Twisted homology of configuration spaces

Martin Palmer // 20 August 2013

Abstract

Fix a connected open manifold M and a path-connected space X. Then the sequence $C_n(M,X)$ of configuration spaces of n distinct unordered points in M equipped with labels from X is known to be *homologically stable*: in each degree, the integral homology is eventually independent of n. In this note we prove that this is also true for homology with twisted coefficients. Obviously one cannot choose local coefficients randomly for each space in the sequence and expect stability: what is needed is a so-called finite-degree twisted coefficient system for $\{C_n(M,X)\}$, which we begin by explaining in detail. We then use the untwisted homological stability result to deduce twisted homological stability in this setting. The result and the methods are generalisations of those of Betley [Bet02] in the case of the symmetric groups.

Contents

§1. Introduction

For a pair of spaces M and X, the *configuration space of n unordered points in M with labels in X* is defined by

$$C_n(M,X) \coloneqq (\mathrm{Emb}(n,M)\times X^n)/\Sigma_n.$$

Here n is the discrete space of cardinality n, so $\mathrm{Emb}(n,M)$ is the subspace of M^n where no two points coincide. The symmetric group Σ_n acts diagonally, permuting the points and the list of labels, so an element of $C_n(M,X)$ is a subset of M of cardinality n, together with an element of X "attached" to each point. More generally, one could define a configuration space associated to a fibre bundle $\pi\colon E\to M$ by

$$C_n(M,\pi) \coloneqq \{(e_1,\dots,e_n)\in E^n \mid \pi(e_i)\neq\pi(e_j) \text{ for } i\neq j\}/\Sigma_n.$$

An element of $C_n(M,\pi)$ is thus a subset of M of cardinality n, together with an element of $\pi^{-1}(p)$ "attached" to each point p of this subset. However, in this note we will restrict our attention to configuration spaces with labels in a fixed label-space X, corresponding to the trivial bundle $M\times X\to M$.

Assumption 1.1 From now on we always assume that M is an open, connected manifold of dimension at least 2, and that X is a path-connected space.

Since M is open, there is a well-defined "stabilisation map" $C_n(M,X)\to C_{n+1}(M,X)$, defined in §2.2 below, so called because the sequence of spaces $\{C_n(M,X)\}$ is homologically stable with respect to it:

2010 *Mathematics Subject Classification*: Primary 55R80; secondary 57N65.
Key words and phrases: Configuration spaces, homological stability, twisted coefficients.

Theorem 1.2 ([Seg73, McD75, Seg79, RW13]) *Under the conditions on M and X assumed above, the map $C_n(M,X) \to C_{n+1}(M,X)$ induces an isomorphism on integral homology in degrees $* \leqslant \frac{n}{2}$, and is split-injective on homology in all degrees.*

Twisted coefficients. Several other families of groups or spaces which are homologically stable are also known to have homological stability for *twisted coefficients*. For example general linear groups [Dwy80], mapping class groups of surfaces [Iva93, CM09, Bol12] and the symmetric groups [Bet02] are known to satisfy this phenomenon.

The minimum data needed for the question of twisted homological stability for a sequence $\{Y_n\}$ to be defined at all is a functor $\pi_1(\{Y_n\}) \to \mathsf{Ab}$. By $\pi_1(\{Y_n\})$ we mean the category (groupoid) where the objects are the natural numbers, all morphisms are automorphisms, and $\mathrm{Aut}(n) = \pi_1(Y_n)$. In other words this is just a choice of $\pi_1(Y_n)$-module for each n. Of course there is no chance of stability with respect to such a general "twisted coefficient system", since the $\pi_1(Y_n)$-modules for differing n have absolutely no relation to each other.

To get a notion of twisted coefficient system that has a chance of stability one needs to add some (non-endo)morphisms to $\pi_1(\{Y_n\})$ and require that the functor from this new category to Ab satisfy some finiteness conditions defined in terms of the new morphisms. The correct way to do this depends on the particular context one is working in.

In §§2,3 below we will define a *twisted coefficient system of degree d* for the sequence $\{C_n(M,X)\}$ to be a functor from a certain category $\mathcal{B}(M,X)$ to Ab satisfying a certain finiteness condition. To state the main result, it is enough to mention that it includes the data of a $\pi_1 C_n(M,X)$-module T_n for each n, and the stabilisation map induces a natural map

$$H_*(C_n(M,X);T_n) \to H_*(C_{n+1}(M,X);T_{n+1}). \tag{1.1}$$

The main result of this note is the following:

Theorem 1.3 *Under Assumption 1.1, if T is a twisted coefficient system for $\{C_n(M,X)\}$ of degree d, then the map (1.1) is an isomorphism in degrees $* \leqslant \frac{n-d}{2}$, and is split-injective in all degrees.*

This is a generalisation of the result of [Bet02], where twisted homological stability is proved for the symmetric groups $\{\Sigma_n\}$, which is the case $M = \mathbb{R}^\infty$ and $X = *$.

Remark 1.4 The split-injectivity statement of this theorem is fairly easy, and has essentially the same proof as in the untwisted case. It is proved separately in §7. We remark that it does not in fact depend on the twisted coefficient system being of finite degree.

Remark 1.5 If $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ is a *rational* twisted coefficient system, i.e. its image lies in the subcategory $\mathsf{Vect}_{\mathbb{Q}} < \mathsf{Ab}$ of rational vector spaces, then the stability range in Theorem 1.3 can be improved to $* < n-d$, at least when M is either orientable or at least 3-dimensional. See Remark 6.5 after the proof of Theorem 1.3 in §6.

Some special cases of Theorem 1.3 are as follows:

Corollary 1.6 *There are isomorphisms*

$$\begin{aligned} H_*\big(C_n(M,X);\mathbb{Z}\big[\Sigma_n/(\Sigma_k\times\Sigma_{n-k})\big]\big) &\cong H_*\big(C_{n+1}(M,X);\mathbb{Z}\big[\Sigma_{n+1}/(\Sigma_k\times\Sigma_{n+1-k})\big]\big),\\ H_*\big(C_n(M,X);\mathbb{Z}\big[\Sigma_n/\Sigma_{n-k}\big]\big) &\cong H_*\big(C_{n+1}(M,X);\mathbb{Z}\big[\Sigma_{n+1}/\Sigma_{n+1-k}\big]\big),\\ H_*\big(C_n(M,X);H_q(Z^n;F)\big) &\cong H_*\big(C_{n+1}(M,X);H_q(Z^{n+1};F)\big), \end{aligned}$$

in degrees $* \leqslant \frac{n-d}{2}$, where $d = k$ and $d = \lfloor \frac{q}{h+1} \rfloor$ respectively. On the third line F is a field and Z is a based space with $\widetilde{H}_i(Z) = 0$ for all $i \leqslant h$.

Proof. These follow from Theorem 1.3 and the examples of twisted coefficient systems in §4. ∎

Remark 1.7 There is a sequence of $\pi_1(C_n(M,X))$-modules which does not fit into the framework of this note (it doesn't even form a twisted coefficient system, let alone a finite-degree one), but which nevertheless does exhibit homological stability. Every loop in $C_n(M,X)$ induces a permutation of its base configuration, so there is a natural map $\pi_1 C_n(M,X) \to \Sigma_n$, which we can compose with the sign homomorphism to give a map $\pi_1 C_n(M,X) \to \mathbb{Z}/2$. This makes $\mathbb{Z}[\mathbb{Z}/2]$ into a $\pi_1 C_n(M,X)$-module, and its kernel corresponds to a double cover $C_n^+(M,X) \to C_n(M,X)$. The space $C_n^+(M,X)$ is the "oriented configuration space" where each configuration is additionally equipped with an ordering of its points up to even permutations. One can easily see that

$$H_*(C_n^+(M,X);\mathbb{Z}) \cong H_*(C_n(M,X);\mathbb{Z}[\mathbb{Z}/2]). \tag{1.2}$$

In [Pal13] the author proved that the sequence of spaces $C_n^+(M,X)$, with analogous stabilisation maps, is homologically stable as $n \to \infty$, in the range $* \leqslant \frac{n-5}{3}$. Via the identification (1.2) this is twisted homological stability w.r.t. the sequence of local coefficients $\mathbb{Z}[\mathbb{Z}/2]$.

A note on terminology. To keep our terminology from becoming ambiguous, we will always use "local coefficient system" and "twisted coefficient system" as follows. For a space Y, a *local coefficient system* for Y will have its usual meaning as a $\pi_1(Y)$-module (or functor from the fundamental groupoid of Y to Ab, or a bundle of abelian groups over Y). The phrase *twisted coefficient system* will always be used in the sense of Definition 2.2 below; in particular it applies to a *sequence* of spaces.

Acknowledgements. The content of this note appeared, in a slightly different form, as part of the author's PhD thesis in 2012, and he would like to thank his supervisor, Ulrike Tillmann, for her invaluable advice and guidance throughout his PhD.

§2. Twisted coefficient systems

§2.1. Setup. First we fix some data: Let M be the interior of a smooth connected manifold-with-boundary $\bar{M}$ of dimension $d \geqslant 2$ and let X be a path-connected space with basepoint x_0. Choose a point $a \in \partial\bar{M}$, and let U be a coordinate neighbourhood of a with an identification $U \cong \mathbb{R}^d_+ = \{x \in \mathbb{R}^d \mid x_1 \geqslant 0\}$ which sends a to 0. Also choose a self-embedding $e\colon \bar{M} \hookrightarrow \bar{M}$ which is isotopic to the identity, is *equal* to the identity outside U, and such that $e(a) \in M$ (in the *interior*). Moreover, we choose an isotopy $I\colon e \simeq \mathrm{id}_{\bar{M}}$. We obtain a sequence of points in M by defining

$$a_1 \coloneqq e(a) \qquad\qquad a_n \coloneqq e(a_{n-1}) \text{ for } n \geqslant 2.$$

The isotopy I provides us with canonical paths $p_n\colon [0,1] \to M$ between a_n and a_{n+1}.

§2.2. The configuration space and the stabilisation map. Recall that the configuration space of n unordered points in M with labels in X is defined to be

$$C_n(M,X) \coloneqq ((M^n \smallsetminus \Delta) \times X^n)/\Sigma_n = (\mathrm{Emb}(n,M) \times X^n)/\Sigma_n,$$

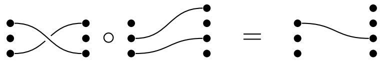
where $\Delta = \{(p_1,\ldots,p_n) \in M^n \mid p_i = p_j \text{ for some } i \neq j\}$ is the so-called *fat diagonal* of M^n, and the symmetric group Σ_n acts diagonally, permuting the points of M along with their labels in X. Thus a labelled configuration is an unordered set of ordered pairs in $M \times X$, generically denoted by $\{(p_1,x_1),\ldots,(p_n,x_n)\}$. When X is a point we will also write $C_n(M) = C_n(M,*)$.

Definition 2.1 The *stabilisation map* $s_n\colon C_n(M,X) \to C_{n+1}(M,X)$ is defined by

$$\{(p_1,x_1),\ldots,(p_n,x_n)\} \;\mapsto\; \{(e(p_1),x_1),\ldots,(e(p_n),x_n),(a_1,x_0)\}.$$

Essentially, the existing configuration is "pushed" further into the interior of the manifold by e, and the new configuration point a_1 added in the newly vacated space. Up to homotopy, the only "extra data" that this map depends on is the component of $\bar{M}$ containing a.

§2.3. Twisted coefficient systems. We define the category $\mathcal{B}(M,X)$ to have $\coprod_{n\geqslant 0} X^n$ as its set of objects, and a morphism from $(x_1,\ldots,x_m)$ to $(y_1,\ldots,y_n)$ is a choice of $k \leqslant \min\{m,n\}$ and a path in $C_k(M,X)$ from a k-element subset of $\{(a_1,x_1),\ldots,(a_m,x_m)\}$ to a k-element subset of $\{(a_1,y_1),\ldots,(a_m,y_n)\}$ up to endpoint-preserving homotopy. The identity is given by $k = m = n$ and the constant path. Composition of two morphisms is given by concatenating paths and deleting configuration points for which the concatenated path is defined only half-way. For example (omitting the labels in X):

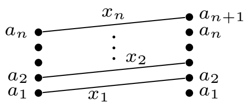

(2.1)

Again, when X is a point we will also write $\mathcal{B}(M) = \mathcal{B}(M, X)$.

Definition 2.2 A twisted coefficient system, associated to the direct system of spaces $\{C_n(M, X)\}$, is a functor from $\mathcal{B}(M, X)$ to the category Ab of abelian groups.

For each n, take $\{(a_1, x_0), \ldots, (a_n, x_0)\}$ as the basepoint of $C_n(M, X)$. Then the automorphism group of the object $(x_0)^n = (x_0, \ldots, x_0)$ (a tuple of length n) in $\mathcal{B}(M, X)$ is precisely the fundamental group $\pi_1 C_n(M, X)$. So if we are given a functor $T\colon \mathcal{B}(M, X) \to \mathsf{Ab}$ this induces an action of $\pi_1 C_n(M, X)$ on $T_n \coloneqq T((x_0)^n)$, and we can define the local homology $H_*(C_n(M, X); T_n)$.

For every object $\underline{x} = (x_1, \ldots, x_n)$ of $\mathcal{B}(M, X)$ there is a natural morphism $\iota_{\underline{x}}\colon (x_1, \ldots, x_n) \to (x_0, x_1, \ldots, x_n)$ as follows. It is represented by the path in $C_n(M, X)$ from $\{(a_1, x_1), \ldots, (a_n, x_n)\}$ to $\{(a_2, x_1), \ldots, (a_{n+1}, x_n)\}$ where each configuration point a_i travels along the path p_i (see §2.1) and the labels x_i stay constant. Schematically, this may be pictured as:



(2.2)

When $\underline{x} = (x_0)^n$ we will write $\iota_{\underline{x}} =: \iota_n$ for this canonical morphism $(x_0)^n \to (x_0)^{n+1}$. For any $\gamma \in \pi_1 C_n(M, X) = \mathrm{Aut}_{\mathcal{B}(M,X)}((x_0)^n)$ it is easy to check that

$$\iota_n \circ \gamma = (s_n)_*(\gamma) \circ \iota_n,$$

so for any T the map $T\iota_n\colon T_n \to T_{n+1}$ is equivariant with respect to the group-homomorphism $(s_n)_*\colon \pi_1 C_n(M, X) \to \pi_1 C_{n+1}(M, X)$. Hence we have an induced map

$$(s_n; T\iota_n)_*\colon H_*(C_n(M, X); T_n) \to H_*(C_{n+1}(M, X); T_{n+1}).$$

This is the map (1.1) which induces the isomorphism in Theorem 1.3.

Notation 2.3 From now on, by abuse of notation, we will denote the induced map $T\iota_n\colon T_n \to T_{n+1}$ also by $\iota_n\colon T_n \to T_{n+1}$. Similarly for the left-inverse $\pi_n\colon (x_0)^{n+1} \to (x_0)^n$ of ι_n (see §3.1): we denote its image under T also by $\pi_n\colon T_{n+1} \to T_n$.

§2.4. A special case. Let $X = *$ and assume that M is simply-connected and of dimension $d \geqslant 3$. Since X is just a point the objects of $\mathcal{B}(M, X) = \mathcal{B}(M)$ are (in canonical bijection with) the natural numbers $\mathbb{N}$ (including zero). The conditions on M imply that $\pi_1 C_n(M) \cong \Sigma_n$, in other words a path in $C_n(M)$ from the basepoint $\{a_1, \ldots, a_n\}$ to itself is determined by the permutation it induces on the set $\{a_1, \ldots, a_n\}$. More generally, a morphism from $\{a_1, \ldots, a_m\}$ to $\{a_1, \ldots, a_n\}$ in $\mathcal{B}(M)$ is determined by the partially-defined injection $\{a_1, \ldots, a_m\} \dashrightarrow \{a_1, \ldots, a_n\}$ it induces. Hence there is a canonical isomorphism of categories $\mathcal{B}(M) \cong \Sigma$, where Σ is the category defined as follows. Its object set is $\mathbb{N}$ (including zero), a morphism in $\Sigma(m, n)$ is a partially-defined injection $m \dashrightarrow n$, and composition is composition of partially-defined functions (where the composite function is defined exactly where it *can* be).[1]

In particular this is true for $M = \mathbb{R}^\infty$. Of course, $\mathbb{R}^\infty$ is not a finite-dimensional manifold, as was assumed of M, but the definitions make sense for arbitrary spaces M and X, and $C_n(\mathbb{R}^\infty)$ is the colimit of the spaces $C_n(\mathbb{R}^d)$ under the obvious inclusions. The space $\mathrm{Emb}(n, \mathbb{R}^\infty)$ is a contractible space on which the natural action of Σ_n is free, so its quotient $C_n(\mathbb{R}^\infty)$ is a model for the classifying space $B\Sigma_n$.

Now, every smooth manifold M admits a unique-up-to-isotopy embedding $M \hookrightarrow \mathbb{R}^\infty$. From the construction of $\mathcal{B}(M, X)$ one can see that this embedding, together with the map $X \to *$,

[1] It is a subcategory of the category with objects $\mathbb{N}$ and morphisms partially-defined *functions*, which is precisely Γ^{op}, (a skeleton of) the category of finite pointed sets.

induces a canonical functor $\mathcal{B}(M,X) \to \mathcal{B}(\mathbb{R}^\infty) \cong \Sigma$. Another description of this functor is that it forgets both the labels of the paths and the paths themselves, remembering only the partially-defined injection induced by the paths.

In particular this means that any twisted coefficient system $\Sigma \to \mathsf{Ab}$ canonically induces a twisted coefficient system $\mathcal{B}(M,X) \to \Sigma \to \mathsf{Ab}$.

§2.5. A more general case. Rather than configuration spaces of points, one can consider more generally configuration spaces of embedded submanifolds, for example as follows. Fix an open connected manifold M as before, and also a closed manifold P. Choose an embedding $e\colon M \hookrightarrow M$ which is isotopic to the identity and not surjective, so that we can choose an embedding $\iota_1\colon P \hookrightarrow M \smallsetminus e(M)$. We then get a sequence of embeddings of P by defining $\iota_n := e^{n-1} \circ \iota_1\colon P \hookrightarrow M$. Writing the disjoint union $P \sqcup \cdots \sqcup P$ of n copies of P as nP for short, define $C_{nP}(M)$ to be the path-component of $\mathrm{Emb}(nP,M)/\mathrm{Diff}(nP)$ containing $[\iota_1 \sqcup \cdots \sqcup \iota_n]$. The stabilisation map $C_{nP}(M) \to C_{(n+1)P}(M)$ can then be defined by sending $[\phi_1 \sqcup \cdots \sqcup \phi_n]$ to $[(e \circ \phi_1) \sqcup \cdots \sqcup (e \circ \phi_n) \sqcup \iota_1]$.

Everything in this note generalises to this setting, including an analogous notion of *twisted coefficient system* for $\{C_{nP}(M,X)\}$ (one can also define labelled configuration spaces of submanifolds). In a forthcoming paper [Pal] we will prove (untwisted) homological stability for these more general kinds of configuration spaces, and the arguments in this note therefore immediately give a twisted homological stability result for these spaces too.

§3. Height and degree of a twisted coefficient system

§3.1. Degree. First we will define the *degree* of a functor $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$. Recall from §2.3 the natural morphisms $\iota_{\underline{x}}\colon \underline{x} \to (x_0, \underline{x})$. The adjective "natural" suggests that they should form a natural transformation, and in fact they do: For every morphism ϕ of $\mathcal{B}(M,X)$ we have a commutative square

$$\begin{tikzcd}
(x_1,\ldots,x_m) \arrow[r, "\iota_{\bar{x}}"] \arrow[d, "\phi"'] & (x_0,x_1,\ldots,x_m) \arrow[d, "S\phi"] \\
(y_1,\ldots,y_n) \arrow[r, "\iota_{\bar{y}}"'] & (x_0,y_1,\ldots,y_n)
\end{tikzcd} \tag{3.1}$$

where the morphism $S\phi$ is defined as follows: if ϕ is represented by a path p in $C_k(M,X)$ for some $k \leqslant \min\{m,n\}$, then $S\phi$ is represented by the path $s_k \circ p$ in $C_{k+1}(M,X)$. Thus we have an endofunctor $S\colon \mathcal{B}(M,X) \to \mathcal{B}(M,X)$ (which could be called the *stabilisation endofunctor*) and a natural transformation $\iota\colon \mathrm{id} \Rightarrow S$. Note that each $\iota_{\underline{x}}$ has an obvious left-inverse $\pi_{\underline{x}}$, and these morphisms fit together to form a left-inverse $\pi\colon S \Rightarrow \mathrm{id}$ for ι.

So, given any $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ we get a natural transformation $T \circ \iota\colon T \Rightarrow T \circ S$, or in other words a morphism in the abelian category $\mathsf{Ab}^{\mathcal{B}(M,X)}$. Denote its cokernel by $\Delta T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$.

Definition 3.1 The *degree* of a functor $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ is defined recursively by

$$\deg(0) = -1 \qquad\qquad \deg(T) = \deg(\Delta T) + 1,$$

where 0 is the identically-zero functor.

Example 3.2 The degree of T is $\leqslant 0$ exactly when $\Delta T = 0$, i.e. when $T\iota_{\underline{x}}\colon T\underline{x} \to T(x_0, \underline{x})$ is an isomorphism for all $\underline{x} \in \mathcal{B}(M,X)$. But the category $\mathcal{B}(M,X)$ is generated by the morphisms $\iota_{\underline{x}}$, their left-inverses $\pi_{\underline{x}}$ and isomorphisms. Hence $\deg(T) \leqslant 0$ if and only if T is constant in the sense that it sends every morphism to an isomorphism.

See §4 for some less trivial examples.

§3.2. Height. Denote the homomorphism $\pi_1 C_n(M,X) \to \Sigma_n$ which only remembers the permutation of the basepoint configuration by u (this is part of the canonical functor $\mathcal{B}(M,X) \to \Sigma$ from §2.4). Write $G_n := \pi_1 C_n(M,X)$ and define $G_n^k := u^{-1}(\Sigma_{n-k} \times \Sigma_k)$. To define the *height* of a functor $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ we need the following decomposition result:

Proposition 3.3 *Let $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ be any functor, and recall that we write $T_n \coloneqq T((x_0)^n)$. Then for $k = 0, \ldots, n$ there is a direct summand (as abelian groups) T_n^k of T_n such that the action of $G_n^k \leqslant G_n$ on T_n preserves it: so it is also a direct summand as a G_n^k-module. Moreover, there is a decomposition of T_n as a G_n-module:*

$$T_n \cong \bigoplus_{k=0}^{n} \left(\mathbb{Z}G_n \otimes_{\mathbb{Z}G_n^k} T_n^k\right). \tag{3.2}$$

This identification is natural in the sense that $\iota_n\colon T_n \to T_{n+1}$ sends T_n^k into T_{n+1}^k, and the map of the right-hand side induced by ι_n and $(s_n)_*$ corresponds under (3.2) to ι_n on the left-hand side.

This is very similar to the cross-effect decomposition of a functor from a pointed monoidal category[2] to an abelian category, which was proved in this generality in [HPV12, Proposition 2.4], and goes back to Eilenberg and MacLane [EML54, §9]. However, our category $\mathcal{B}(M,X)$ is *not* in general monoidal (although it is when M is of the form $\mathbb{R} \times N$), so we will give a complete proof of this decomposition in our case here.[3] This is a little technical, so the reader may wish to skip directly to Definition 3.10 at this point.

Definition 3.4 For $S \subseteq \{1, \ldots, n\} \eqqcolon \underline{n}$ let $f_S\colon (x_0)^n \to (x_0)^n$ be the endomorphism in $\mathcal{B}(M,X)$ given by the constant path in $C_{|S|}(M,X)$ on the configuration $\{(a_i, x_0) \mid i \in \underline{n} \smallsetminus S\}$. So this is the endomorphism which "forgets" the points a_i for $i \in S$ and is the identity elsewhere.

Definition 3.5 For $p \geqslant 0$ and $\{S_1, \ldots, S_p\}$ a partition of $S \subseteq \underline{n}$ define

$$T_n[S_1|\cdots|S_p] \;\coloneqq\; \mathrm{im}(Tf_{\underline{n}\smallsetminus S}) \cap \bigcap_{i=1}^{p} \ker(Tf_{S_i}).$$

Note that the induced maps $Tf_S\colon T_n \to T_n$ are not in general G_n-module homomorphisms, so these are subgroups but not sub-G_n-modules.

We will write S^δ for the *discrete* partition of S, and define

$$T_n^k \;\coloneqq\; T_n[\{n{-}k{+}1, \ldots, n\}^\delta].$$

Remark 3.6 A few immediate observations are the following: Each $Tf_S\colon T_n \to T_n$ is idempotent. The composition of Tf_{S_1} and Tf_{S_2} is $Tf_{S_1 \cup S_2}$, so in particular the Tf_S for $S \subseteq \underline{n}$ all pairwise commute. By definition $T_n[\,] = \mathrm{im}(Tf_{\underline{n}})$, and since $f_\varnothing = \mathrm{id}$ we also have $T_n[\underline{n}] = \mathrm{im}(Tf_\varnothing) \cap \ker(Tf_{\underline{n}}) = \ker(Tf_{\underline{n}})$, so:

$$T_n = \mathrm{im}(Tf_{\underline{n}}) \oplus \ker(Tf_{\underline{n}}) = T_n[\,] \oplus T_n[\underline{n}]. \tag{3.3}$$

The following lemma is less immediate but can be proved by some diagram-chasing and drawing little cartoons like (2.1) and (2.2). We will give a proof in symbols.

Lemma 3.7 *For $k \leqslant m \leqslant n$, the map*

$$\iota_m^n \coloneqq \iota_{n-1} \circ \cdots \circ \iota_m \colon T_m \to T_n$$

is split-injective and sends T_m^k into T_n^k. Moreover, its restriction to a map $T_m^k \to T_n^k$ is a bijection. Hence any left-inverse for ι_m^n restricts to a bijection $T_n^k \to T_m^k$.

Proof. As mentioned in §3.1, each $\iota_{\underline{x}}$ has a natural left-inverse $\pi_{\underline{x}}$ – these compose to give a left-inverse π_m^n for ι_m^n. Just as for ι_n and π_n, by an abuse of notation we will denote the induced map $Tf_S\colon T_n \to T_n$ also by f_S.

We now show that $\iota_m(T_m^k) \subseteq T_{m+1}^k$, and hence by induction that $\iota_m^n(T_m^k) \subseteq T_n^k$. Suppose $x = \iota_m(y)$ for $y \in T_m^k$. Then by definition $y = f_{\{1,\ldots,m-k\}}(z)$ for some $z \in T_m$. Since $\pi_m\colon T_{m+1} \to T_m$ is split-surjective we have $z = \pi_m(w)$ for some $w \in T_{m+1}$. Hence

$$x = \iota_m \circ f_{\{1,\ldots,m-k\}} \circ \pi_m(w) = f_{\{1,\ldots,m-k+1\}}(w). \tag{3.4}$$

[2] A monoidal category whose unit object is also initial and terminal.
[3] The proof we give here was informed in part by reading [CDG11].

For any $m-k+2 \leqslant i \leqslant m+1$ we have

$$f_{\{i\}}(x) = f_{\{i\}} \circ \iota_m(y) = \iota_m \circ f_{\{i-1\}}(y) = \iota_m(0) = 0, \tag{3.5}$$

since $y \in T_m^k$. The two properties (3.4) and (3.5) verify that $x \in T_{m+1}^k$.

Now we show that the restriction of ι_m to $T_m^k \to T_{m+1}^k$ is a bijection, and hence by induction that the restriction of ι_m^n to $T_m^k \to T_n^k$ is a bijection. Suppose $x \in T_{m+1}^k$, and define $z \coloneqq \pi_m(x) \in T_m$. Then

$$\iota_m(z) = \iota_m \circ \pi_m(x) = f_{\{1\}}(x).$$

But note that $x = f_{\{1,\dots,m-k+1\}}(y)$ for some $y \in T_{m+1}$, so

$$\begin{aligned} f_{\{1\}}(x) &= f_{\{1\}} \circ f_{\{1,\dots,m-k+1\}}(y) \\ &= f_{\{1,\dots,m-k+1\}}(y) \qquad \text{(by Remark 3.6 and since } k \leqslant m\text{)} \\ &= x. \end{aligned}$$

So it remains to prove that $z \in T_m^k$. Firstly,

$$z = \pi_m \circ f_{\{1,\dots,m-k+1\}}(y) = f_{\{1,\dots,m-k\}} \circ \pi_m(y).$$

Secondly, for any $m-k+1 \leqslant i \leqslant m$, we have

$$\iota_m \circ f_{\{i\}}(z) = f_{\{i+1\}} \circ \iota_m(z) = f_{\{i+1\}}(x) = 0,$$

since $x \in T_{m+1}^k$. But ι_m is split-injective, so $f_{\{i\}}(z) = 0$. These two facts verify that $z \in T_m^k$. ☐

The following lemma will allow us to construct the required decomposition by induction:

Lemma 3.8 *For all $\{S_1, \dots, S_p\}$ partitioning $S \subseteq \underline{n}$ with $p \geqslant 2$, there is a split short exact sequence*

$$0 \to T_n[S_1|\cdots|S_p] \hookrightarrow T_n[S_1 \sqcup S_2|\cdots|S_p] \twoheadrightarrow T_n[S_1|S_3|\cdots|S_p] \oplus T_n[S_2|\cdots|S_p] \to 0.$$

The first map is the inclusion, and a section of the second map is given by the inclusion of each of the two factors. So in other words we have a decomposition

$$T_n[S_1 \sqcup S_2|\cdots|S_p] \;=\; T_n[S_1|\cdots|S_p] \;\oplus\; T_n[S_2|\cdots|S_p] \;\oplus\; T_n[S_1|S_3|\cdots|S_p].$$

Proof. One can check from the definitions that the following facts are true:

1. Tf_{S_2} restricts to a map $T_n[S_1 \sqcup S_2|\cdots|S_p] \to T_n[S_1|S_3|\cdots|S_p]$, and similarly Tf_{S_1} restricts to a map $T_n[S_1 \sqcup S_2|\cdots|S_p] \to T_n[S_2|\cdots|S_p]$.
2. $T_n[S_1|S_3|\cdots|S_p]$ and $T_n[S_2|\cdots|S_p]$ are contained in $T_n[S_1 \sqcup S_2|\cdots|S_p]$.
3. For $\{i, j\} \subseteq \{1, 2\}$ if $x \in T_n[S_i|S_3|\cdots|S_p]$, then $Tf_{S_j}(x)$ is x when $i \neq j$ and 0 when $i = j$.

These facts imply that the map (Tf_{S_2}, Tf_{S_1}) restricts to the required split surjection (with a section given by inclusion of each factor). The kernel of this is

$$\begin{aligned} &T_n[S_1 \sqcup S_2|S_3|\cdots|S_p] \cap \ker(Tf_{S_1}) \cap \ker(Tf_{S_2}) \\ &= \operatorname{im}(Tf_{\underline{n} \smallsetminus S}) \cap \bigcap_{i=3}^{p} \ker(Tf_{S_i}) \cap \ker(Tf_{S_1 \sqcup S_2}) \cap \ker(Tf_{S_1}) \cap \ker(Tf_{S_2}) \\ &= T_n[S_1|\cdots|S_p], \end{aligned}$$

since $\ker(Tf_{S_1}) \subseteq \ker(Tf_{S_1 \sqcup S_2})$. ☐

We can now use this to inductively prove a more general decomposition:

Lemma 3.9 *For any* $\varnothing \neq S \subseteq \underline{n}$ *and* $R \subseteq \underline{n} \smallsetminus S$ *there is a decomposition*

$$T_n[S|R^\delta] = \bigoplus_{\varnothing \neq Q \subseteq S} T_n[(Q \sqcup R)^\delta]. \tag{3.6}$$

As before, Q^δ *denotes the discrete partition of the set* Q*, so for example* $T_n[\{1,2\}|\{3,4,5\}^\delta]$ *means* $T_n[\{1,2\}|\{3\}|\{4\}|\{5\}]$*. Note that this decomposition is an equality of subgroups, not just an abstract isomorphism of groups.*

Proof. The $|S| = 1$ case is obvious, so we assume that $|S| \geqslant 2$ and assume the theorem for smaller values of $|S|$ by induction. Pick an element $s \in S$. Then by Lemma 3.8,

$$T_n[S|R^\delta] \;=\; T_n[S \smallsetminus \{s\}|(R \sqcup \{s\})^\delta] \;\oplus\; T_n[S \smallsetminus \{s\}|R^\delta] \;\oplus\; T_n[\{s\}|R^\delta].$$

Apply the inductive hypothesis to the right-hand side. The proposition then follows from the observation that for $\varnothing \neq Q \subseteq S$, exactly one of the following holds: (i) $s \in Q$ but $Q \neq \{s\}$; (ii) $s \notin Q$; (iii) $Q = \{s\}$. □

We can now use this to deduce the decomposition we want:

Proof of Proposition 3.3. Combining (3.6) (setting $R \coloneqq \varnothing$ and $S \coloneqq \underline{n}$) with (3.3) we obtain:

$$T_n = \bigoplus_{k=0}^{n} \bigoplus_{\substack{Q \subseteq \underline{n} \\ |Q|=k}} T_n[Q^\delta]. \tag{3.7}$$

The action of G_n on T_n permutes the summands via the projection $G_n \to \Sigma_n$ and the obvious action of Σ_n on subsets of $\underline{n}$. So:

- $T_n^k = T_n[\{n-k+1, \ldots, n\}^\delta]$ is preserved by the action of $G_n^k \leqslant G_n$ on T_n.
- The G_n-action on T_n preserves the outer direct sum.
- The inner direct sum is the induced module $\mathrm{Ind}_{G_n^k}^{G_n} T_n^k = \mathbb{Z}G_n \otimes_{\mathbb{Z}G_n^k} T_n^k$.

This proves the decomposition of G_n-modules (3.2). We proved in Lemma 3.7 above that $\iota_n \colon T_n \to T_{n+1}$ sends T_n^k into T_{n+1}^k, and the naturality statement is clear. □

Having established this decomposition we can now define the *height* of a twisted coefficient system:

Definition 3.10 The *height* of a functor $T \colon \mathcal{B}(M,X) \to \mathsf{Ab}$ is the height at which the decomposition (3.2) is truncated. More precisely, we define $\mathrm{height}(T)$ by: $\mathrm{height}(T) \leqslant h$ if and only if $T_n^k = 0$ for all $k > h$ and all n. (So in particular $\mathrm{height}(T) = -1$ if and only if $T = 0$.)

§3.3. Height and degree. These two notions are related as follows:

Lemma 3.11 *For any functor* $T \colon \mathcal{B}(M,X) \to \mathsf{Ab}$*,* $\mathrm{height}(T) \leqslant \deg(T)$*.*

This inequality is useful because having an upper bound on the *height* of a twisted coefficient system is what is needed to prove Theorem 1.3, whereas it is often easier to find an upper bound on the *degree* in examples.

Proof. We will use induction on d to prove the statement

$$\deg(T) \leqslant d \;\Rightarrow\; \mathrm{height}(T) \leqslant d \tag{IH$_d$}$$

for all $d \geqslant -1$, using the decomposition (3.7) above, which we restate as:

$$T_n \;=\; \bigoplus_{S \subseteq \underline{n}} T_n[S^\delta]. \tag{3.8}$$

In this notation the height of T is determined by $\mathrm{height}(T) \leqslant d$ if and only if $T_n[S^\delta] = 0$ for all $|S| > d$ and all n.

When $d = -1$ the definitions of height and degree coincide. This deals with the base case, so let $d \geqslant 0$ and assume that (IH_{d-1}) holds. For all n we have a split short exact sequence $0 \to T_n \to T_{n+1} \to \Delta T_n \to 0$. Applying (3.8), this is

$$0 \to \bigoplus_{S \subseteq \underline{n}} T_n[S^\delta] \longrightarrow \bigoplus_{R \subseteq \underline{n+1}} T_{n+1}[R^\delta] \longrightarrow \bigoplus_{Q \subseteq \underline{n}} \Delta T_n[Q^\delta] \to 0.$$

Analysing the maps carefully we see that

(a) $T_n[S^\delta]$ is sent isomorphically onto $T_{n+1}[(S+1)^\delta]$ by the first map.
(b) $T_{n+1}[(Q \sqcup \{1\})^\delta]$ is sent isomorphically onto $\Delta T_n[(Q-1)^\delta]$ by the second map.

Suppose that $\deg(T) \leqslant d$. Then $\deg(\Delta T) \leqslant d-1$ by the definition of degree, and so by the inductive hypothesis (IH_{d-1}), $\mathrm{height}(\Delta T) \leqslant d-1$. By fact (b) above this implies that

$$T_{n+1}[R^\delta] = 0 \text{ whenever } |R| > d \text{ and } 1 \in R. \tag{3.9}$$

For any fixed k, the subgroups $\{T_{n+1}[R^\delta] \mid |R| = k\}$ are all abstractly isomorphic via the action of G_{n+1} on T_{n+1}. Also note that $d \geqslant 0$, so that $|R| > 0$, i.e. $R \neq \varnothing$. Hence:

$$T_{n+1}[R^\delta] = 0 \text{ for } \textit{all } |R| > d. \tag{3.10}$$

Therefore by (a), $T_n[S^\delta] = 0$ for all $|S| > d$; in other words, $\mathrm{height}(T) \leqslant d$. □

Remark 3.12 To prove that $\mathrm{height}(T) = \deg(T)$, one could try to reverse the argument above to get the other inequality. This goes wrong in one place though: Above we were able to deduce (3.10) from (3.9) because for every $|R| > d$, there is an R' of the same cardinality which contains 1. However, for the converse we would need to deduce (3.10) from:

$$T_{n+1}[R^\delta] = 0 \text{ whenever } |R| > d \text{ and } 1 \notin R. \tag{3.11}$$

Now there *is* a subset $R \subseteq \underline{n+1}$ for which there does not exist $R' \subseteq \underline{n+1}$ of the same cardinality and not containing 1 – namely $\underline{n+1}$ itself. This is the basic asymmetry which prevented us from proving an *equality* between height and degree.

Remark 3.13 The notion of *height* in this chapter is the same as the notion of degree in [Bet02] (for twisted coefficient systems for symmetric groups) and [Dwy80] (for general linear groups), whereas the notion of *degree* in this chapter is in the same spirit as the notion of degree in [Iva93], [CM09] and [Bol12] (for mapping class groups of surfaces). Hence Lemma 3.11 provides a link between these two notions of degree.

We finish this section with a few immediate facts about the degree of a twisted coefficient system.

Lemma 3.14 *For twisted coefficient systems $T, T' \colon \mathcal{B}(M,X) \to \mathsf{Ab}$ and a fixed abelian group A,*

(a) $\deg(T \oplus T') = \max\{\deg(T), \deg(T')\}$,
(b) $\deg(T \otimes A) \leqslant \deg(T)$,
and more generally, for $\deg(T)$ *and* $\deg(T')$ *non-negative,*
(c) $\deg(T \otimes T') \leqslant \deg(T) + \deg(T')$,

where $\oplus$ and $\otimes$ are defined objectwise.

Proof. Fact (a) follows by induction from the fact that $\Delta(T \oplus T') = \Delta T \oplus \Delta T'$. Fact (b) follows from the fact that $\Delta(T \otimes A) = \Delta T \otimes A$, which is true because tensoring a *split* short exact sequence with A preserves split-exactness. Fact (c) is proved by induction with base case (b), and inductive step using the fact that

$$\Delta(T \otimes T') \;=\; (T \otimes \Delta T') \oplus (\Delta T \otimes T') \oplus (\Delta T \otimes \Delta T'). \qquad \square$$

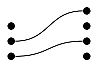
§4. Examples of twisted coefficient systems

Recall that the category Σ has objects the natural numbers including zero, and morphisms the partially-defined injections. We will give some examples of functors $T\colon \Sigma \to \mathsf{Ab}$, which are twisted coefficient systems for the special case $M = \mathbb{R}^\infty$ and $X = *$ (since $\mathcal{B}(\mathbb{R}^\infty) \cong \Sigma$). However, recall (§2.4) that there is a canonical functor $U\colon \mathcal{B}(M,X) \to \Sigma$ for each (M,X), so these examples also give twisted coefficient systems in general. Moreover, one can check (see §3 for notation) that $\Delta(T \circ U) = \Delta T \circ U$, so by induction $\deg(T \circ U) = \deg(T)$, and also $(T \circ U)^k_n = T^k_n$, so $\mathrm{height}(T \circ U) = \mathrm{height}(T)$.

Example 4.1 Fix a path-connected based space $(Z,*)$, an integer $q \geqslant 0$ and a field F. The functor $\hat{T}_Z\colon \Sigma \to \mathsf{Top}$ is defined on objects by $n \mapsto Z^n$, and on morphisms as follows: given a partially-defined injection $j\colon \{1,\ldots,m\} \dashrightarrow \{1,\ldots,n\}$ in Σ, define $\hat{T}_Z(j)\colon Z^m \to Z^n$ to be the map

$$(z_1,\ldots,z_m) \mapsto (z_{j^{-1}(1)},\ldots,z_{j^{-1}(n)}),$$

where $z_\varnothing$ is taken to mean the basepoint $*$, for example

 $\quad : \quad (z_1,z_2,z_3) \mapsto (*,z_1,*,z_2).$

The functor $T_{Z,q,F}\colon \Sigma \to \mathsf{Ab}$ is then the composite functor $H_q(-;F) \circ \hat{T}_Z$.

Lemma 4.2 *The twisted coefficient system $T_{Z,q,F}$ has degree at most $\lfloor \frac{q}{h+1} \rfloor$, where for a path-connected space Z,*

$$h = hconn_F(Z) \coloneqq \max\{k \geqslant 0 \mid \widetilde{H}_i(Z;F) = 0 \textit{ for all } i \leqslant k\} \geqslant 0.$$

Proof. First note that the Künneth theorem gives us natural split short exact sequences

$$0 \to H_q(Z^n;F) \longrightarrow H_q(Z^{n+1};F) \longrightarrow \bigoplus_{i=1}^{q} H_{q-i}(Z^n;F) \otimes H_i(Z;F) \to 0,$$

which together with the fact that $H_i(Z;F) = 0$ for $1 \leqslant i \leqslant h$ implies that

$$\Delta T_{Z,q,F} = \bigoplus_{i=h+1}^{q} T_{Z,q-i,F} \otimes H_i(Z;F).$$

So by Lemma 3.14 above, $\deg(T_{Z,q,F}) \leqslant 1 + \max\{\deg(T_{Z,q-i,F}) \mid h+1 \leqslant i \leqslant q\}$. Abbreviating $\deg(T_{Z,q,F})$ to t_q, we have the recurrence inequality

$$t_q \;\leqslant\; 1 + \max\{t_0,\ldots,t_{q-h-1}\}. \tag{4.1}$$

Note that $H_0(Z^n;F) \to H_0(Z^{n+1};F)$ is the identity map $F \to F$ for all n, so $\Delta T_{Z,0,F} = 0$, and hence $\deg(T_{Z,0,F}) = 0$. Also note that for $1 \leqslant q \leqslant h$, $hconn_F(Z) \geqslant q$ implies that $hconn_F(Z^n) \geqslant q$ for all n (by the Künneth theorem), so $T_{Z,q,F}(n) = H_q(Z^n;F) = 0$, and hence $\deg(T_{Z,q,F}) = -1 \leqslant 0$. So we also have the initial conditions

$$t_0, t_1, \ldots, t_h \leqslant 0. \tag{4.2}$$

It now remains to prove that the recurrence inequality (4.1) and the initial conditions (4.2) imply that $t_q \leqslant \lfloor \frac{q}{h+1} \rfloor$ for all $q \geqslant 0$. This will be done by induction on q. The base case is $0 \leqslant q \leqslant h$ which is covered by the initial conditions (4.2). Assume that $q \geqslant h+1$. Then:

$$\begin{aligned} t_q &\leqslant 1 + \max\{t_0,\ldots,t_{q-h-1}\} \\ &\leqslant 1 + \lfloor \tfrac{q-h-1}{h+1} \rfloor \\ &= \lfloor \tfrac{q}{h+1} \rfloor \end{aligned}$$ □

Remark 4.3 See also [Han09a, Proposition 12], where it is proved (in the terminology of this note) that the *height* of $T_{Z,q,F}$ is at most q.

Example 4.4 Let $f\mathrm{Set}_*$ be the category of finite sets and partially-defined functions, equivalently the category of finite pointed sets. There is a free functor $\mathbb{Z}-\colon f\mathrm{Set}_* \to \mathsf{Ab}$ taking S to $\mathbb{Z}S$ and taking $j\colon S \dashrightarrow R$ to the homomorphism $\sum_{s\in S} n_s s \mapsto \sum_{s\in S} n_s j(s)$, where $j(s)$ means $0 \in \mathbb{Z}R$ if j is undefined on s. So any functor $\Sigma \to f\mathrm{Set}_*$ gives a twisted coefficient system for Σ by composing with $\mathbb{Z}-$.

For example one can just take $\Sigma \hookrightarrow f\mathrm{Set}_*$ to be the inclusion as a subcategory. More generally, for $0 \leqslant a \leqslant b$ one can take the functor $P_{a,b}\colon \Sigma \to f\mathrm{Set}_*$ which on objects is

$$\underline{n} \;\mapsto\; P_{a,b}(\underline{n}) = \{S \subseteq \underline{n} \mid a \leqslant |S| \leqslant b\}$$

and which takes $j\colon \{1,\ldots,m\} \dashrightarrow \{1,\ldots,n\}$ to

$$S \;\mapsto\; \begin{cases} j(S) & \text{if } a \leqslant |j(S)| \leqslant b \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Denote the composite functor $\Sigma \to \mathsf{Ab}$ by $\mathbb{Z}P_{a,b}$. Note that $\mathbb{Z}P_{b,b}(n) \cong \mathbb{Z}\big[\Sigma_n/(\Sigma_b \times \Sigma_{n-b})\big]$ as Σ_n-modules.

From the definitions one can check that $\Delta\mathbb{Z}P_{0,0} = 0$, $\Delta\mathbb{Z}P_{0,b} = \mathbb{Z}P_{0,b-1}$ for $b \geqslant 1$, and $\Delta\mathbb{Z}P_{a,b} = \mathbb{Z}P_{a-1,b-1}$ for $a \geqslant 1$. Hence by induction,

$$\deg(\mathbb{Z}P_{a,b}) = b.$$

(With a bit more work, one can check directly that the height of $\mathbb{Z}P_{a,b}$ is also exactly b.)

There is also an ordered version of this. The functor $\widetilde{P}_{a,b}\colon \Sigma \to f\mathrm{Set}_*$ takes $\underline{n}$ to the *ordered* subsets of $\underline{n}$ with cardinality between a and b, and it is defined on morphisms as above, where $j(S)$ inherits its ordering from S. Again, denote the composite functor $\Sigma \to \mathsf{Ab}$ by $\mathbb{Z}\widetilde{P}_{a,b}$. Note that $\mathbb{Z}\widetilde{P}_{b,b}(n) \cong \mathbb{Z}\big[\Sigma_n/\Sigma_{n-b}\big]$ as Σ_n-modules.

To find the degree of $\mathbb{Z}\widetilde{P}_{a,b}$ we need to consider something slightly more general. For $0 \leqslant a \leqslant b$ and a finite set R disjoint from $\{1,2,3,\ldots\}$, let $\widetilde{P}^R_{a,b}$ be the functor $\Sigma \to f\mathrm{Set}_*$ which takes $\underline{n}$ to the set of subsets $S \subseteq \underline{n}$ of cardinality between a and b, each equipped with an ordering of $S \sqcup R$. Then one can check from the definitions that $\Delta\mathbb{Z}\widetilde{P}^R_{0,0} = 0$, $\Delta\mathbb{Z}\widetilde{P}^R_{0,b} = \mathbb{Z}\widetilde{P}^{R_+}_{0,b-1}$ for $b \geqslant 1$, and $\Delta\mathbb{Z}\widetilde{P}^R_{a,b} = \mathbb{Z}\widetilde{P}^{R_+}_{a-1,b-1}$ for $a \geqslant 1$, where $R_+ = R \sqcup \{*\}$. Hence by induction on b,

$$\deg(\mathbb{Z}\widetilde{P}^R_{a,b}) = b.$$

§5. A twisted Serre spectral sequence

To prove Theorem 1.3 we will need a generalisation of the basic Serre spectral sequence, allowing the base space to be equipped with a local coefficient system. It is a special case of (the homology version of) an *equivariant* generalisation of the Serre spectral sequence constructed by Moerdijk and Svensson in [MS93]. This section gives a brief description of their spectral sequence and deduces the particular case that we will need.

We start by describing an alternative basepoint-independent viewpoint on (co)homology with local coefficients (in the non-equivariant setting).

Definition 5.1 For a space Y let $\Delta(Y)$ be the category whose objects are all singular simplices in Y, and whose morphisms are simplicial operations (generated by face and degeneracy maps). Denote the fundamental groupoid of Y by $\pi(Y)$, and the standard n-simplex by Δ^n. There is a canonical functor $v_Y\colon \Delta(Y) \to \pi(Y)$ which takes a singular simplex $\Delta^n \to Y$ to the image of its barycentre b_n. A morphism $\Delta^k \xrightarrow{\alpha} \Delta^n \to Y$ is taken to the image of the straight-line path in Δ^n from $\alpha(b_k)$ to b_n.

A covariant (resp. contravariant) functor $\Delta(Y) \to \mathsf{Ab}$ is a *coefficient system* for homology (resp. cohomology); it is a *local coefficient system* if it factors up to natural isomorphism through v_Y.

The functor $v_Y\colon \Delta(Y) \to \pi(Y)$ encapsulates most of the combinatorics needed to define (co)homology with local coefficients. The definition makes sense for any (not necessarily local) coefficient system, but it is only homotopy-invariant for local coefficient systems.

Definition 5.2 (Homology) Given a space Y and coefficient system $M\colon \Delta(Y) \to \mathsf{Ab}$, the homology $H_*(Y; M)$ is the homology of the chain complex $C_*(\Delta(Y); M)$:

$$\xrightarrow{\partial_{n+1}} \bigoplus_{\sigma \in N_n\Delta(Y)} M(\sigma_0) \xrightarrow{\partial_n} \bigoplus_{\tau \in N_{n-1}\Delta(Y)} M(\tau_0) \xrightarrow{\partial_{n-1}}$$

where $N_\bullet\Delta(Y)$ denotes the nerve of the category $\Delta(Y)$, and for a chain of singular simplices $\sigma = (\Delta^{k_0} \to \Delta^{k_1} \to \cdots \to \Delta^{k_n} \to Y)$ of $N_n\Delta(Y)$, the 0th one $\Delta^{k_0} \to Y$ is denoted by σ_0. The map ∂_n is the alternating sum of maps ∂_n^i which are defined using the ith face map of $N_\bullet\Delta(Y)$.[4]

Definition 5.3 (Cohomology) Given a space Y and coefficient system $M\colon \Delta(Y)^{\mathrm{op}} \to \mathsf{Ab}$, the cohomology $H^*(Y; M)$ is the homology of the cochain complex $C^*(\Delta(Y); M)$:

$$\xrightarrow{\delta_{n-1}} \prod_{\sigma \in N_n\Delta(Y)} M(\sigma_0) \xrightarrow{\delta_n} \prod_{\tau \in N_{n+1}\Delta(Y)} M(\tau_0) \xrightarrow{\delta_{n+1}}$$

where the map δ_n is the alternating sum of maps δ_n^i which are defined using the ith face map of $N_\bullet\Delta(Y)$.[5]

This reduces to ordinary (untwisted) homology and cohomology when M is constant. (Although it does not reduce to the usual singular (co)chain complex, one can show that it does compute the same homology as it; cf. [MS93, Theorem 2.2].)

In [MS93] the above is generalised to the equivariant setting: they define $v_Y\colon \Delta_G(Y) \to \pi_G(Y)$ for a G-space Y, and equivariant twisted cohomology $H^*_G(Y; M)$ for any coefficient system $\Delta_G(Y)^{\mathrm{op}} \to \mathsf{Ab}$. Again a coefficient system is *local* if it factors up to natural isomorphism through v_Y. Cohomology with respect to local coefficient systems is G-homotopy invariant [MS93, Theorem 2.3]. Their main theorem is the existence of a twisted equivariant Serre spectral sequence:

Theorem 5.4 ([MS93, Theorem 3.2]) *For any G-fibration $f\colon Y \to X$ (i.e. $Y^H \to X^H$ is a fibration for all $H \leqslant G$) and any local coefficient system M on Y, there is a local coefficient system $H^q_G(f; M)$ on X for each $q \geqslant 0$ and a spectral sequence*

$$E_2^{p,q} = H^p_G\big(X; H^q_G(f; M)\big) \;\Rightarrow\; H^*_G(Y; M) \tag{5.1}$$

with the usual cohomological grading.

Remark 5.5 We will describe the local coefficient system $H^q(f; M)$ in the non-equivariant case. As a functor $\Delta(X)^{\mathrm{op}} \to \mathsf{Ab}$ it does the following. A singular simplex $\Delta^k \xrightarrow{\sigma} X$ is taken to the cohomology $H^q(\sigma^*(Y); M)$, where $\sigma^*(Y)$ is the pullback of σ and f, and we denote any pullback of the coefficients M also by M. A morphism $\Delta^l \xrightarrow{\alpha} \Delta^k \xrightarrow{\sigma} X$ induces a map of pullbacks $(\sigma \circ \alpha)^*(Y) \to \sigma^*(Y)$ and hence a map on cohomology.

It is a *local* coefficient system since it factors up to natural isomorphism through v_X by the following functor $\pi(X)^{\mathrm{op}} \to \mathsf{Ab}$. A point $x \in X$ is taken to $H^q(f^{-1}(x); M)$. Given a homotopy class $[I \xrightarrow{p} X]$ of paths from x to y, there are induced maps of pullbacks $f^{-1}(x) \hookrightarrow p^*(Y) \hookleftarrow f^{-1}(y)$. These induce maps on cohomology, and since they are *isomorphisms*[6] the first one can be inverted to get a composite map $H^q(f^{-1}(x); M) \to H^q(f^{-1}(y); M)$. One can check that this map is independent of the choice of representing path p.

In [MS93] the authors point out that there is an analogous version of the spectral sequence (5.1) for homology. We will only need the non-equivariant (but twisted) version, which is:[7]

[4] For $\sigma \in N_n\Delta(Y)$, let τ be its ith face. There is a canonical map $\sigma_0 \to \tau_0$ (which is the identity except when $i = 0$) inducing a map $M(\sigma_0) \to M(\tau_0)$. The direct sum of these maps is ∂_n^i.

[5] Given an element $\{g_\sigma \in M(\sigma_0) \mid \sigma \in N_n\Delta(Y)\}$, we need to choose an element of $M(\tau_0)$ for each $\tau \in N_{n+1}\Delta(Y)$. Let σ be the ith face of τ, which has a canonical map $\tau_0 \to \sigma_0$ (which is the identity except when $i = 0$). Apply M to get a map $M(\sigma_0) \to M(\tau_0)$ and take the image of g_σ under this map.

[6] The inclusion $\{0\} \hookrightarrow [0,1]$ is an acyclic cofibration, so its pullback along the fibration f is again an acyclic cofibration, in particular a weak equivalence.

[7] This was also stated (referencing [MS93]) as Theorem 4.1 of [Han09b].

Theorem 5.6 *For any fibration $f\colon Y \to X$ and any local coefficient system M on Y, there is a local coefficient system $H_q(f;M)$ on X for each $q \geqslant 0$ and a spectral sequence*

$$E^2_{p,q} = H_p\big(X; H_q(f;M)\big) \;\Rightarrow\; H_*(Y;M) \tag{5.2}$$

with the usual homological grading.

The description of the local coefficient systems $H_q(f;M)$ is the same as above, replacing cohomology with homology. When the local coefficient system M on Y is pulled back from the base X, they are built out of the *untwisted* homology of each fibre.

We now return to the viewpoint of local coefficient systems as an action of the fundamental group on an abelian group. In the special case where the local coefficient system on Y is a pullback of one on X the above can be rephrased as:

Corollary 5.7 *For any fibration $f\colon Y \to X$ with fibre F over the basepoint $x_0 \in X$, and any $\pi_1(X)$-module M, there is a spectral sequence*

$$E^2_{p,q} = H_p\big(X; H_q(F;M)\big) \;\Rightarrow\; H_*(Y;M) \tag{5.3}$$

with the usual homological grading. Here the action of $\pi_1(Y)$ on M is pulled back from that of $\pi_1(X)$ via f_* and the action of $\pi_1(F)$ on M is trivial. The action of $\pi_1(X)$ on $H_q(F;M)$ is induced by its diagonal action on the chain complex $S_*(X) \otimes_{\mathbb{Z}} M$.

This is natural for maps of fibrations in the obvious way:

Proposition 5.8 *Suppose we have a map of fibrations (the vertical maps are fibrations, and the square commutes on the nose):*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}$$

and a $\pi_1(X')$-module M. Denote the fibres over the basepoints by F and F' respectively. Then there is a map of spectral sequences (5.3) where:

- *The map $F \to F'$ induces a map of untwisted homology $H_q(F;M) \to H_q(F';M)$, which is equivariant w.r.t. the homomorphism $\pi_1(X) \to \pi_1(X')$, so it induces a map of twisted homology $H_p(X;H_q(F;M)) \to H_p(X';H_q(F';M))$. This is the map on the E^2 pages.*
- *The action of $\pi_1(Y)$ on M is the pullback of the action of $\pi_1(Y')$ on M, so the map $Y \to Y'$ induces a map of twisted homology $H_*(Y;M) \to H_*(Y';M)$. This is the map in the limit.*

§6. Proof of twisted homological stability

We now use the twisted Serre spectral sequence of the previous section to prove Theorem 1.3. We first record another fact we will use:

Lemma 6.1 (Shapiro for covering spaces) *Suppose we have a based space X which is locally nice enough to have a universal cover, a subgroup H of $\pi_1(X)$ and an H-module A. Let $\hat{X}$ be the (based) covering space corresponding to H. Then*

$$H_*(\hat{X};A) \cong H_*(X;\mathbb{Z}\pi_1(X) \otimes_{\mathbb{Z}H} A). \tag{6.1}$$

Moreover, given a map of the above data, namely a (based) map $f\colon X \to X'$ such that $f_*(H) \subseteq H'$ (so that there is a unique based lift $\hat{f}\colon \hat{X} \to \hat{X}'$) and a map $\phi\colon A \to A'$ which is equivariant w.r.t. f_*, the identification (6.1) is natural in the sense that

$$\begin{tikzcd} H_*(\hat{X};A) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_*(\hat{X}';A') \arrow[d, phantom, "\cong" rotate=90] \\ H_*(X;\mathbb{Z}\pi_1(X) \otimes_{\mathbb{Z}H} A) \arrow[r] & H_*(X';\mathbb{Z}\pi_1(X') \otimes_{\mathbb{Z}H'} A') \end{tikzcd} \tag{6.2}$$

commutes.

Proof. Denote the singular chain complex functor by $S_*(\ \ )$ and the universal cover of X by $\widetilde{X}$. Then we have an isomorphism of chain complexes

$$S_*(\widetilde{X}) \otimes_{\mathbb{Z}H} A \longrightarrow S_*(\widetilde{X}) \otimes_{\mathbb{Z}\pi_1(X)} \mathbb{Z}\pi_1(X) \otimes_{\mathbb{Z}H} A$$

given by $\sigma \otimes a \mapsto \sigma \otimes [c_x] \otimes a$, where c_x is the constant loop at the basepoint x of X. Taking homology gives the identification (6.1). Let $\widetilde{f}$ denote the unique (based) lift of f to $\widetilde{X} \to \widetilde{X}'$. The diagram (6.2) is induced by

$$\begin{tikzcd}
S_*(\widetilde{X}) \otimes_{\mathbb{Z}H} A \arrow[r] \arrow[d, "\cong"'] & S_*(\widetilde{X}') \otimes_{\mathbb{Z}H'} A' \arrow[d, "\cong"] \\
S_*(\widetilde{X}) \otimes_{\mathbb{Z}\pi_1(X)} \mathbb{Z}\pi_1(X) \otimes_{\mathbb{Z}H} A \arrow[r] & S_*(\widetilde{X}') \otimes_{\mathbb{Z}\pi_1(X')} \mathbb{Z}\pi_1(X') \otimes_{\mathbb{Z}H'} A'
\end{tikzcd}$$

and one can check that both routes around the square send $\sigma \otimes a$ to $\widetilde{f}_\sharp(\sigma) \otimes [c_{x'}] \otimes \phi(a)$. □

This will be applied to the following covering spaces of configuration spaces:

Definition 6.2 The configuration space $C_{(k,n-k)}(M,X)$ of k red and $n-k$ green points in M with labels in X is defined to be

$$(\mathrm{Emb}(n,M) \times X^n)/(\Sigma_{n-k} \times \Sigma_k),$$

and we give it the basepoint $\{(a_1,x_0),\ldots,(a_n,x_0)\}$ with the points $a_1,\ldots,a_{n-k}$ coloured green and the points $a_{n-k+1},\ldots,a_n$ coloured red. There is also a stabilisation map $s_n^k\colon C_{(k,n-k)}(M,X) \to C_{(k,n-k+1)}(M,X)$, which is defined exactly as in §2.2, and adds a new green point to the configuration.

Definition 6.3 Let $f\colon C_{(k,n-k)}(M,X) \to C_k(M,X)$ be the map which forgets the green points. We will also need the following two maps for technical reasons: Define $p\colon C_k(M,X) \to C_k(M,X)$ to be the self-homotopy-equivalence induced by the self-embedding $e|_M\colon M \hookrightarrow M$ (see §2.1). Choose a self-diffeomorphism of M which is isotopic to the identity and which takes a_i to $a_{i+n-k+1}$ for $i = 1,\ldots,k$. Denote by ϕ the self-homeomorphism $C_k(M,X) \to C_k(M,X)$ induced by this.

The forgetful maps f are locally trivial fibre bundles, so we have a map of fibrations:

$$\begin{tikzcd}
C_{(k,n-k)}(M,X) \arrow[r, "s_n^k"] \arrow[d, "f"'] & C_{(k,n-k+1)}(M,X) \arrow[d, "\phi^{-1} \circ f"] \\
C_k(M,X) \arrow[r, "\phi^{-1} \circ p"'] & C_k(M,X)
\end{tikzcd} \tag{6.3}$$

The p is there to ensure that it commutes on the nose, and the ϕ^{-1} is there to deal with basepoints: on the bottom-left we have to give $C_k(M,X)$ the basepoint $\{(a_{n-k+1},x_0),\ldots,(a_n,x_0)\}$, but on the bottom-right we can give it its usual basepoint of $\{(a_1,x_0),\ldots,(a_k,x_0)\}$.

The map s_n^k restricted to the fibres over the basepoints is a map

$$C_{n-k}(M \smallsetminus \{a_{n-k+1},\ldots,a_n\},X) \to C_{n-k+1}(M \smallsetminus \{a_{n-k+2},\ldots,a_{n+1}\},X),$$

but this can be identified, up to homeomorphism, with the stabilisation map $s_{n-k}\colon C_{n-k}(M_k,X) \to C_{n-k+1}(M_k,X)$, where M_k is M with a subset of $M \smallsetminus U$ of size k removed (see §2.1 for notation).

Finally, before beginning the proof proper, we mention how a certain local coefficient system pulls back along the maps in (6.3). The covering space $C_{(k,n-k)}(M,X) \to C_n(M,X)$ corresponds to the subgroup $G_n^k \leqslant G_n = \pi_1 C_n(M,X)$. Recall from Proposition 3.3 that T_n^k is a G_n^k-module (it is a sub-G_n^k-module of T_n), so it is a local coefficient system for $C_{(k,n-k)}(M,X)$.

Lemma 6.4 *The local coefficient system T_k^k on the right-hand base space pulls back to the local coefficient systems T_n^k and T_{n+1}^k on the total spaces of* (6.3).

Proof. By Lemma 3.7, the left-inverse π_k^n of $\iota_k^n \colon T_k \to T_n$ restricts to a bijection $T_n^k \to T_k^k$. So this is an isomorphism of abelian groups, and it is enough to check that it is equivariant w.r.t. the map on π_1 induced by the composite $\phi^{-1} \circ p \circ f$ in (6.3). This is true because both $e|_M \colon M \hookrightarrow M$ (which induces p) and the diffeomorphism which induces ϕ are isotopic to the identity. Exactly the same argument works for the right-hand side. ☐

Proof of Theorem 1.3 (except the split-injectivity claim). We need to show that the map

$$H_*(C_n(M,X);T_n) \longrightarrow H_*(C_{n+1}(M,X);T_{n+1}) \tag{6.4}$$

induced by s_n and ι_n is an isomorphism in the range $* \leqslant \frac{n-d}{2}$. By the decomposition (3.2) of Proposition 3.3, and the fact that T has degree d, this is the same as the map

$$\bigoplus_{k=0}^{d} H_*(C_n(M,X);\mathbb{Z}G_n \otimes_{\mathbb{Z}G_n^k} T_n^k) \longrightarrow \bigoplus_{k=0}^{d} H_*(C_{n+1}(M,X);\mathbb{Z}G_{n+1} \otimes_{\mathbb{Z}G_{n+1}^k} T_{n+1}^k) \tag{6.5}$$

induced by s_n, ι_n and $(s_n)_*$. By Shapiro's Lemma for covering spaces (Lemma 6.1) this is isomorphic to the map

$$\bigoplus_{k=0}^{d} H_*(C_{(k,n-k)}(M,X);T_n^k) \longrightarrow \bigoplus_{k=0}^{d} H_*(C_{(k,n-k+1)}(M,X);T_{n+1}^k) \tag{6.6}$$

induced by s_n^k and ι_n. The map of fibrations (6.3) gives the following map of twisted Serre spectral sequences (Corollary 5.7, Proposition 5.8 and Lemma 6.4):

$$\begin{array}{ccc} E^2_{p,q} = H_p(C_k(M,X);H_q(C_{n-k}(M_k,X);T_k^k)) & \Rightarrow & H_*(C_{(k,n-k)}(M,X);T_n^k) \\ \downarrow & & \downarrow \\ E^2_{p,q} = H_p(C_k(M,X);H_q(C_{n-k+1}(M_k,X);T_k^k)) & \Rightarrow & H_*(C_{(k,n-k+1)}(M,X);T_{n+1}^k). \end{array} \tag{6.7}$$

The map in the limit is the kth summand of (6.6), and the map on E^2 pages is induced by the stabilisation map s_{n-k} on the fibres and the homotopy-equivalence $\phi^{-1} \circ p$ on the base. Note that T_k^k is a *constant* coefficient system once it has been pulled back to the fibres $C_{n-k}(M_k,X)$ and $C_{n-k+1}(M_k,X)$, since it was originally pulled back from the base.

Hence, by *untwisted* homological stability for configuration spaces (Theorem 1.2) and the universal coefficient theorem, the map on E^2 pages is an isomorphism for $q \leqslant \frac{n-k}{2}$ (and all $p \geqslant 0$). By the Zeeman comparison theorem[8] it is therefore an isomorphism in the limit for $* \leqslant \frac{n-k}{2}$. So in the range $* \leqslant \frac{n-d}{2}$ each summand in (6.6) is an isomorphism, so (6.4) is an isomorphism. ☐

Remark 6.5 When M is orientable [Chu12, Corollary 3] or at least 3-dimensional [RW13, Theorem B], the stabilisation map $C_n(M,X) \to C_{n+1}(M,X)$ is an isomorphism on *rational* homology in the larger range $* < n$. If $T \colon \mathcal{B}(M,X) \to \mathsf{Vect}_{\mathbb{Q}}$ is a *rational* twisted coefficient system of degree d, then the constant coefficients T_k^k appearing in (6.7) above are rational vector spaces, and hence the same proof tells us that the map

$$H_*(C_n(M,X);T_n) \to H_*(C_{n+1}(M,X);T_{n+1})$$

is an isomorphism in the range $* < n-d$.

§7. Split-injectivity

To prove the split-injectivity part of Theorem 1.3 we will use the following lemma which was used implicitly by Nakaoka in [Nak60] and later written down explicitly by Dold in [Dol62]:

[8] The required implication is contained in the proof of Theorem 1 of [Zee57], although stronger hypotheses are stated there. An explicit statement of the comparison theorem which applies to our case is Theorem 1.2 of [Iva93]. It is also written in Remarque 1.8 of [CDG11].

Lemma 7.1 ([Dol62, Lemma 2]) *Given a sequence* $0 \to A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots$ *of abelian groups and homomorphisms, the following is sufficient to imply that each of the maps* ϕ_i *is split-injective: There exist maps* $\tau_{k,n}\colon A_n \to A_k$ *for* $1 \leqslant k \leqslant n$ *with* $\tau_{n,n} = \mathrm{id}$ *such that*

$$\mathrm{im}(\tau_{k,n} - \tau_{k,n+1} \circ \phi_n) \leqslant \mathrm{im}(\phi_{k-1}). \tag{7.1}$$

Let $U_n(M,X)$ be the universal cover of $C_n(M,X)$. One can think of its elements as n-strand "open-ended braids" in $M \times [0,1]$ (n pairwise disjoint paths in $M \times [0,1]$ which are the identity in the second coordinate and start at $\{(a_1,0),\ldots,(a_n,0)\}$, up to endpoint-preserving homotopy) with each strand labelled by the based path space PX. Let $\widetilde{s}_n\colon U_n(M,X) \to U_{n+1}(M,X)$ be the lift of the stabilisation map which applies $e|_M \times \mathrm{id}_{[0,1]}$ to the braid and adds a vertical strand at a_1 labelled by the constant path c_{x_0}.

As before, denote $\pi_1 C_n(M,X)$ by G_n, and denote the singular chain complex of a space by $S_*(\ )$. Let $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ be any twisted coefficient system (we do not assume finite-degree in this section). Then the map

$$(s_n;\iota_n)_*\colon H_*(C_n(M,X);T_n) \longrightarrow H_*(C_{n+1}(M,X);T_{n+1}). \tag{7.2}$$

is induced by the map of chain complexes

$$(\widetilde{s}_n)_\sharp \otimes \iota_n\colon S_*(U_n(M,X)) \otimes_{\mathbb{Z}G_n} T_n \longrightarrow S_*(U_{n+1}(M,X)) \otimes_{\mathbb{Z}G_{n+1}} T_{n+1}.$$

Proof of Theorem 1.3 (split-injectivity claim). We want to prove that (7.2) is split-injective for all $*$ and n. By Dold's Lemma 7.1, it is sufficient to construct chain maps

$$t_{k,n}\colon S_*(U_n(M,X)) \otimes_{\mathbb{Z}G_n} T_n \longrightarrow S_*(U_k(M,X)) \otimes_{\mathbb{Z}G_k} T_k$$

for $1 \leqslant k \leqslant n$ such that $t_{n,n} = \mathrm{id}$ and

$$t_{k,n} \simeq t_{k,n+1} \circ ((\widetilde{s}_n)_\sharp \otimes \iota_n) - ((\widetilde{s}_{k-1})_\sharp \otimes \iota_{k-1}) \circ t_{k-1,n}. \tag{7.3}$$

Let $S \subseteq \{1,\ldots,n\}$. There is a unique partially-defined injection $\{1,\ldots,n\} \dashrightarrow \{1,\ldots,|S|\}$ which is order-preserving and is defined precisely on S. This is a morphism $n \to |S|$ in the category Σ. Let $\pi_{S,n}$ be the lift along $\mathcal{B}(M,X) \to \Sigma$ to a morphism $(x_0)^n \to (x_0)^{|S|}$ given by travelling along the paths p_i (see §2.1) and keeping the labels constant. By our standard abuse of notation we will denote its image under T also by $\pi_{S,n}\colon T_n \to T_{|S|}$.

We also define a map $p_{S,n}\colon U_n(M,X) \to U_{|S|}(M,X)$ as follows. Given an open-ended braid in $U_n(M,X)$, forget the strands which start at $(a_i,0)$ for $i \in \{1,\ldots,n\} \smallsetminus S$, and then concatenate this with the reverse of $\pi_{S,n}\colon (x_0)^n \to (x_0)^{|S|}$ to get an open-ended braid in $U_{|S|}(M,X)$.

Directly from these definitions one can check (where the notation $(S-1)$ means $\{s-1 \mid s \in S\}$):

(a) If $1 \notin S$ then $\pi_{S,n+1} \circ \iota_n = \pi_{(S-1),n}$ and $p_{S,n+1} \circ \widetilde{s}_n \simeq p_{(S-1),n}$.
(b) If $1 \in S$ then $\pi_{S,n+1} \circ \iota_n = \iota_{|S|-1} \circ \pi_{(S\smallsetminus\{1\}-1),n}$ and $p_{S,n+1} \circ \widetilde{s}_n = \widetilde{s}_{|S|-1} \circ p_{(S\smallsetminus\{1\}-1),n}$.

We now define $t_{k,n}$ to be the following chain map:

$$\sigma \otimes x \ \mapsto \sum_{S\subseteq\{1,\ldots,n\},\, |S|=k} (p_{S,n})_\sharp(\sigma) \otimes \pi_{S,n}(x).$$

Clearly $t_{n,n} = \mathrm{id}$, so we just need to check the identity (7.3). The right-hand side of this is:

$$\begin{aligned} \sigma \otimes x \quad \mapsto & \sum_{S\subseteq\{1,\ldots,n+1\},\, |S|=k} \big((p_{S,n+1})_\sharp \circ (\widetilde{s}_n)_\sharp(\sigma)\big) \otimes \big(\pi_{S,n+1} \circ \iota_n(x)\big) \\ & - \sum_{R\subseteq\{1,\ldots,n\},\, |R|=k-1} \big((\widetilde{s}_{k-1})_\sharp \circ (p_{R,n})_\sharp(\sigma)\big) \otimes \big(\iota_{k-1} \circ \pi_{R,n}(x)\big). \end{aligned} \tag{7.4}$$

Using (a) and (b) above, we see that the top line of this decomposition is chain-homotopic to:

$$\begin{aligned} \sigma \otimes x \ \mapsto & \sum_{S\subseteq\{1,\ldots,n+1\},\, |S|=k,\, 1\in S} \big((\widetilde{s}_{k-1})_\sharp \circ (p_{(S\smallsetminus\{1\}-1),n})_\sharp(\sigma)\big) \otimes \big(\iota_{k-1} \circ \pi_{(S\smallsetminus\{1\}-1),n}(x)\big) \\ & + \sum_{S\subseteq\{1,\ldots,n+1\},\, |S|=k,\, 1\notin S} (p_{(S-1),n})_\sharp(\sigma) \otimes \pi_{(S-1),n}(x). \end{aligned} \tag{7.5}$$

The first line of (7.5) cancels with the second line of (7.4), leaving just the second line of (7.5), which is precisely $t_{k,n}$, as required. □

References

[Bet02] Stanislaw Betley. *Twisted homology of symmetric groups*. Proc Amer Math Soc, vol. 130, no. 12, 3439–3445 (electronic), 2002. [cited on pp. 1, 2, 9]

[Bol12] Søren K. Boldsen. *Improved homological stability for the mapping class group with integral or twisted coefficients*. Math Z, vol. 270, no. 1-2, 297–329, 2012. [cited on pp. 2, 9]

[CDG11] Gaël Collinet, Aurélien Djament and James Griffin. *Stabilité homologique pour les groupes d'automorphismes des produits libres*, 2011. ArXiv:1109.2686 (to appear in IMRN). [cited on pp. 6, 15]

[Chu12] Thomas Church. *Homological stability for configuration spaces of manifolds*. Invent Math, vol. 188, no. 2, 465–504, 2012. [cited on p. 15]

[CM09] Ralph L. Cohen and Ib Madsen. *Surfaces in a background space and the homology of mapping class groups*. In *Algebraic geometry—Seattle 2005. Part 1*, vol. 80 of *Proc. Sympos. Pure Math.*, pp. 43–76. Amer. Math. Soc., Providence, RI, 2009. [cited on pp. 2, 9]

[Dol62] Albrecht Dold. *Decomposition theorems for $S(n)$-complexes*. Ann of Math (2), vol. 75, 8–16, 1962. [cited on pp. 15, 16]

[Dwy80] W. G. Dwyer. *Twisted homological stability for general linear groups*. Ann of Math (2), vol. 111, no. 2, 239–251, 1980. [cited on pp. 2, 9]

[EML54] Samuel Eilenberg and Saunders Mac Lane. *On the groups $H(\Pi, n)$. II. Methods of computation*. Ann of Math (2), vol. 60, 49–139, 1954. [cited on p. 6]

[Han09a] Elizabeth Hanbury. *Homological stability of non-orientable mapping class groups with marked points*. Proc Amer Math Soc, vol. 137, no. 1, 385–392, 2009. [cited on p. 10]

[Han09b] Elizabeth Hanbury. *An open-closed cobordism category with background space*. Algebr Geom Topol, vol. 9, no. 2, 833–863, 2009. [cited on p. 12]

[HPV12] Manfred Hartl, Teimuraz Pirashvili and Christine Vespa. *Polynomial functors from Algebras over a set-operad and non-linear Mackey functors*, 2012. ArXiv:1209.1607. [cited on p. 6]

[Iva93] Nikolai V. Ivanov. *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, vol. 150 of *Contemp. Math.*, pp. 149–194. Amer. Math. Soc., Providence, RI, 1993. [cited on pp. 2, 9, 15]

[McD75] Dusa McDuff. *Configuration spaces of positive and negative particles*. Topology, vol. 14, 91–107, 1975. [cited on p. 2]

[MS93] I. Moerdijk and J.-A. Svensson. *The equivariant Serre spectral sequence*. Proc Amer Math Soc, vol. 118, no. 1, 263–278, 1993. [cited on pp. 11, 12]

[Nak60] Minoru Nakaoka. *Decomposition theorem for homology groups of symmetric groups*. Ann of Math (2), vol. 71, 16–42, 1960. [cited on p. 15]

[Pal] Martin Palmer. *Homological stability for configuration spaces of disconnected submanifolds*. In preparation. [cited on p. 5]

[Pal13] Martin Palmer. *Homological stability for oriented configuration spaces*. Trans Amer Math Soc, vol. 365, no. 7, 3675–3711, 2013. [cited on p. 3]

[RW13] Oscar Randal-Williams. *Homological stability for unordered configuration spaces*. Q J Math, vol. 64, no. 1, 303–326, 2013. [cited on pp. 2, 15]

[Seg73] Graeme Segal. *Configuration-spaces and iterated loop-spaces*. Invent Math, vol. 21, 213–221, 1973. [cited on p. 2]

[Seg79] Graeme Segal. *The topology of spaces of rational functions*. Acta Math, vol. 143, no. 1-2, 39–72, 1979. [cited on p. 2]

[Zee57] E. C. Zeeman. *A proof of the comparison theorem for spectral sequences*. Proc Cambridge Philos Soc, vol. 53, 57–62, 1957. [cited on p. 15]

Mathematisches Institut, Einsteinstraße 62, 48149 Münster, Germany
mpalm_01@uni-muenster.de