem 1.6. We have a square of maps

$$
\begin{array}{ccc}
C(M) & \longrightarrow & \Gamma(M) \\
\pi \downarrow & & \downarrow \pi' \\
C(M, \mathring{B}) & \longrightarrow & \Gamma(M, \mathring{B})
\end{array}
$$

which commutes up to homotopy. Give $C(M)$ the empty configuration as its basepoint; this determines basepoints for the other spaces in the square. The set-theoretic fiber of π is $C(\mathring{B})$ and that of π' is $\Gamma(\mathring{B})$. There is a self-map-of-diagrams of this square, given by the stabilization maps on $C(M)$ and on $\Gamma(M)$, and the identity maps on the bottom two spaces. Taking an infinite sequence of copies of this self-map, and taking mapping telescopes objectwise, we obtain

---

[11]Actually it only needs to induce isomorphisms on homology w.r.t. abelian coefficients systems that are pulled back from a certain other space, but it will actually be true for all abelian coefficient systems.

$$\begin{CD}
\mathbb{Z} \times C_\infty(M) @>>> \Gamma(M) \\
@V{\pi_\infty}VV @VV{\pi'}V \\
C(M, \mathring{B}) @>>> \Gamma(M, \mathring{B})
\end{CD}$$

in which the top-right space is still homotopy-equivalent to $\Gamma(M)$ since the stabilization maps $\Gamma(M) \to \Gamma(M)$ are all homotopy equivalences. The ladder of maps whose homotopy colimit is π_∞, namely

$$\begin{CD}
C(M) @>{t}>> C(M) @>>> \cdots \\
@V{\pi}VV @VV{\pi}V \\
C(M, \mathring{B}) @>{\mathrm{id}}>> C(M, \mathring{B}) @>>> \cdots
\end{CD}$$

commutes on the nose since we chose B such that the stabilization map does not affect the configuration in $M \smallsetminus B$. Hence the set-theoretic fiber of π_∞ is $\mathbb{Z} \times C_\infty(\mathring{B})$. Putting this together we have the large diagram shown in Figure 3.5, where s_B, s_M denote the scanning maps for $\mathring{B}$ and M respectively, and $s_{(M,B)}$ denotes the relative scanning map.

$$\begin{array}{ccccc}
\mathit{fib}(\pi_\infty) = & \mathbb{Z} \times C_\infty(\mathring{B}) & \xrightarrow{\;\;s_B\;\;} & \Gamma(\mathring{B}) & = \mathit{fib}(\pi') \\
& \Big\downarrow{\scriptstyle i_\infty} & & \Big\downarrow{\scriptstyle i'} & \\
\mathit{hofib}(\hat{s}) = \mathit{hofib}(\hat{\pi}) \to & \mathit{hofib}(\pi_\infty) & \xrightarrow{\;\;\hat{s}\;\;} & \mathit{hofib}(\pi') & \\
{\scriptstyle (*)}\Big\downarrow & \Big\downarrow & & \Big\downarrow & \\
\mathit{hofib}(s_M) & \longrightarrow \mathbb{Z} \times C_\infty(M) & \xrightarrow{\;\;s_M\;\;} & \Gamma(M) & \\
{\scriptstyle \hat{\pi}}\Big\downarrow & {\scriptstyle \pi_\infty}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi'} & \\
\mathit{hofib}(s_{(M,B)}) & \longrightarrow C(M, \mathring{B}) & \xrightarrow{\;s_{(M,B)}\;} & \Gamma(M, \mathring{B}) &
\end{array}$$

Figure 3.5   The large diagram from the proof of Theorem 1.6. The important square is the bottom right one; the rest of the diagram just keeps track of the fibers and homotopy fibers of the maps in this square.

The map $s_{(M,B)}$ is a weak equivalence by [Böd87, Proposition 2], and hence the map $(*)$ in the diagram is also a weak equivalence. The map π' is a fibration [Böd87], so i' is also a weak equivalence. By Lemma 3.4, the map π_∞ is an abelian (Serre) homology fibration, so i_∞ is an abelian homology equivalence.

Now we know that the map s_B is acyclic by the previous section, since $\mathring{B} \cong \mathbb{R}^d$ and $d \geq 2$. Note that $\Gamma(\mathring{B})$ is an H-space so it has abelian fundamental group,

and hence so does $hofib(\pi')$, as i' is a weak equivalence. Suppose we are given a coefficient system $\mathcal{F}$ on $hofib(\pi')$. The composition $i' \circ s_B$ is acyclic, so it induces an isomorphism on homology with coefficients pulled back from $\mathcal{F}$. By commutativity, this means that the composition

$$H_*(\mathbb{Z} \times C_\infty(\mathring{B}); i_\infty^* \hat{s}^* \mathcal{F}) \xrightarrow{(i_\infty)_*} H_*(hofib(\pi_\infty); \hat{s}^* \mathcal{F}) \xrightarrow{\hat{s}_*} H_*(hofib(\pi'); \mathcal{F}) \quad (3.4)$$

is an isomorphism. The coefficient system $\hat{s}^* \mathcal{F}$ is abelian and i_∞ is an abelian homology equivalence, so the first map in (3.4) is an isomorphism. Therefore so is the second, and $\mathcal{F}$ was arbitrary, so we have proved that $\hat{s}$ is acyclic. One of several equivalent characterizations of acyclicity (mentioned in the introduction) is that the reduced integral homology of $hofib(\hat{s})$ is trivial. Since $(*)$ is a weak equivalence this is also true of $hofib(s_M)$, and therefore s_M is acyclic. $\qquad\square$

Using homological stability for oriented configuration spaces, we can now draw conclusions about the lift $\widetilde{s}: C_k^+(M) \to \Gamma_k^+(M)$ of the scanning map when M admits boundary. Namely, we have the following corollary.

Corollary 3.5. *If M admits boundary, the lift of the scanning map $\widetilde{s}_k: C_k^+(M) \to \Gamma_k^+(M)$ induces an isomorphism on $H_*(-;\mathbb{Z})$ in the range $* \le (k-5)/3$ and a surjection for $* \le (k-2)/3$.*

Proof. Consider the following commutative diagram:

$$
\begin{array}{ccc}
C_k^+(M) & \xhookrightarrow{\;\imath_k\;} & hocolim_j(C_j^+(M)) = C_\infty^+(M) \\
{\scriptstyle \widetilde{s}_k}\downarrow & & \downarrow{\scriptstyle \widetilde{s}_\infty} \\
\Gamma_k^+(M) & \xhookrightarrow{\;\jmath_k\;} & hocolim_j(\Gamma_j^+(M)) = \Gamma_\infty^+(M)
\end{array}
$$

The map $\jmath_k$ is a homotopy equivalence, since the stabilization maps $T: \Gamma_k^+(M) \to \Gamma_{k+1}^+(M)$ are all homotopy equivalences. By Theorem 1.6, the scanning map $s_\infty: C_\infty(M) \to \Gamma_\infty(M)$ is acyclic, so its pullback $\widetilde{s}_\infty$ is also acyclic, in particular a homology equivalence. Hence the maps $\widetilde{s}_k$ and $\imath_k$ are the same on homology, and the map $\imath_k$ is an isomorphism (resp. surjective) in the claimed range by [Pal13] (which was stated as Theorem 1.4 in the introduction). $\qquad\square$

3.4 Manifolds not admitting boundary

In this subsection, we describe how to generalize Corollary 3.5 to the case of manifolds that do not admit a boundary. This is based on the arguments used by McDuff to prove Theorem 1.1 of [McD75]. Pick a metric d on the manifold M (we always assume our manifolds are paracompact) and a smooth function

$\varepsilon\colon M \to \mathbb{R}_{>0}$. Denote the ball in M of radius $r > 0$ about a point $p \in M$ by $B_r(p)$.

Definition 3.6. Let ${}^\varepsilon C_k(M)$ be the subspace of $C_k(M)$ consisting of configurations $\{p_1, \ldots, p_k\}$ such that $B_{\varepsilon(p_i)}(p_i)$ and $B_{\varepsilon(p_j)}(p_j)$ are disjoint for $i \neq j$. Define ${}^\varepsilon C_k^+(M) \subseteq C_k^+(M)$ analogously.

Remark 3.7. The inclusion ${}^\varepsilon C_k(M) \hookrightarrow C_k(M)$ is a homotopy equivalence for sufficiently small ε (taking $\varepsilon < \frac{1}{k}\{\text{injectivity radius}\}$ should do), as is the inclusion ${}^\varepsilon C_k^+(M) \hookrightarrow C_k^+(M)$.

We now prove Theorem 1.5, that the scanning map induces an isomorphism $\widetilde{s}_*\colon H_*(C_k^+(M)) \to H_*(\Gamma_k^+(M))$ in the range $* \leq (k-5)/3$, without assuming that the manifold M admits boundary. We also prove that the scanning map $s\colon C_k(M) \to \Gamma_k(M)$ induces an isomorphism in homology with $\mathbb{Z}^{(-1)}$ coefficients in a similar range.

Proof of Theorem 1.5. Choose an open subset $U \subseteq M$ such that $\varepsilon \equiv 1$ on U, and $U \cong (-\ell, \ell)^d$ as metric spaces for some $\ell \gg 0$. (The metric and the function ε are just auxiliary data, so we may choose them so that this exists.) Since the unit ball $B = B_1(0)$ may contain at most one point of a configuration in ${}^\varepsilon C_k^+(M)$, we have a surjective map

$$\pi\colon {}^\varepsilon C_k^+(M) \longrightarrow M/N \cong S^d$$

forgetting everything outside this ball, where we have denoted the complement $M \smallsetminus B$ by N.

Over each point $p \in B \subset S^d$, the fiber of π is precisely ${}^\varepsilon C_{k-1}^+(M \smallsetminus B_2(p))$, and these fibers fit together to make $\pi|_{\pi^{-1}(B)}\colon \pi^{-1}(B) \to B$ into a fiber bundle. Since the base is contractible, this is bundle-isomorphic to the trivial bundle over B with fiber ${}^\varepsilon C_{k-1}^+(M \smallsetminus B_2(0)) \cong {}^\varepsilon C_{k-1}^+(N)$.

Actually the identification of the fiber in the above paragraph is only valid when $k \geq 3$; the answer is slightly different when $k \leq 2$ due to the extra "ordering up to even permutations" data. However, the statement of Theorem 1.5 is vacuously true for $k \leq 4$, so we may as well assume that $k \geq 5$ for this proof.

Over the point in S^d corresponding to N, the fiber of π is ${}^\varepsilon C_k^+(N)$.[12] Hence we have a homeomorphism

$$ {}^\varepsilon C_k^+(M)/{}^\varepsilon C_k^+(N) \;\cong\; \Sigma^d({}^\varepsilon C_{k-1}^+(N)_+). $$

---

[12] Actually, this is not quite true on the nose. The fiber in question consists of configurations of k points in $M \smallsetminus B$ such that the ε-balls — in M — centered at the points are pairwise disjoint, whereas ${}^\varepsilon C_k^+(M \smallsetminus B)$ consists of configurations of k points in $M \smallsetminus B$ such that the ε-balls — in $M \smallsetminus B$ — centered at the points are pairwise disjoint. The latter is a strictly weaker condition, so the fiber is a proper subset of ${}^\varepsilon C_k^+(M \smallsetminus B)$. However, the inclusion is a homotopy equivalence.

Looking at the section space side: this time the projection $\Gamma_k^+(M) \to \Gamma(M, N)$ is a fiber bundle with each fiber homeomorphic to $\Gamma_k^+(N)$. From Definition 3.3, we see that in this case $\Gamma(M, N)$ consists of sections of the trivial bundle $B \times S^d \to B$ whose support is contained in a compact subset of M. But the closure of B in M is compact, so this is a vacuous condition, and $\Gamma(M, N) \cong \mathrm{Map}(B, S^d) \simeq S^d$.

Now, if the base were *homeomorphic* to S^d we could say that $\Gamma_k^+(M)/\Gamma_k^+(N)$ is homeomorphic to $\Sigma^d(\Gamma_k^+(N)_+)$, similarly to above. Although this time the base is only homotopy equivalent to S^d, it is nevertheless not hard to show that $\Gamma_k^+(M)/\Gamma_k^+(N)$ is at least *homology equivalent* to $\Sigma^d(\Gamma_k^+(N)_+)$.[13]

The scanning map ${}^\varepsilon C_k^+(M) \to \Gamma_k^+(M)$ is a map of pairs

$$({}^\varepsilon C_k^+(M), {}^\varepsilon C_k^+(N)) \longrightarrow (\Gamma_k^+(M), \Gamma_k^+(N)),$$

and therefore induces a map of long exact sequences in homology (the lower one is the Wang sequence for the bundle $\Gamma_k^+(M) \to \Gamma(M, N) \simeq S^d$):

$$
\begin{array}{ccccccc}
H_{i+1-d}({}^\varepsilon C_{k-1}^+(N)) & \longrightarrow & \widetilde{H}_i({}^\varepsilon C_k^+(N)) & \longrightarrow & \widetilde{H}_i({}^\varepsilon C_k^+(M)) & \longrightarrow & H_{i-d}({}^\varepsilon C_{k-1}^+(N)) \\
\downarrow {\scriptstyle (s \circ t)_*} & & \downarrow {\scriptstyle s_*} & & \downarrow {\scriptstyle s_*} & & \downarrow {\scriptstyle (s \circ t)_*} \\
H_{i+1-d}(\Gamma_k^+(N)) & \longrightarrow & \widetilde{H}_i(\Gamma_k^+(N)) & \longrightarrow & \widetilde{H}_i(\Gamma_k^+(M)) & \longrightarrow & H_{i-d}(\Gamma_k^+(N)).
\end{array}
$$

(The stabilization map t appears when we do the identifications from the above two paragraphs because the inclusion of the fiber ${}^\varepsilon C_{k-1}^+(N) \hookrightarrow {}^\varepsilon C_k^+(M)$ adds a point to the configuration.)

Since N admits a boundary, we can apply Theorem 1.4 and Corollary 3.5 to conclude that the first, second, fourth and fifth (not shown) vertical maps above are isomorphisms in the ranges $3i \leq k + 3d - 9$, $3i \leq k - 5$, $3i \leq k + 3d - 6$ and $3i \leq k - 2$ respectively. Since $d \geq 2$, the range $3i \leq k - 5$ is sufficient. The theorem now follows from the five-lemma. □

Using Theorem 1.5, we can now prove a similar theorem regarding the scanning map $C_k(M) \to \Gamma_k(M)$ on twisted homology $H_*(-; \mathbb{Z}^{(-1)})$. Recall that $R^{(-1)}$, for a ring R, is the $R[\mathbb{Z}/2]$-module where the generator of $\mathbb{Z}/2$ acts by multiplication by -1. The fundamental groups $\pi_1(C_k(M))$ and $\pi_1(\Gamma_k(M))$ have natural maps to $\mathbb{Z}/2$ described in §3.1, so $R^{(-1)}$ becomes a module over their group-rings too. See the introduction or [BCT89] for a discussion of the relationship between $H_*(C_k(M); \mathbb{Z}^{(-1)})$ and the homology of the spaces appearing in the generalized Snaith splitting introduced in [Böd87].

---

[13]This is essentially the construction of the Wang sequence for a fibration with base homotopy equivalent to S^d.

Corollary 3.8. *The scanning map induces an isomorphism $H_*(C_k(M); \mathbb{Z}^{(-1)})) \to H_*(\Gamma_k(M); \mathbb{Z}^{(-1)})$ in the range $* \leq (k-8)/3$ and a surjection for $* \leq (k-5)/3$.*

Proof. We will say that a ring R satisfies Condition $\circledast$ if the scanning map

$$s \colon H_*(C_k(M); R^{(-1)}) \to H_*(\Gamma_k(M); R^{(-1)})$$

is an isomorphism in the range $* \leq (k-5)/3$ and a surjection for $* \leq (k-2)/3$. We will first prove that $\mathbb{Q}$ and $\mathbb{Z}_p$, for all primes p, satisfy this condition. (For ease of notation we will use $\mathbb{Z}_m$ to denote the ring $\mathbb{Z}/m\mathbb{Z}$ in this proof.) Since $-1 = 1$ in $\mathbb{Z}_2$, this case follows immediately from Theorem 1.3. Now let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}_p$ with p odd. Since 2 is invertible in $\mathbb{F}$, we have that the system of coefficients $\mathbb{F}[\mathbb{Z}/2]$ described in the introduction is isomorphic to $\mathbb{F} \oplus \mathbb{F}^{(-1)}$. Thus $H_*(C_k^+(M); \mathbb{F}) \cong H_*(C_k(M); \mathbb{F} \oplus \mathbb{F}^{(-1)})$ and similarly for $\Gamma_k^+(M)$, and so by Theorem 1.5,

$$s_* \colon H_*(C_k(M); \mathbb{F} \oplus \mathbb{F}^{-1}) \to H_*(\Gamma_k(M); \mathbb{F} \oplus \mathbb{F}^{-1})$$

is an isomorphism for $* \leq (k-5)/3$ and a surjection for $* \leq (k-2)/3$.

The long exact sequence in homology associated to a short exact sequence of coefficients gives the following commutative diagram:

$$
\begin{array}{cccc}
H_q(C_k(M); \mathbb{F}) \to H_q(C_k(M); \mathbb{F} \oplus \mathbb{F}^{(-1)}) \to H_q(C_k(M); \mathbb{F}^{(-1)}) \to H_{q-1}(C_k(M); \mathbb{F}) \\
\downarrow{s_*} \qquad\qquad \downarrow{s_*} \qquad\qquad \downarrow{s_*} \qquad\qquad \downarrow{s_*} \\
H_q(\Gamma_k(M); \mathbb{F}) \to H_q(\Gamma_k(M); \mathbb{F} \oplus \mathbb{F}^{(-1)}) \to H_q(\Gamma_k(M); \mathbb{F}^{(-1)}) \to H_{q-1}(\Gamma_k(M); \mathbb{F}).
\end{array}
$$

By the five-lemma, Theorem 1.3 and Theorem 1.5, we conclude that $\mathbb{F}$ satisfies Condition $\circledast$.

Next we show that for all n, the ring $\mathbb{Z}_{p^n}$ satisfies Condition $\circledast$. This can be seen by induction, using the five-lemma and the long exact sequence in homology associated to the short exact sequences of coefficients:

$$0 \to \mathbb{Z}_p^{(-1)} \to \mathbb{Z}_{p^{n+1}}^{(-1)} \to \mathbb{Z}_{p^n}^{(-1)} \to 0.$$

Here it is essential for keeping the induction going that, in the proof of the five-lemma for

$$
\begin{array}{ccccccccc}
\cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\varepsilon} \\
\cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot
\end{array}
$$

only *surjectivity* of α is used to prove that γ is an isomorphism, and nothing about α is used to prove surjectivity of γ.

Thus $\mathbb{Z}_{p^\infty} = colim_n(\mathbb{Z}_{p^n})$ also satisfies Condition $\circledast$, since homology commutes with direct limits of (twisted) coefficients. Since $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_i \mathbb{Z}_{p_i^\infty}$, there is a short exact sequence:

$$0 \to \mathbb{Z}^{(-1)} \to \mathbb{Q}^{(-1)} \to \bigoplus_i \mathbb{Z}_{p_i^\infty}^{(-1)} \to 0.$$

The corollary now follows from the long exact sequence in homology associated to this short exact sequence of coefficients. $\square$

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