Twisted homological stability for configuration spaces

Martin Palmer // 17 December 2014

Abstract

Let M be an open, connected manifold. A classical theorem of McDuff and Segal states that the sequence $\{C_n(M)\}$ of configuration spaces of n unordered, distinct points in M is *homologically stable* with coefficients in $\mathbb{Z}$ – in each degree, the integral homology is eventually independent of n. The purpose of this note is to prove that this phenomenon also holds for homology with twisted coefficients. We first define an appropriate notion of *finite-degree twisted coefficient system* for $\{C_n(M)\}$ and then use a spectral sequence argument to deduce the result from the untwisted homological stability result of McDuff and Segal. The result and the methods are generalisations of those of Betley [Bet02] for the symmetric groups.

Contents

1. Introduction

For a pair of spaces M and X, the *configuration space of n unordered points in M with labels in X* is defined by

$$C_n(M,X) \coloneqq (\mathrm{Emb}(n,M) \times X^n)/\Sigma_n.$$

Here n is the discrete space of cardinality n, so $\mathrm{Emb}(n,M)$ is the subspace of M^n where no two points coincide. The symmetric group Σ_n acts diagonally, permuting the points and the list of labels, so an element of $C_n(M,X)$ is a subset of M of cardinality n, together with an element of X "attached" to each point. More generally, one could define a configuration space associated to a fibre bundle $\pi\colon E \to M$ by

$$C_n(M,\pi) \coloneqq \{(e_1,\dots,e_n) \in E^n \mid \pi(e_i) \neq \pi(e_j) \text{ for } i \neq j\}/\Sigma_n.$$

An element of $C_n(M,\pi)$ is thus a subset of M of cardinality n, together with an element of $\pi^{-1}(p)$ "attached" to each point p of this subset. However, with the exception of Remark 1.9, we will restrict our attention to configuration spaces with labels in a fixed label-space X, corresponding to the trivial bundle $M \times X \to M$.

Assumption 1.1 Henceforth we assume that M is an open, connected manifold with $\dim(M) \geqslant 2$, and that X is a path-connected space. To be precise, by an *open* manifold we mean a manifold with empty boundary, each of whose (path-)components is non-compact but paracompact.

Since M is open, there are well-defined "stabilisation maps" $C_n(M,X) \to C_{n+1}(M,X)$, which we define precisely in §2.2 below. They are so called because the sequence of spaces $\{C_n(M,X)\}$ is homologically stable with respect to them:

2010 *Mathematics Subject Classification*: Primary 55R80; secondary 57N65.
Key words and phrases: Configuration spaces, homological stability, polynomial twisted coefficients.
Address: Mathematisches Institut, Einsteinstraße 62, 48149 Münster, Germany
Email address: mpalm_01@uni-muenster.de

Theorem 1.2 ([Seg73, McD75, Seg79, RW13]) *Under the conditions on M and X assumed above, the map $C_n(M,X) \to C_{n+1}(M,X)$ induces an isomorphism on integral homology in degrees $* \leqslant \frac{n}{2}$, and is split-injective on homology in all degrees.*

Twisted coefficients. Several other families of groups or spaces which are homologically stable are also known to have homological stability for *twisted coefficients*. For example general linear groups [Dwy80], mapping class groups of surfaces [Iva93, CM09, Bol12] and the symmetric groups [Bet02] are known to satisfy this phenomenon. A machine for proving twisted homological stability for many natural families of groups is constructed in [Wah14], and in particular covers the cases of mapping class groups of non-orientable surfaces and orientable 3-manifolds.

The minimum data required in order to pose the question of twisted homological stability for a sequence of based, path-connected spaces $\{Y_n\}$ is a functor $\pi_1(\{Y_n\}) \to \mathsf{Ab}$, where the source is the category (groupoid) where the objects are the natural numbers, all morphisms are automorphisms and $\mathrm{Aut}(n) = \pi_1(Y_n)$. In other words, this is just a choice of $\pi_1(Y_n)$-module for each n. There is of course no chance of stability with respect to such a general "twisted coefficient system", as the $\pi_1(Y_n)$-modules for differing n may be completely unrelated.

To obtain a notion of *twisted coefficient system* with a chance of stability, one needs to add some (non-endo)morphisms to $\pi_1(\{Y_n\})$ and require that the functor from this new source category to Ab satisfy some finiteness conditions defined in terms of the new morphisms. The correct way to do this depends on the particular context one is working in (although a very general context for classifying spaces of discrete groups is introduced in [Wah14]).

In §§2,3 below we will define a *twisted coefficient system of degree d* for the sequence $\{C_n(M,X)\}$ to be a functor from a certain category $\mathcal{B}(M,X)$ to Ab satisfying a certain finiteness condition. To state the main result, it is enough to mention that it includes the data of a $\pi_1 C_n(M,X)$-module T_n for each n, and that the stabilisation map induces a natural map

$$H_*(C_n(M,X);T_n) \longrightarrow H_*(C_{n+1}(M,X);T_{n+1}). \tag{1.1}$$

The main result of this note is the following:

Theorem 1.3 *Under Assumption 1.1, if T is a twisted coefficient system for $\{C_n(M,X)\}$ of degree d, then the map (1.1) is an isomorphism in degrees $* \leqslant \frac{n-d}{2}$, and is split-injective in all degrees.*

This is a generalisation of the result of [Bet02], where twisted homological stability is proved for the symmetric groups $\{\Sigma_n\}$, corresponding to the case $M = \mathbb{R}^\infty$ and $X = *$.

Remark 1.4 (*Split-injectivity*) The split-injectivity statement of this theorem is fairly easy, and has essentially the same proof as in the untwisted case. It is proved separately in §7, and its proof does not depend on the twisted coefficient system being of finite degree – this assumption is only required for surjectivity in the stable range.

Remark 1.5 (*When $\cdot 2$ is invertible*) If $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ is a twisted coefficient system of $\mathbb{Z}[\frac{1}{2}]$-modules, i.e. its image lies in the subcategory $\mathbb{Z}[\frac{1}{2}]$-mod of Ab, then the stability range in Theorem 1.3 can be improved to $* \leqslant n-d$, as long as M is at least 3-dimensional. When M is a surface, a similar improvement is possible if T is a *rational* twisted coefficient system, i.e. its image lies in the subcategory $\mathsf{Vect}_{\mathbb{Q}}$ of Ab. The improved range in this case is $* \leqslant n-d$ when M is non-orientable and $* < n-d$ when M is orientable. This uses the improved homological stability ranges, for untwisted coefficients, obtained in [Chu12, RW13, KM14b, Knu14]. See Remark 6.5 after the proof of Theorem 1.3 in §6.

Remark 1.6 (*Related results*) Some other examples of twisted homological stability theorems for configuration spaces are as follows. Firstly, there is the stability result [Bet02, Theorem 4.3], of which the present note is a generalisation, corresponding to taking $M = \mathbb{R}^\infty$ and $X = *$ in Theorem 1.3. Another example is [CF13, Corollary 4.4], which concerns the braid groups $\beta_n = \pi_1(C_n(\mathbb{R}^2))$. The local coefficient systems in this case are the rational β_n-representations $V_{\lambda[n]}$. Here, λ is a fixed Young diagram with $|\lambda|$ boxes, $\lambda[n]$ is the Young diagram obtained by adding a new row of length $n - |\lambda|$ to the top of λ and $V_{\lambda[n]}$ is the irreducible Σ_n-representation corresponding to $\lambda[n]$, viewed as a β_n-representation via the projection $\beta_n \to \Sigma_n$. Another example concerning the braid groups is [Wah14, Example 5.3 and Theorem 6.13], which proves homological stability for

β_n with coefficients in the Burau representations $\beta_n \to \mathrm{Aut}(\mathbb{Z}[t^{\pm 1}]^n)$. Note that in this example the local coefficient systems do not factor through the projection $\beta_n \to \Sigma_n$. In fact, the homology of β_n with coefficients in the *reduced rational* Burau representations $\beta_n \to \mathrm{Aut}(\mathbb{Q}[t^{\pm 1}]^{n-1})$ is explicitly computed in [Che14], and one can directly read off a stable range from the computation. Interestingly, the stable range obtained in [Che14] for the reduced *rational* Burau representations has slope 1, whereas the stable range obtained in [Wah14] for the *integral* Burau representations has slope $\frac{1}{2}$. This parallels the improvements to the range of Theorem 1.3 discussed in Remark 1.5 above. See also Remark 4.10, which explains that the Burau representations do not fit into the framework of the present note.

Some special cases of Theorem 1.3 are as follows. Fix a principal ideal domain R and a path-connected based space Z with $H_*(Z;R)$ flat over R in all degrees. For example we could take R to be a field, or we could take $R = \mathbb{Z}$ and assume that the integral homology of Z is torsion-free. Also choose non-negative integers q, h and suppose that $\widetilde{H}_*(Z;R) = 0$ in the range $* \leqslant h$. The homology group $H_q(Z^n;R)$ is a $\mathbb{Z}[\Sigma_n]$-module given by permuting the factors of Z^n, and hence also a $\mathbb{Z}[\pi_1(C_n(M,X))]$-module via the projection $\pi_1(C_n(M,X)) \to \Sigma_n$.

Corollary 1.7 *There are isomorphisms*

$$H_*\big(C_n(M,X); H_q(Z^n;R)\big) \;\cong\; H_*\big(C_{n+1}(M,X); H_q(Z^{n+1};R)\big)$$

in the range $* \leqslant \frac{1}{2}\big(n - \lfloor \frac{q}{h+1} \rfloor\big)$. If we take $R = \mathbb{Q}$, or R is a ring in which 2 is invertible and M is at least 3-dimensional, then this holds in the larger range $* \leqslant n - \lfloor \frac{q}{h+1} \rfloor$ (except in the case where M is an orientable surface, in which case the larger range is $* < n - \lfloor \frac{q}{h+1} \rfloor$).

Proof. The first statement follows directly from Theorem 1.3 applied to Example 4.1, using Lemma 4.2 and Remark 4.4 to compute the degree of the twisted coefficient system in this case. The improved ranges follow from Remark 1.5 above. ☐

For an ordered partition $\mu = (\mu_1, \ldots, \mu_k)$ of $|\mu| = \mu_1 + \cdots + \mu_k$, denote by Σ_μ the product of symmetric groups $\Sigma_{\mu_1} \times \cdots \times \Sigma_{\mu_k}$, which is naturally a subgroup of $\Sigma_{|\mu|}$. Fix an ordered partition λ, and assume that $n \geqslant |\lambda|$, so that there is an induced ordered partition $\lambda[n] \coloneqq (n - |\lambda|, \lambda_1, \ldots, \lambda_k)$ of n. Then $\Sigma_n/\Sigma_{\lambda[n]}$ is a (transitive) Σ_n-set, so that $R[\Sigma_n/\Sigma_{\lambda[n]}]$ is a $\pi_1(C_n(M,X))$-module via the projection $\pi_1(C_n(M,X)) \to \Sigma_n$ for any ring R.

Corollary 1.8 *There are isomorphisms*

$$H_*\big(C_n(M,X); R\big[\Sigma_n/\Sigma_{\lambda[n]}\big]\big) \;\cong\; H_*\big(C_{n+1}(M,X); R\big[\Sigma_{n+1}/\Sigma_{\lambda[n+1]}\big]\big)$$

in the range $* \leqslant \frac{1}{2}\big(n - |\lambda|\big)$. If we take $R = \mathbb{Q}$, or R is a ring in which 2 is invertible and M is at least 3-dimensional, then this holds in the larger range $* \leqslant n - |\lambda|$ (except in the case where M is an orientable surface, in which case the larger range is $* < n - |\lambda|$).

In particular this includes stability for coefficients in $\mathbb{Z}[\Sigma_n/\Sigma_{n-k}]$ or in $\mathbb{Z}[\Sigma_n/(\Sigma_k \times \Sigma_{n-k})]$ in the range $* \leqslant \frac{n-k}{2}$ by taking λ to be $(1, \ldots, 1)$ or (k) respectively.

Proof. The first statement follows directly from Theorem 1.3 applied to Example 4.6, using Lemma 4.7 and Remark 4.8 to compute the degree of the twisted coefficient system in this case. The improved ranges follow from Remark 1.5 above. ☐

Remark 1.9 (*Configurations with twisted labels*) A consequence of Corollary 1.7 is (untwisted) homological stability for configuration spaces $C_n(M,\pi)$ with labels in a fibre bundle $\pi\colon E \to M$ with path-connected fibres, defined at the very beginning of this note. This uses the Serre spectral sequence for the fibre bundle

$$C_n(M,\pi) \longrightarrow C_n(M)$$

that forgets the labels, and which has E^2 page isomorphic to the twisted homology groups of $C_n(M)$ with coefficients in the homology groups of F^n, where F is the typical fibre of π. Corollary 1.7 says that the stabilisation maps induce a map of spectral sequences which is an isomorphism in a range on the E^2 page, as long as we take field coefficients. One can then reconstruct an integral homological stability result from the fields $\mathbb{F}_p$ and $\mathbb{Q}$. This is proved in detail in Appendix B of [CP14] and also in Appendix A of [KM14a]. We note that this result can alternatively be proved using a generalisation of the proof of [RW13], which is concerned with configuration spaces with labels in a fixed space. This alternative proof is also sketched in Appendix A of [KM14a].

Remark 1.10 (*Coloured configuration spaces*) Corollary 1.8 may in fact be deduced quickly from untwisted homological stability, as follows. First note that

$$H_*\big(C_n(M,X); R\big[\Sigma_n/\Sigma_{\lambda[n]}\big]\big) \;\cong\; H_*(C_{\lambda[n]}(M,X); R),$$

where the *coloured configuration space* $C_{\lambda[n]}(M,X)$ is defined to be the covering space of $C_n(M,X)$ with $\left|\Sigma_n/\Sigma_{\lambda[n]}\right| = \binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2}\cdots\binom{n-\lambda_1-\cdots-\lambda_{k-1}}{\lambda_k}$ sheets, in which the n points are coloured according to the partition $\lambda[n]$. There is a stabilisation map

$$C_{\lambda[n]}(M,X) \longrightarrow C_{\lambda[n+1]}(M,X)$$

given by adding a point of the first colour to a coloured configuration (similarly to the stabilisation map defined in Definition 2.1). This commutes up to homotopy with the projections to $C_\lambda(M,X)$, which are fibre bundles, and the map of fibres is the ordinary stabilisation map $C_{n-|\lambda|}(M_{|\lambda|},X) \to C_{n+1-|\lambda|}(M_{|\lambda|},X)$, where $M_{|\lambda|}$ denotes the manifold M with $|\lambda|$ points removed. The result then follows by applying the relative Serre spectral sequence associated to this map of fibre bundles over $C_\lambda(M,X)$.

Remark 1.11 (*Representation stability*) Let $F_n(M,X)$ denote the configuration space of n distinct, *ordered* points in M labelled by X. This is an $(n!)$-sheeted covering space of $C_n(M,X)$, and in the notation of the previous remark it may also be written as $C_{(1,\ldots,1)}(M,X)$, where the partition contains n instances of the number 1. The sequence of graded $\mathbb{Q}[\Sigma_n]$-modules $\{H^*(F_n(M,X);\mathbb{Q})\}$ satisfies *representation stability*, a notion introduced in [CF13] and proved in this case by [Chu12]. Roughly, this says that for each fixed degree $*$ and Young diagram λ, the number of copies of the irreducible Σ_n-representation $V_{\lambda[n]}$ in the nth term $H^*(F_n(M,X);\mathbb{Q})$ of the sequence is eventually independent of n. See Remark 1.6 for an explanation of this notation. Moreover, the stability in this case is *uniform*: the bound on "eventually" depends only on $*$ and not on λ.

The rational homology of $F_n(M,X)$ is related to the groups appearing in Corollary 1.8 as follows:

$$H_*(F_n(M,X);\mathbb{Q}) \otimes_{\mathbb{Q}[\Sigma_n]} \mathbb{Q}[\Sigma_n/\Sigma_{\lambda[n]}] \;\cong\; H_*(C_n(M,X);\mathbb{Q}[\Sigma_n/\Sigma_{\lambda[n]}]). \tag{1.2}$$

This follows from the collapse of the Künneth spectral sequence for the singular chain complex $C_*(F_n(M,X);\mathbb{Q})$ and the module $\mathbb{Q}[\Sigma_n/\Sigma_{\lambda[n]}]$ over the ring $\mathbb{Q}[\Sigma_n]$. By Corollary 1.8 (c.f. also the previous remark), this sequence of graded groups is stable in the range $* \leqslant n-|\lambda|$ (or the range $* < n-|\lambda|$ if M is an orientable surface). There is a proof due to Søren Galatius [personal communication], involving only the representation theory of symmetric groups, that takes stability of the left-hand side of (1.2) as input and proves representation stability for $\{H^*(F_n(M,X);\mathbb{Q})\}$. This therefore reveals a link between twisted homological stability (via Corollary 1.8) and representation stability.

More quantitatively, the argument of Galatius proves representation stability in the range $n \geqslant 2.\max\{|\lambda|,*\}$ (or $n \geqslant 2.\max\{|\lambda|,*+1\}$ in the case of orientable surfaces). For comparison, the range obtained in [Chu12] is $n \geqslant 2*$ for manifolds of dimension at least 3 and $n \geqslant 4*$ for surfaces. So the range obtained by Galatius' argument improves the range of [Chu12] for surfaces when $|\lambda| \leqslant 2*$ (and also works equally well for non-orientable manifolds). If we define the *complexity* of a Young diagram μ to be the number of boxes below the first row, $\kappa(\mu) = |\mu| - \mu_1$, then we can say that the range is improved for Young diagrams with low complexity, since $\kappa(\lambda[n]) = |\lambda|$. However, it does not recover *uniform* representation stability, as the range depends on λ as well as on $*$.

Remark 1.12 (*Alternating coefficients*) There is a sequence of $\pi_1(C_n(M,X))$-modules that does not fit into the framework of this note (it does not form a twisted coefficient system at all, let alone a finite-degree one), but which nevertheless does exhibit homological stability. Every loop in $C_n(M,X)$ induces a permutation of its base configuration, so there is a natural projection map $\pi_1(C_n(M,X)) \to \Sigma_n$, which we can compose with the sign homomorphism to obtain a map

$\pi_1(C_n(M,X)) \to \mathbb{Z}/2$. This makes $\mathbb{Z}[\mathbb{Z}/2]$ into a $\pi_1(C_n(M,X))$-module, and its kernel corresponds to a double cover $C_n^+(M,X) \to C_n(M,X)$. The space $C_n^+(M,X)$ is the "oriented configuration space" where each configuration is additionally equipped with an ordering of its points up to even permutations. One can easily see that

$$H_*(C_n^+(M,X);\mathbb{Z}) \cong H_*(C_n(M,X);\mathbb{Z}[\mathbb{Z}/2]). \tag{1.3}$$

In [Pal13] the author proved that the sequence of spaces $C_n^+(M,X)$, with analogous stabilisation maps, is homologically stable as $n \to \infty$, in the range $* \leqslant \frac{n-5}{3}$. Via the identification (1.3) this is twisted homological stability for $C_n(M,X)$ with respect to the sequence of local coefficients $\mathbb{Z}[\mathbb{Z}/2]$.

Remark 1.13 (*Future generalisations*) It would be very interesting to generalise this twisted homological stability result to a setting which admits more examples. As mentioned before the statement of Theorem 1.3, a twisted coefficient system for $\{C_n(M,X)\}$ is a functor from a certain category $\mathcal{B}(M,X)$ (the category of "partial braids" on M) to the category of abelian groups. One way to encompass more examples would be to replace $\mathcal{B}(M,X)$ by the subcategory $\mathcal{B}_1(M,X)$ of "full braids" on M. This is analogous to the category of finite sets and partially-defined injections, and its subcategory of everywhere-defined injections. Twisted coefficient systems in this more general sense would include for example the Burau representations when $M = \mathbb{R}^2$ and $X = *$ (see Remark 4.10). The category $\mathcal{B}(M,X)$ has a zero object, which is only initial (and not terminal) in the subcategory $\mathcal{B}_1(M,X)$. However, the methods developed in [Wah14] for proving twisted homological stability for families of groups involve functors from a source category which is only required to have an initial object (see also [DV13, §1.2]). This suggests that it is reasonable to hope for a stability result for twisted coefficient systems indexed by the category $\mathcal{B}_1(M,X)$.

A note on terminology. To keep our terminology from becoming ambiguous, we will always use the terms "local coefficient system" and "twisted coefficient system" as follows. For a space Y, a *local coefficient system* for Y will have its usual meaning as a bundle of abelian groups over Y, or a functor from the fundamental groupoid of Y to Ab, or (when Y is based and path-connected) a $\pi_1(Y)$-module. The phrase *twisted coefficient system* will always be used in the sense of Definition 2.2 below; in particular it applies to a *sequence* of spaces.

Acknowledgements. The content of this note appeared, in a slightly different form, as part of the author's PhD thesis in 2012, and he would like to thank his supervisor, Ulrike Tillmann, for her invaluable advice and guidance throughout his PhD. He would also like to thank Søren Galatius for sharing his proof of representation stability for ordered configuration spaces, cf. Remark 1.11, and Aurélien Djament for a discussion of cross-effect decompositions, cf. Remark 3.5.

2. Twisted coefficient systems

2.1. Setup. First we fix some data. Recall from Assumption 1.1 that M is an open, connected manifold of dimension at least 2 and X is a path-connected space. This assumption on M means that we may pick a connected manifold $\overline{M}$ with non-empty boundary $\partial\overline{M}$ whose interior is M (although we must allow $\partial\overline{M}$ to be non-compact in general). Also choose a basepoint x_0 for X. Choose a point $a \in \partial\overline{M}$, and let U be a coordinate neighbourhood of a with an identification $U \cong \mathbb{R}^d_+ = \{x \in \mathbb{R}^d \mid x_1 \geqslant 0\}$ which sends a to 0. Also choose a self-embedding $e\colon \overline{M} \hookrightarrow \overline{M}$ which is isotopic to the identity, is *equal* to the identity outside U, and such that $e(a) \in M$ (i.e. in the interior of $\overline{M}$). Moreover, we choose an isotopy $I\colon e \simeq \mathrm{id}_{\overline{M}}$. We obtain a sequence of points in M by defining

$$a_1 \coloneqq e(a) \qquad\qquad a_n \coloneqq e(a_{n-1}) \text{ for } n \geqslant 2.$$

The isotopy I provides us with canonical paths $p_n\colon [0,1] \to M$ between a_n and a_{n+1}.

2.2. The configuration space and the stabilisation map. Recall that the configuration space of n unordered points in M with labels in X is defined to be

$$C_n(M,X) \coloneqq ((M^n \smallsetminus \Delta) \times X^n)/\Sigma_n = (\mathrm{Emb}(n,M) \times X^n)/\Sigma_n,$$

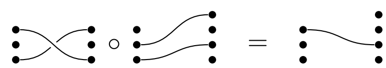
where $\Delta = \{(p_1, \ldots, p_n) \in M^n \mid p_i = p_j \text{ for some } i \neq j\}$ is the so-called *fat diagonal* of M^n, and the symmetric group Σ_n acts diagonally, permuting the points of M along with their labels in X. Thus a labelled configuration is an unordered set of ordered pairs in $M \times X$, generically denoted by $\{(p_1, x_1), \ldots, (p_n, x_n)\}$. When X is a point we will also write $C_n(M) = C_n(M, *)$.

Definition 2.1 The *stabilisation map* $s_n \colon C_n(M, X) \to C_{n+1}(M, X)$ is defined by

$$\{(p_1, x_1), \ldots, (p_n, x_n)\} \;\mapsto\; \{(e(p_1), x_1), \ldots, (e(p_n), x_n), (a_1, x_0)\}.$$

Essentially, the existing configuration is "pushed" further into the interior of the manifold by e, and the new configuration point a_1 added in the newly vacated space. Up to homotopy, the only "extra data" that this map depends on is the component of $\overline{M}$ containing a.

2.3. Twisted coefficient systems. We define the category $\mathcal{B}(M, X)$ to have $\coprod_{n \geqslant 0} X^n$ as its set of objects, and a morphism from $(x_1, \ldots, x_m)$ to $(y_1, \ldots, y_n)$ is a choice of $k \leqslant \min\{m, n\}$ and a path in $C_k(M, X)$ from a k-element subset of $\{(a_1, x_1), \ldots, (a_m, x_m)\}$ to a k-element subset of $\{(a_1, y_1), \ldots, (a_m, y_n)\}$ up to endpoint-preserving homotopy. The identity is given by $k = m = n$ and the constant path. Composition of two morphisms is given by concatenating paths and deleting configuration points for which the concatenated path is defined only half-way. For example (omitting the labels in X):

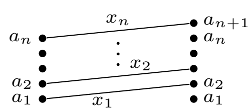

(2.1)

Again, when X is a point we will also write $\mathcal{B}(M) = \mathcal{B}(M, X)$.

Definition 2.2 A twisted coefficient system, associated to the direct system of spaces $\{C_n(M, X)\}$, is a functor from $\mathcal{B}(M, X)$ to the category Ab of abelian groups.

For each n, take $\{(a_1, x_0), \ldots, (a_n, x_0)\}$ as the basepoint of $C_n(M, X)$. Then the automorphism group of the object $(x_0)^n = (x_0, \ldots, x_0)$ (a tuple of length n) in $\mathcal{B}(M, X)$ is precisely the fundamental group $\pi_1 C_n(M, X)$. So if we are given a functor $T \colon \mathcal{B}(M, X) \to \mathsf{Ab}$ this induces an action of $\pi_1 C_n(M, X)$ on $T_n \coloneqq T((x_0)^n)$, and we can define the local homology $H_*(C_n(M, X); T_n)$.

For every object $\underline{x} = (x_1, \ldots, x_n)$ of $\mathcal{B}(M, X)$ there is a natural morphism $\iota_{\underline{x}} \colon (x_1, \ldots, x_n) \to (x_0, x_1, \ldots, x_n)$ as follows. It is represented by the path in $C_n(M, X)$ from $\{(a_1, x_1), \ldots, (a_n, x_n)\}$ to $\{(a_2, x_1), \ldots, (a_{n+1}, x_n)\}$ where each configuration point a_i travels along the path p_i (see §2.1) and the labels x_i stay constant. Schematically, this may be pictured as:



(2.2)

When $\underline{x} = (x_0)^n$ we will write $\iota_{\underline{x}} =: \iota_n$ for this canonical morphism $(x_0)^n \to (x_0)^{n+1}$. For any $\gamma \in \pi_1 C_n(M, X) = \mathrm{Aut}_{\mathcal{B}(M,X)}((x_0)^n)$ it is easy to check that

$$\iota_n \circ \gamma = (s_n)_*(\gamma) \circ \iota_n,$$

so for any T the map $T\iota_n \colon T_n \to T_{n+1}$ is equivariant with respect to the group-homomorphism $(s_n)_* \colon \pi_1 C_n(M, X) \to \pi_1 C_{n+1}(M, X)$. Hence we have an induced map

$$(s_n; T\iota_n)_* \colon H_*(C_n(M, X); T_n) \to H_*(C_{n+1}(M, X); T_{n+1}).$$

This is the map (1.1) which induces the isomorphism in Theorem 1.3.

Notation 2.3 From now on, by abuse of notation, we will denote the induced map $T\iota_n \colon T_n \to T_{n+1}$ also by $\iota_n \colon T_n \to T_{n+1}$. Similarly for the left-inverse $\pi_n \colon (x_0)^{n+1} \to (x_0)^n$ of ι_n (see §3.1): we denote its image under T also by $\pi_n \colon T_{n+1} \to T_n$.

2.4. A special case. Let $X = *$ and assume that M is simply-connected and of dimension at least 3. Since X is just a point, the objects of $\mathcal{B}(M,X) = \mathcal{B}(M)$ are (in canonical bijection with) the non-negative integers. The conditions on M imply that $\pi_1 C_n(M) \cong \Sigma_n$, in other words a path in $C_n(M)$ from the basepoint $\{a_1,\ldots,a_n\}$ to itself is determined by the permutation it induces on the set $\{a_1,\ldots,a_n\}$. More generally, a morphism from $\{a_1,\ldots,a_m\}$ to $\{a_1,\ldots,a_n\}$ in $\mathcal{B}(M)$ is determined by the partially-defined injection $\{a_1,\ldots,a_m\} \dashrightarrow \{a_1,\ldots,a_n\}$ it induces. Hence there is a canonical isomorphism of categories $\mathcal{B}(M) \cong \Sigma$, where Σ is the category defined as follows:

Definition 2.4 The category Σ has objects $\{0,1,2,\ldots\}$, and a morphism from m to n in Σ is a partially-defined injection $m \dashrightarrow n$. Composition is then composition of partially-defined functions (where the composite function is defined exactly where it is possible to define it). Note that Σ is an *inverse category*, i.e. every morphism f has a morphism g such that $fgf = f$ and $gfg = g$. It is a subcategory of the category with objects $\{0,1,2,\ldots\}$ and morphisms partially-defined *functions*, which is precisely Γ^{op}, a skeleton of the category $\mathsf{Set}_*^{\mathsf{fin}}$ of finite pointed sets. Partially-defined injections are also sometimes called partially-defined bijections. The category Σ is also sometimes called **finPInj** [Heu09], FI♯ [CEF12] or Θ [CDG13].

In particular we have $\mathcal{B}(\mathbb{R}^\infty) \cong \Sigma$. Of course, $\mathbb{R}^\infty$ is not a finite-dimensional manifold, as was assumed of M, but the definitions make sense for arbitrary spaces M and X, and $C_n(\mathbb{R}^\infty)$ is the colimit of the spaces $C_n(\mathbb{R}^d)$ under the obvious inclusions. The space $\mathrm{Emb}(n,\mathbb{R}^\infty)$ is a contractible Hausdorff space on which the natural action of Σ_n is free, so its quotient $C_n(\mathbb{R}^\infty)$ is a model for the classifying space $B\Sigma_n$.

Any embedding $\overline{M} \hookrightarrow \overline{N}$ taking $\partial\overline{M}$ into $\partial\overline{N}$, together with a continuous map $X \to Y$, induces a functor $\mathcal{B}(M,X) \to \mathcal{B}(N,Y)$. Any manifold $\overline{M}$ has a unique-up-to-isotopy embedding into B^∞, the closed unit ball in $\mathbb{R}^\infty$. This embedding, together with the map $X \to *$, induces a canonical functor $\mathcal{B}(M,X) \to \mathcal{B}(\mathring{B}^\infty) \cong \mathcal{B}(\mathbb{R}^\infty) \cong \Sigma$. Another description of this functor is that it forgets both the labels of the paths and the paths themselves, remembering only the partially-defined injection induced by the paths.

In particular this means that any twisted coefficient system $\Sigma \to \mathsf{Ab}$ canonically induces a twisted coefficient system $\mathcal{B}(M,X) \to \Sigma \to \mathsf{Ab}$.

2.5. A more general case. Instead of configurations of points (closed 0-dimensional submanifolds), one may consider configurations of closed submanifolds of higher dimension. Let $\overline{M}$ be a connected manifold with non-empty boundary and of dimension at least 2, as before. Also fix a closed manifold P and an embedding $\iota_0 \colon P \hookrightarrow \partial\overline{M}$. Choose an embedding $e \colon \overline{M} \hookrightarrow \overline{M}$ which is isotopic to the identity and such that $e(\overline{M})$ is disjoint from $\iota_0(P)$. We obtain a sequence of pairwise-disjoint embeddings of P into M by defining $\iota_n \coloneqq e^n \circ \iota_0$. Writing the disjoint union $P \sqcup \cdots \sqcup P$ of n copies of P as nP for short, define $C_{nP}(M)$ to be the path-component of $\mathrm{Emb}(nP,M)/\mathrm{Diff}(nP)$ containing $[\iota_1 \sqcup \cdots \sqcup \iota_n]$. A stabilisation map $C_{nP}(M) \to C_{(n+1)P}(M)$ may then be defined by sending $[\phi_1 \sqcup \cdots \sqcup \phi_n]$ to $[(e\circ\phi_1) \sqcup \cdots \sqcup (e\circ\phi_n) \sqcup \iota_1]$. One may also define more complicated versions of this setup, in which the submanifolds in $C_{nP}(M)$ are parametrised modulo a subgroup of $\mathrm{Diff}(P)$ and come equipped with labels in some bundle over $\mathrm{Emb}(P,M)$.

Everything in this note generalises to this setting, including an analogous notion of *twisted coefficient system* for $\{C_{nP}(M)\}$. In an article in preparation [Pal] we prove (untwisted) homological stability for these more general kinds of configuration spaces, as long as $\dim(P) \leqslant \frac{1}{2}(\dim(M)-3)$. The arguments of this note then immediately imply a twisted homological stability result for these spaces too.

3. Height and degree of a twisted coefficient system

3.1. Degree. First we will define the *degree* of a functor $T \colon \mathcal{B}(M,X) \to \mathsf{Ab}$. Recall from §2.3 the natural morphisms $\iota_{\underline{x}} \colon \underline{x} \to (x_0,\underline{x})$. The adjective "natural" suggests that they should form a natural transformation, and in fact they do: For every morphism ϕ of $\mathcal{B}(M,X)$ we have a commutative square

$$\begin{array}{ccc} (x_1,\dots,x_m) & \xrightarrow{\iota_{\bar{x}}} & (x_0,x_1,\dots,x_m) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle S\phi} \\ (y_1,\dots,y_n) & \xrightarrow[\iota_{\bar{y}}]{} & (x_0,y_1,\dots,y_n) \end{array} \tag{3.1}$$

where the morphism $S\phi$ is defined as follows: if ϕ is represented by a path p in $C_k(M,X)$ for some $k \leqslant \min\{m,n\}$, then $S\phi$ is represented by the path $s_k \circ p$ in $C_{k+1}(M,X)$. Thus we have an endofunctor $S\colon \mathcal{B}(M,X) \to \mathcal{B}(M,X)$ (which could be called the *stabilisation endofunctor*) and a natural transformation $\iota\colon \mathrm{id} \Rightarrow S$. Note that each $\iota_{\underline{x}}$ has an obvious left-inverse $\pi_{\underline{x}}$, and these morphisms fit together to form a left-inverse $\pi\colon S \Rightarrow \mathrm{id}$ for ι.

So, given any $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ we get a natural transformation $T \circ \iota\colon T \Rightarrow T \circ S$, or in other words a morphism in the abelian category $\mathsf{Ab}^{\mathcal{B}(M,X)}$. Denote its cokernel by $\Delta T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$.

Definition 3.1 The *degree* of a functor $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ is defined recursively by

$$\deg(0) = -1 \qquad\qquad \deg(T) = \deg(\Delta T) + 1,$$

where 0 is the identically-zero functor.

Example 3.2 The degree of T is $\leqslant 0$ exactly when $\Delta T = 0$, i.e. when $T\iota_{\underline{x}}\colon T\underline{x} \to T(x_0,\underline{x})$ is an isomorphism for all $\underline{x} \in \mathcal{B}(M,X)$. But the category $\mathcal{B}(M,X)$ is generated by the morphisms $\iota_{\underline{x}}$, their left-inverses $\pi_{\underline{x}}$ and isomorphisms. Hence $\deg(T) \leqslant 0$ if and only if T is constant in the sense that it sends every morphism to an isomorphism.

See §4 for some less trivial examples.

3.2. Height. Denote the homomorphism $\pi_1 C_n(M,X) \to \Sigma_n$ which only remembers the permutation of the basepoint configuration by u (this is part of the canonical functor $\mathcal{B}(M,X) \to \Sigma$ from §2.4). Write $G_n \coloneqq \pi_1 C_n(M,X)$ and define $G_n^k \coloneqq u^{-1}(\Sigma_{n-k} \times \Sigma_k)$. To define the *height* of a functor $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ we need the following decomposition result:

Proposition 3.3 *Let $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ be any functor, and recall that we write $T_n \coloneqq T((x_0)^n)$. Then for $k = 0,\dots,n$ there is a direct summand (as abelian groups) T_n^k of T_n such that the action of $G_n^k \leqslant G_n$ on T_n preserves it: so it is also a direct summand as a G_n^k-module. Moreover, there is a decomposition of T_n as a G_n-module:*

$$T_n \cong \bigoplus_{k=0}^{n} \left(\mathbb{Z}G_n \otimes_{\mathbb{Z}G_n^k} T_n^k \right). \tag{3.2}$$

This identification is natural in the sense that $\iota_n\colon T_n \to T_{n+1}$ sends T_n^k into T_{n+1}^k, and the map of the right-hand side induced by ι_n and $(s_n)_*$ corresponds under (3.2) to ι_n on the left-hand side.

Remark 3.4 (*Related decompositions*) This is similar to the cross-effect decomposition of a functor from a pointed monoidal category (a monoidal category whose unit object is also initial and terminal) to an abelian category, which appears in [HPV12, Proposition 2.4] (see also [DV13, Proposition 1.6]), and the idea of which goes back to Eilenberg and MacLane [EML54, §9]. However, our category $\mathcal{B}(M,X)$ is not in general monoidal (it is when M is of the form $\mathbb{R} \times N$), so this setup does not cover our situation. A similar cross-effect decomposition appears in [HV11, Proposition 1.4] for functors from a source category which has finite coproducts – however, $\mathcal{B}(M,X)$ also does not have finite coproducts. Yet another similar decomposition appears in [CDG13, Lemme 2.7(3)] for functors from a source category which is a wreath product $\mathcal{C} \wr \Lambda$, where $\mathcal{C}$ is any category and $\Sigma \leqslant \Lambda \leqslant \mathsf{Se}^{\mathsf{fin}}$. Here, $\mathsf{Se}^{\mathsf{fin}}$ is the category of finite sets and partially-defined functions and Σ is its subcategory of partially-defined injections, as in Definition 2.4. Our category $\mathcal{B}(M,X)$ may be written as a wreath product $PX \wr \mathcal{B}(M)$, where PX is the path category of X and the wreath product is defined using the projection $\mathcal{B}(M) \to \Sigma$. This is however not of the form considered in [CDG13], unless M is simply-connected and of dimension at least 3 (see §2.4).

Since none of the existing decompositions in the literature covers the general case that we require, we give a complete proof of the decomposition (3.2) in our situation (i.e. Proposition 3.3). This is a little technical, so the reader may wish to skip directly to Definition 3.12 at this point. Before embarking upon the proof of Proposition 3.3, we point out a correction.

Remark 3.5 (*A correction*) We should mention that the proof of the decomposition in Lemme 2.7(3) of [CDG13] contains an error. We will briefly explain the error and sketch a corrected proof of their decomposition. See [CDG13, §2.1] for any unexplained notation. The first part of their proof establishes a decomposition

$$T(C) = \bigoplus_{P \subseteq \mathcal{P}(E)} \bigcap_{A \in P} T_{A,M}(C), \tag{3.3}$$

where $T_{A,S}(C)$ is defined to be $\ker(T(d_{C,A})) \cap \mathrm{im}(T(d_{C,S}))$, $M = M_P$ is defined to be $\bigcap(\mathcal{P}(E) \smallsetminus P)$ and the notation $\mathcal{P}(E)$ means the power set of E.[1] The aim is then to show that this is equal to

$$\bigoplus_{S \subseteq E} \bigcap_{A \in Q_S} T_{A,S}(C), \tag{3.4}$$

where we define $Q_S \coloneqq \{A \in \mathcal{P}(E) \mid A \subsetneq S\}$. Define also $R_S \coloneqq \{A \in \mathcal{P}(E) \mid A \not\supseteq S\}$ and note that $Q_S \subseteq R_S$ with equality exactly when $S = E$. They state that $T_{A,S}(C) = 0$ whenever $A \notin Q_S$, but in fact this is only true under the stronger assumption that $A \notin R_S$. We may therefore restrict the direct sum in (3.3) to those P such that $P \subseteq R_{M_P}$ (rather than $P \subseteq Q_{M_P}$, as claimed). The P with this property are precisely the subsets R_S for $S \subseteq E$. Moreover, the function $R \colon \mathcal{P}(E) \to \mathcal{P}(\mathcal{P}(E))$ given by $S \mapsto R_S$ is injective (in contrast to the function Q), so we see that (3.3) is equal to

$$\bigoplus_{S \subseteq E} \bigcap_{A \in R_S} T_{A,S}(C). \tag{3.5}$$

The final step of the proof is to show that restricting each intersection to the subset Q_S of R_S does not change it. The subset Q_S is coinitial in R_S, but the function $\mathcal{P}(E) \to \mathcal{P}(T(C))$ given by $A \mapsto T_{A,S}(C)$ is non-*increasing*, so this does not help us. Instead, this follows from the facts that $T_{A,S}(C) = T_{A\cap S,S}(C)$ and $\{A \cap S \mid A \in R_S\} = Q_S$.

An alternative correction to the proof of Lemme 2.7(3) of [CDG13] was pointed out to us later by Aurélien Djament, which we also briefly sketch. The decomposition (3.3) arises from the family of pairwise-commuting idempotents $\{T(d_{C,S}) \mid S \subseteq E\}$ of $T(C)$. If we instead consider the subfamily $\{T(d_{C,E\smallsetminus\{s\}}) \mid s \in E\}$, the corresponding decomposition is

$$T(C) = \bigoplus_{S \subseteq E} \bigcap_{s \in S} T_{E\smallsetminus\{s\},S}(C). \tag{3.6}$$

Using the fact that $T_{A,S}(C) = T_{A\cap S,S}(C)$, we may replace $T_{E\smallsetminus\{s\},S}(C)$ with $T_{S\smallsetminus\{s\},S}(C)$ on the right-hand side. Note that $\{S \smallsetminus \{s\} \mid s \in S\}$ is cofinal in Q_S and the function $\mathcal{P}(E) \to \mathcal{P}(T(C))$ given by $A \mapsto T_{A,S}(C)$ is non-increasing, so this is equal to (3.4).

We now prove Proposition 3.3, for which we will need the following definitions.

Definition 3.6 For $S \subseteq \{1,\ldots,n\} \eqqcolon \underline{n}$ let $f_S \colon (x_0)^n \to (x_0)^n$ be the endomorphism in $\mathcal{B}(M,X)$ given by the constant path in $C_{|S|}(M,X)$ on the configuration $\{(a_i,x_0) \mid i \in \underline{n} \smallsetminus S\}$. So this is the endomorphism which "forgets" the points a_i for $i \in S$ and is the identity elsewhere.

Definition 3.7 For $p \geqslant 0$ and $\{S_1,\ldots,S_p\}$ a partition of $S \subseteq \underline{n}$ define

$$T_n[S_1|\cdots|S_p] \;\coloneqq\; \mathrm{im}(Tf_{\underline{n}\smallsetminus S}) \cap \bigcap_{i=1}^{p} \ker(Tf_{S_i}).$$

Note that the induced maps $Tf_S \colon T_n \to T_n$ are not in general G_n-module homomorphisms, so these are subgroups but not sub-G_n-modules.

We will write S^δ for the *discrete* partition of S, and define

$$T_n^k \;\coloneqq\; T_n[\{n{-}k{+}1,\ldots,n\}^\delta].$$

[1] There is a typo in [CDG13], where M is incorrectly defined to be $\bigcap P$, rather than $\bigcap(\mathcal{P}(E) \smallsetminus P)$.

Remark 3.8 A few immediate observations are the following: Each $Tf_S\colon T_n \to T_n$ is idempotent. The composition of Tf_{S_1} and Tf_{S_2} is $Tf_{S_1 \cup S_2}$, so in particular the Tf_S for $S \subseteq \underline{n}$ all pairwise commute. By definition $T_n[\,] = \mathrm{im}(Tf_{\underline{n}})$, and since $f_\varnothing = \mathrm{id}$ we also have $T_n[\underline{n}] = \mathrm{im}(Tf_\varnothing) \cap \ker(Tf_{\underline{n}}) = \ker(Tf_{\underline{n}})$, so:

$$T_n = \mathrm{im}(Tf_{\underline{n}}) \oplus \ker(Tf_{\underline{n}}) = T_n[\,] \oplus T_n[\underline{n}]. \tag{3.7}$$

The following lemma is less immediate but can be proved by some diagram-chasing and drawing little cartoons like (2.1) and (2.2). We will give a proof in symbols.

Lemma 3.9 *For $k \leqslant m \leqslant n$, the map*

$$\iota_m^n := \iota_{n-1} \circ \cdots \circ \iota_m \colon T_m \to T_n$$

is split-injective and sends T_m^k into T_n^k. Moreover, its restriction to a map $T_m^k \to T_n^k$ is a bijection. Hence any left-inverse for ι_m^n restricts to a bijection $T_n^k \to T_m^k$.

Proof. As mentioned in §3.1, each $\iota_{\underline{x}}$ has a natural left-inverse $\pi_{\underline{x}}$ – these compose to give a left-inverse π_m^n for ι_m^n. Just as for ι_n and π_n, by an abuse of notation we will denote the induced map $Tf_S\colon T_n \to T_n$ also by f_S.

We now show that $\iota_m(T_m^k) \subseteq T_{m+1}^k$, and hence by induction that $\iota_m^n(T_m^k) \subseteq T_n^k$. Suppose $x = \iota_m(y)$ for $y \in T_m^k$. Then by definition $y = f_{\{1,\ldots,m-k\}}(z)$ for some $z \in T_m$. Since $\pi_m\colon T_{m+1} \to T_m$ is split-surjective we have $z = \pi_m(w)$ for some $w \in T_{m+1}$. Hence

$$x = \iota_m \circ f_{\{1,\ldots,m-k\}} \circ \pi_m(w) = f_{\{1,\ldots,m-k+1\}}(w). \tag{3.8}$$

For any $m-k+2 \leqslant i \leqslant m+1$ we have

$$f_{\{i\}}(x) = f_{\{i\}} \circ \iota_m(y) = \iota_m \circ f_{\{i-1\}}(y) = \iota_m(0) = 0, \tag{3.9}$$

since $y \in T_m^k$. The two properties (3.8) and (3.9) verify that $x \in T_{m+1}^k$.

Now we show that the restriction of ι_m to $T_m^k \to T_{m+1}^k$ is a bijection, and hence by induction that the restriction of ι_m^n to $T_m^k \to T_n^k$ is a bijection. Suppose $x \in T_{m+1}^k$, and define $z := \pi_m(x) \in T_m$. Then

$$\iota_m(z) = \iota_m \circ \pi_m(x) = f_{\{1\}}(x).$$

But note that $x = f_{\{1,\ldots,m-k+1\}}(y)$ for some $y \in T_{m+1}$, so

$$\begin{aligned} f_{\{1\}}(x) &= f_{\{1\}} \circ f_{\{1,\ldots,m-k+1\}}(y) \\ &= f_{\{1,\ldots,m-k+1\}}(y) \qquad \text{(by Remark 3.8 and since } k \leqslant m) \\ &= x. \end{aligned}$$

So it remains to prove that $z \in T_m^k$. Firstly,

$$z = \pi_m \circ f_{\{1,\ldots,m-k+1\}}(y) = f_{\{1,\ldots,m-k\}} \circ \pi_m(y).$$

Secondly, for any $m-k+1 \leqslant i \leqslant m$, we have

$$\iota_m \circ f_{\{i\}}(z) = f_{\{i+1\}} \circ \iota_m(z) = f_{\{i+1\}}(x) = 0,$$

since $x \in T_{m+1}^k$. But ι_m is split-injective, so $f_{\{i\}}(z) = 0$. These two facts verify that $z \in T_m^k$. $\square$

The following lemma will allow us to construct the required decomposition by induction:

Lemma 3.10 *For all $\{S_1, \ldots, S_p\}$ partitioning $S \subseteq \underline{n}$ with $p \geqslant 2$, there is a split short exact sequence*

$$0 \to T_n[S_1|\cdots|S_p] \hookrightarrow T_n[S_1 \sqcup S_2|\cdots|S_p] \twoheadrightarrow T_n[S_1|S_3|\cdots|S_p] \oplus T_n[S_2|\cdots|S_p] \to 0.$$

The first map is the inclusion, and a section of the second map is given by the inclusion of each of the two factors. So in other words we have a decomposition

$$T_n[S_1 \sqcup S_2|\cdots|S_p] = T_n[S_1|\cdots|S_p] \oplus T_n[S_2|\cdots|S_p] \oplus T_n[S_1|S_3|\cdots|S_p].$$

Proof. One can check from the definitions that the following facts are true:

1. Tf_{S_2} restricts to a map $T_n[S_1 \sqcup S_2|\cdots|S_p] \to T_n[S_1|S_3|\cdots|S_p]$, and similarly Tf_{S_1} restricts to a map $T_n[S_1 \sqcup S_2|\cdots|S_p] \to T_n[S_2|\cdots|S_p]$.
2. $T_n[S_1|S_3|\cdots|S_p]$ and $T_n[S_2|\cdots|S_p]$ are contained in $T_n[S_1 \sqcup S_2|\cdots|S_p]$.
3. For $\{i,j\} \subseteq \{1,2\}$ if $x \in T_n[S_i|S_3|\cdots|S_p]$, then $Tf_{S_j}(x)$ is x when $i \neq j$ and 0 when $i = j$.

These facts imply that the map (Tf_{S_2}, Tf_{S_1}) restricts to the required split surjection (with a section given by inclusion of each factor). The kernel of this is

$$\begin{aligned} &T_n[S_1 \sqcup S_2|S_3|\cdots|S_p] \cap \ker(Tf_{S_1}) \cap \ker(Tf_{S_2}) \\ &= \mathrm{im}(Tf_{\underline{n} \smallsetminus S}) \cap \bigcap_{i=3}^{p} \ker(Tf_{S_i}) \cap \ker(Tf_{S_1 \sqcup S_2}) \cap \ker(Tf_{S_1}) \cap \ker(Tf_{S_2}) \\ &= T_n[S_1|\cdots|S_p], \end{aligned}$$

since $\ker(Tf_{S_1}) \subseteq \ker(Tf_{S_1 \sqcup S_2})$. □

We can now use this to inductively prove a more general decomposition:

Lemma 3.11 *For any $\varnothing \neq S \subseteq \underline{n}$ and $R \subseteq \underline{n} \smallsetminus S$ there is a decomposition*

$$T_n[S|R^\delta] = \bigoplus_{\varnothing \neq Q \subseteq S} T_n[(Q \sqcup R)^\delta]. \tag{3.10}$$

As before, Q^δ denotes the discrete partition of the set Q, so for example $T_n[\{1,2\}|\{3,4,5\}^\delta]$ means $T_n[\{1,2\}|\{3\}|\{4\}|\{5\}]$. Note that this decomposition is an equality of subgroups, not just an abstract isomorphism of groups.

Proof. The $|S| = 1$ case is obvious, so we assume that $|S| \geqslant 2$ and assume the theorem for smaller values of $|S|$ by induction. Pick an element $s \in S$. Then by Lemma 3.10,

$$T_n[S|R^\delta] \;=\; T_n[S \smallsetminus \{s\}|(R \sqcup \{s\})^\delta] \;\oplus\; T_n[S \smallsetminus \{s\}|R^\delta] \;\oplus\; T_n[\{s\}|R^\delta].$$

Apply the inductive hypothesis to the right-hand side. The proposition then follows from the observation that for $\varnothing \neq Q \subseteq S$, exactly one of the following holds: (i) $s \in Q$ but $Q \neq \{s\}$; (ii) $s \notin Q$; (iii) $Q = \{s\}$. □

We can now use this to deduce the decomposition we want:

Proof of Proposition 3.3. Combining (3.10) (setting $R := \varnothing$ and $S := \underline{n}$) with (3.7) we obtain:

$$T_n = \bigoplus_{k=0}^{n} \bigoplus_{\substack{Q \subseteq \underline{n} \\ |Q| = k}} T_n[Q^\delta]. \tag{3.11}$$

The action of G_n on T_n permutes the summands via the projection $G_n \to \Sigma_n$ and the obvious action of Σ_n on subsets of $\underline{n}$. So:

- $T_n^k = T_n[\{n-k+1, \ldots, n\}^\delta]$ is preserved by the action of $G_n^k \leqslant G_n$ on T_n.
- The G_n-action on T_n preserves the outer direct sum.
- The inner direct sum is the induced module $\mathrm{Ind}_{G_n^k}^{G_n} T_n^k = \mathbb{Z}G_n \otimes_{\mathbb{Z}G_n^k} T_n^k$.

This proves the decomposition of G_n-modules (3.2). We proved in Lemma 3.9 above that $\iota_n \colon T_n \to T_{n+1}$ sends T_n^k into T_{n+1}^k, and the naturality statement is clear. □

Having established this decomposition we can now define the *height* of a twisted coefficient system:

Definition 3.12 The *height* of a functor $T \colon \mathcal{B}(M,X) \to \mathsf{Ab}$ is the height at which the decomposition (3.2) is truncated. More precisely, we define $\mathrm{height}(T)$ by: $\mathrm{height}(T) \leqslant h$ if and only if $T_n^k = 0$ for all $k > h$ and all n. (So in particular $\mathrm{height}(T) = -1$ if and only if $T = 0$.)

3.3. Height and degree. These two notions are related as follows:

Lemma 3.13 *For any functor* $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$*,* $\mathrm{height}(T) \leqslant \deg(T)$*.*

This inequality is useful because having an upper bound on the *height* of a twisted coefficient system is what is needed to prove Theorem 1.3, whereas it is often easier to find an upper bound on the *degree* in examples.

Proof. We will use induction on d to prove the statement

$$\deg(T) \leqslant d \;\Rightarrow\; \mathrm{height}(T) \leqslant d \tag{IH$_d$}$$

for all $d \geqslant -1$, using the decomposition (3.11) above, which we restate as:

$$T_n \;=\; \bigoplus_{S \subseteq \underline{n}} T_n[S^\delta]. \tag{3.12}$$

In this notation the height of T is determined by $\mathrm{height}(T) \leqslant d$ if and only if $T_n[S^\delta] = 0$ for all $|S| > d$ and all n.

When $d = -1$ the definitions of height and degree coincide. This deals with the base case, so let $d \geqslant 0$ and assume that (IH$_{d-1}$) holds. For all n we have a split short exact sequence $0 \to T_n \to T_{n+1} \to \Delta T_n \to 0$. Applying (3.12), this is

$$0 \to \bigoplus_{S \subseteq \underline{n}} T_n[S^\delta] \longrightarrow \bigoplus_{R \subseteq \underline{n+1}} T_{n+1}[R^\delta] \longrightarrow \bigoplus_{Q \subseteq \underline{n}} \Delta T_n[Q^\delta] \to 0.$$

Analysing the maps carefully we see that
(a) $T_n[S^\delta]$ is sent isomorphically onto $T_{n+1}[(S+1)^\delta]$ by the first map.
(b) $T_{n+1}[(Q \sqcup \{1\})^\delta]$ is sent isomorphically onto $\Delta T_n[(Q-1)^\delta]$ by the second map.
Suppose that $\deg(T) \leqslant d$. Then $\deg(\Delta T) \leqslant d-1$ by the definition of degree, and so by the inductive hypothesis (IH$_{d-1}$), $\mathrm{height}(\Delta T) \leqslant d-1$. By fact (b) above this implies that

$$T_{n+1}[R^\delta] = 0 \text{ whenever } |R| > d \text{ and } 1 \in R. \tag{3.13}$$

For any fixed k, the subgroups $\{T_{n+1}[R^\delta] \mid |R| = k\}$ are all abstractly isomorphic via the action of G_{n+1} on T_{n+1}. Also note that $d \geqslant 0$, so that $|R| > 0$, i.e. $R \neq \varnothing$. Hence:

$$T_{n+1}[R^\delta] = 0 \text{ for } \textit{all } |R| > d. \tag{3.14}$$

Therefore by (a), $T_n[S^\delta] = 0$ for all $|S| > d$; in other words, $\mathrm{height}(T) \leqslant d$. □

Remark 3.14 To prove that $\mathrm{height}(T) = \deg(T)$, one could try to reverse the argument above to get the other inequality. This goes wrong in one place though: Above we were able to deduce (3.14) from (3.13) because for every $|R| > d$, there is an R' of the same cardinality which contains 1. However, for the converse we would need to deduce (3.14) from:

$$T_{n+1}[R^\delta] = 0 \text{ whenever } |R| > d \text{ and } 1 \notin R. \tag{3.15}$$

Now there *is* a subset $R \subseteq \underline{n+1}$ for which there does not exist $R' \subseteq \underline{n+1}$ of the same cardinality and not containing 1 – namely $\underline{n+1}$ itself. This is the basic asymmetry which prevented us from proving an *equality* between height and degree.

Remark 3.15 The notion of *height* in this note is the same as the notion of degree in [Bet02] (for twisted coefficient systems for symmetric groups) and [Dwy80] (for general linear groups), whereas the notion of *degree* in this note is in the same spirit as the notion of degree in [Iva93], [CM09] and [Bol12] (for mapping class groups of surfaces) and [Wah14] (for automorphism groups in a general categorical setting). Hence Lemma 3.13 provides a link between these two notions of degree.

We finish this section with a few immediate facts about the degree of a twisted coefficient system.

Lemma 3.16 *For twisted coefficient systems $T, T' \colon \mathcal{B}(M,X) \to \mathsf{Ab}$ and a fixed abelian group A,*
(a) $\deg(T \oplus T') = \max\{\deg(T), \deg(T')\}$,
(b) $\deg(T \otimes A) \leqslant \deg(T)$,
and more generally, for $\deg(T)$ and $\deg(T')$ non-negative,
(c) $\deg(T \otimes T') \leqslant \deg(T) + \deg(T')$,
where $\oplus$ and $\otimes$ are defined objectwise.

Proof. Fact (a) follows by induction from the fact that $\Delta(T \oplus T') = \Delta T \oplus \Delta T'$. Fact (b) follows from the fact that $\Delta(T \otimes A) = \Delta T \otimes A$, which is true because tensoring a *split* short exact sequence with A preserves split-exactness. Fact (c) is proved by induction with base case (b), and inductive step using the fact that

$$\Delta(T \otimes T') \;=\; (T \otimes \Delta T') \oplus (\Delta T \otimes T') \oplus (\Delta T \otimes \Delta T'). \qquad \square$$

4. Examples of twisted coefficient systems

Recall from Definition 2.4 that the category Σ has objects the natural numbers including zero, and morphisms the partially-defined injections. We will give some examples of functors $T \colon \Sigma \to \mathsf{Ab}$, which are twisted coefficient systems for the special case $M = \mathbb{R}^\infty$ and $X = *$ since $\mathcal{B}(\mathbb{R}^\infty) \cong \Sigma$. However, recall (§2.4) that there is a canonical functor $U \colon \mathcal{B}(M,X) \to \Sigma$ for each (M,X), so these examples also give twisted coefficient systems in general. Moreover, one can check (see §3 for the notation) that $\Delta(T \circ U) = \Delta T \circ U$, so by induction $\deg(T \circ U) = \deg(T)$, and also $(T \circ U)^k_n = T^k_n$, so $\mathrm{height}(T \circ U) = \mathrm{height}(T)$.

Example 4.1 Fix a path-connected based space $(Z,*)$, an integer $q \geqslant 0$ and a field F. The functor $\hat{T}_Z \colon \Sigma \to \mathsf{Top}$ is defined on objects by $n \mapsto Z^n$, and on morphisms as follows: given a partially-defined injection $j \colon \{1,\ldots,m\} \dashrightarrow \{1,\ldots,n\}$ in Σ, define $\hat{T}_Z(j) \colon Z^m \to Z^n$ to be the map

$$(z_1,\ldots,z_m) \mapsto (z_{j^{-1}(1)},\ldots,z_{j^{-1}(n)}),$$

where $z_\varnothing$ is taken to mean the basepoint $*$. For example:

 : $(z_1,z_2,z_3) \mapsto (*,z_1,*,z_2)$.

The functor $T_{Z,q,F} \colon \Sigma \to \mathsf{Ab}$ is then the composite functor $H_q(-;F) \circ \hat{T}_Z$.

Lemma 4.2 *The twisted coefficient system $T_{Z,q,F}$ has degree at most $\lfloor \frac{q}{h+1} \rfloor$, where for a path-connected space Z,*

$$h = hconn_F(Z) \coloneqq \max\{k \geqslant 0 \mid \widetilde{H}_i(Z;F) = 0 \text{ for all } i \leqslant k\} \geqslant 0.$$

Proof. First note that the Künneth theorem gives us natural split short exact sequences

$$0 \to H_q(Z^n;F) \longrightarrow H_q(Z^{n+1};F) \longrightarrow \bigoplus_{i=1}^{q} H_{q-i}(Z^n;F) \otimes_F H_i(Z;F) \to 0, \tag{4.1}$$

which together with the fact that $H_i(Z;F) = 0$ for $1 \leqslant i \leqslant h$ implies that

$$\Delta T_{Z,q,F} = \bigoplus_{i=h+1}^{q} T_{Z,q-i,F} \otimes_F H_i(Z;F). \tag{4.2}$$

So, by Lemma 3.16 above, $\deg(T_{Z,q,F}) \leqslant 1 + \max\{\deg(T_{Z,q-i,F}) \mid h+1 \leqslant i \leqslant q\}$. Abbreviating $\deg(T_{Z,q,F})$ to t_q, we have the recurrence inequality

$$t_q \;\leqslant\; 1 + \max\{t_0,\ldots,t_{q-h-1}\}. \tag{4.3}$$

Note that $H_0(Z^n;F) \to H_0(Z^{n+1};F)$ is the identity map $F \to F$ for all n, so $\Delta T_{Z,0,F} = 0$, and hence $\deg(T_{Z,0,F}) = 0$. Also note that for $1 \leqslant q \leqslant h$, $hconn_F(Z) \geqslant q$ implies that $hconn_F(Z^n) \geqslant q$

for all n (by the Künneth theorem), so $T_{Z,q,F}(n) = H_q(Z^n;F) = 0$, and hence $\deg(T_{Z,q,F}) = -1 \leqslant 0$. So we also have the initial conditions

$$t_0, t_1, \ldots, t_h \leqslant 0. \tag{4.4}$$

It now remains to prove that the recurrence inequality (4.3) and the initial conditions (4.4) imply that $t_q \leqslant \lfloor \frac{q}{h+1} \rfloor$ for all $q \geqslant 0$. This will be done by induction on q. The base case is $0 \leqslant q \leqslant h$ which is covered by the initial conditions (4.4). Assume that $q \geqslant h+1$. Then:

$$\begin{aligned} t_q &\leqslant 1 + \max\{t_0, \ldots, t_{q-h-1}\} \\ &\leqslant 1 + \lfloor \tfrac{q-h-1}{h+1} \rfloor \\ &= \lfloor \tfrac{q}{h+1} \rfloor \end{aligned}$$

□

Remark 4.3 See also [Han09b, Proposition 12], where it is proved (in the terminology of this note) that the *height* of $T_{Z,q,F}$ is at most q.

Remark 4.4 If, in Lemma 4.2, we replace F by a general principal ideal domain R (such as $\mathbb{Z}$), the short exact sequence (4.1) becomes

$$\begin{aligned} 0 \to H_q(Z^n;R) \longrightarrow H_q(Z^{n+1};R) \longrightarrow & \bigoplus_{i=1}^{q} H_{q-i}(Z^n;R) \otimes_R H_i(Z;R) \\ & \oplus \bigoplus_{i=1}^{q} \mathrm{Tor}^R(H_{q-i}(Z^n;R), H_{i-1}(Z;R)) \to 0. \end{aligned} \tag{4.5}$$

Here, as in (4.1), we have used the splitting in the Künneth short exact sequence to move some summands from the left-hand side to the right-hand side. However, this splitting is not always natural, and so (4.5) is not natural for general principal ideal domains R. When $R = F$ is a field, the Tor terms vanish and the Künneth short exact sequence is of the form $0 \to A \to B \to 0 \to 0$, so its splitting is certainly natural in this case.[2] This is the reason why the short exact sequence (4.1) *is* natural – which was necessary to deduce the isomorphism of functors (4.2). More generally, the Tor terms vanish if $H_*(Z;R)$ is flat over R in each degree, so the most general version of Example 4.1 works for a principal ideal domain R and path-connected space Z satisfying this condition. In particular, if $H_*(Z;\mathbb{Z})$ is torsion-free, this example works for homology with integral coefficients.

Notation 4.5 Write $\mathbb{N} = \mathbb{Z}_{>0}$. For $k \in \mathbb{N}$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k$ define $|\lambda| \coloneqq \lambda_1 + \cdots + \lambda_k$. For $\ell \in \mathbb{N}$, define $\lambda \vdash \ell$ to be the statement

$$\lambda \in \mathbb{N}^k \text{ for some } k \in \mathbb{N} \text{ and } |\lambda| = \ell.$$

In words, λ is an *ordered partition* of ℓ of length k. For a set S with $|S| \geqslant \ell$, an *ordered decomposition* of S of type λ is a tuple $(S_1, \ldots, S_k)$ of pairwise disjoint subsets $S_i \subseteq S$ such that $|S_i| = \lambda_i$. Note that this decomposes S into either k or $k+1$ subsets, depending on whether $|S| = \ell$ or $|S| > \ell$. As a final piece of notation, define $\lambda[n] = \lambda$ for $n = \ell$ and

$$\lambda[n] = (n - \ell, \lambda_1, \ldots, \lambda_k)$$

for $n > \ell$, so that $\lambda[n] \vdash n$. We note that this notation has a slightly different meaning compared with its appearance in Remarks 1.6 and 1.11, which involve *unordered* partitions (corresponding to Young diagrams), rather than (as in this section) *ordered* partitions.

Example 4.6 Let $\mathsf{Se}^{\mathsf{fin}}$ be the category of finite sets and partially-defined functions. Note that this is equivalent[3] to the category $\mathsf{Set}_*^{\mathsf{fin}}$ of finite pointed sets. There is a free functor $\mathbb{Z}(-)\colon \mathsf{Se}^{\mathsf{fin}} \to \mathsf{Ab}$ taking S to $\mathbb{Z}S$ and taking a partially-defined function $j\colon S \dashrightarrow R$ to the homomorphism

$$\sum_{s \in S} n_s s \mapsto \sum_{s \in S} n_s j(s), \tag{4.6}$$

[2] This just comes from the fact that a natural transformation is invertible if it is objectwise invertible.
[3] Although not isomorphic, for essentially set-theoretic reasons.

where $j(s)$ means $0 \in \mathbb{Z}R$ if j is undefined on s. So any functor $\Sigma \to \mathsf{Se}^{\mathsf{fin}}$ gives a twisted coefficient system for Σ by composing with $\mathbb{Z}(-)$.

We now define a functor $P_\lambda \colon \Sigma \to \mathsf{Se}^{\mathsf{fin}}$ associated to any $\lambda \vdash \ell$. On objects, it is defined by

$$P_\lambda(\underline{n}) = P_\lambda(n) = \begin{cases} \{\text{ordered decompositions of } \underline{n} \text{ of type } \lambda\} & n \geqslant \ell \\ \varnothing & n < \ell. \end{cases}$$

Given a partially-defined injection $j\colon \{1,\ldots,m\} \dashrightarrow \{1,\ldots,n\}$, we define $P_\lambda(j)\colon P_\lambda(m) \dashrightarrow P_\lambda(n)$ as follows. First, if $m < \ell$ or $n < \ell$ then $P_\lambda(j)$ is the empty function. If $m, n \geqslant \ell$ and $(S_1,\ldots,S_k) \in P_\lambda(m)$, then $P_\lambda(j)$ is defined on $(S_1,\ldots,S_k)$ exactly when j is defined on every element of $\bigcup_{i=1}^k S_i$, in which case its value is $(j(S_1),\ldots,j(S_k)) \in P_\lambda(n)$.

Note that, when $\lambda = 1$, the functor P_λ is simply the inclusion of Σ as a subcategory of $\mathsf{Se}^{\mathsf{fin}}$. There is a natural action of Σ_n on $P_\lambda(n)$, since Σ_n is the automorphism group of $\underline{n}$ in Σ, and an isomorphism

$$\mathbb{Z}P_\lambda(n) \;\cong\; \mathbb{Z}\big[\Sigma_n/\Sigma_{\lambda[n]}\big]$$

of $\mathbb{Z}[\Sigma_n]$-modules, where we write Σ_μ for the subgroup $\Sigma_{\mu_1} \times \cdots \times \Sigma_{\mu_k}$ of $\Sigma_{|\mu|}$. Note that the right-hand side is only defined for $n \geqslant \ell$. In particular, when $\lambda = (1,\ldots,1)$ with $|\lambda| = \ell$, we have $\mathbb{Z}P_\lambda(n) \cong \mathbb{Z}[\Sigma_n/\Sigma_{n-\ell}]$.

We have the following isomorphisms in Ab for $|\lambda| \geqslant 2$:

$$\begin{aligned} \Delta\mathbb{Z}P_\lambda(n) &\cong \mathbb{Z}\big\{(S_1,\ldots,S_k) \in P_\lambda(n+1) \mid 1 \in \textstyle\bigcup_{i=1}^k S_i\big\} \\ &\cong \mathbb{Z}\big(\textstyle\bigsqcup_{i=1}^k P_{\lambda-e_i}(n)\big) \\ &\cong \textstyle\bigoplus_{i=1}^k \mathbb{Z}P_{\lambda-e_i}(n), \end{aligned}$$

where $\lambda - e_i$ is the ordered partition $(\lambda_1,\ldots,\lambda_i - 1,\ldots,\lambda_k)$.[4] The first and third isomorphisms are obviously natural isomorphisms of functors $\Sigma \to \mathsf{Ab}$, and one can also explicitly check that the second isomorphism is natural. Hence we have an isomorphism

$$\Delta\mathbb{Z}P_\lambda \;\cong\; \bigoplus_{i=1}^k \mathbb{Z}P_{\lambda-e_i} \tag{4.7}$$

for $|\lambda| \geqslant 2$. This allows us to prove:

Lemma 4.7 *The twisted coefficient system $\mathbb{Z}P_\lambda$ has degree $|\lambda|$.*

Proof. The proof is by induction on $|\lambda|$. First, if $|\lambda| = 1$ then $\Delta\mathbb{Z}P_\lambda(n) \cong \mathbb{Z}$ for all $n \geqslant 0$. Hence all morphisms in Σ are sent to endomorphisms of $\mathbb{Z}$ in Ab. But all morphisms in Σ have one-sided inverses, so their images in Ab are endomorphisms of $\mathbb{Z}$ admitting one-sided inverses, and hence automorphisms. Thus $\Delta\mathbb{Z}P_\lambda$ has degree 0 (c.f. Example 3.2) and so $\mathbb{Z}P_\lambda$ has degree 1 by definition.

Now assume that $|\lambda| \geqslant 2$. By (4.7), Lemma 3.16 and the inductive hypothesis, we have:

$$\deg(\Delta\mathbb{Z}P_\lambda) = \deg\Big(\bigoplus_{i=1}^k \mathbb{Z}P_{\lambda-e_i}\Big) = \max_{i=1,\ldots,k}(\deg(\mathbb{Z}P_{\lambda-e_i})) = |\lambda| - 1,$$

so $\deg(\mathbb{Z}P_\lambda) = |\lambda|$ by the definition of degree. □

Remark 4.8 Given an arbitrary ring R, there is also a functor $R(-)\colon \mathsf{Se}^{\mathsf{fin}} \to \mathsf{Ab}$ taking a set S to the free R-module generated by S (viewed as an abelian group) and with morphisms defined by the same formula (4.6) as for $\mathbb{Z}(-)$. Thus we have twisted coefficient systems $RP_\lambda\colon \Sigma \to \mathsf{Ab}$ associated to any ring R and ordered partition λ. Just as in the case $R = \mathbb{Z}$, we have isomorphisms $RP_\lambda(n) \cong R[\Sigma_n/\Sigma_{\lambda[n]}]$ of $R[\Sigma_n]$-modules for all n, and the twisted coefficient system RP_λ has degree $|\lambda|$. To see this, we can adapt the proof of Lemma 4.7 directly, as long as we are slightly more careful about the base case. It is not in general true that the monoid $\mathrm{End}_{\mathsf{Ab}}(R)$ has the property that any one-sided inverse is a two-sided inverse (consider $R = \prod^\infty \mathbb{Z}$ for example), so the base case does not come for free. However, one can explicitly compute the maps $\Delta RP_\lambda(m) \to \Delta RP_\lambda(n)$ induced by any $j\colon \{1,\ldots,m\} \dashrightarrow \{1,\ldots,n\}$ in Σ, and see that they are just the identity on R.

[4] And where $(\lambda_1,\ldots,\lambda_k)$ means $(\lambda_1,\ldots,\lambda_{a-1},\lambda_{a+1},\ldots,\lambda_k)$ if $\lambda_a = 0$.

Remark 4.9 Example 4.6 may also be generalised in a different direction. Fix $\ell > 0$ and write

$$\Lambda_\ell = \{\lambda \mid \lambda \in \mathbb{N}^k \text{ for some } k > 0 \text{ and } \lambda \vdash \ell' \text{ for some } \ell' \leqslant \ell\}.$$

This set has a partial order where $(\lambda_1, \ldots, \lambda_k) \leqslant (\mu_1, \ldots, \mu_l)$ if and only if there exists an injection $\alpha\colon \{1, \ldots, k\} \hookrightarrow \{1, \ldots, l\}$ such that $\lambda_i \leqslant \mu_{\alpha(i)}$ for each $i \in \{1, \ldots, k\}$. For $\lambda \leqslant \mu$, write $[\lambda, \mu]$ for the interval $\{\nu \in \Lambda_\ell \mid \lambda \leqslant \nu \leqslant \mu\}$. We may then define a functor $P_{[\lambda,\mu]}\colon \Sigma \to \mathsf{Se}^{\mathsf{fin}}$ as follows. On objects it is defined by

$$P_{[\lambda,\mu]}(\underline{n}) = \begin{cases} \{\text{ordered decompositions of } \underline{n} \text{ with type} \in [\lambda, \mu]\} & n \geqslant |\lambda| \\ \varnothing & n < |\lambda|. \end{cases}$$

Given a partially-defined injection $j\colon \{1, \ldots, m\} \dashrightarrow \{1, \ldots, n\}$, we define $P_{[\lambda,\mu]}(j)$ to be the empty function if either $m < |\lambda|$ or $n < |\lambda|$; otherwise it takes $(S_1, \ldots, S_k)$ to $(j(S_1), \ldots, j(S_k))$ if this is an ordered decomposition with type $\in [\lambda, \mu]$, and is undefined on $(S_1, \ldots, S_k)$ if not.

Again, we may compose this with the functor $R(-)\colon \mathsf{Se}^{\mathsf{fin}} \to \mathsf{Ab}$ for any ring R to obtain a twisted coefficient system $RP_{[\lambda,\mu]}\colon \Sigma \to \mathsf{Ab}$, which has degree $|\mu|$ by a similar argument to above.

Remark 4.10 (*Burau representations*) There is a presentation of the braid category $\mathcal{B}(\mathbb{R}^2)$ with generators $\sigma_{i,n}^\epsilon$, ι_n and π_{n+1} for $n \geqslant 0$, $i \in \{1, \ldots, n-1\}$ and $\epsilon \in \{+1, -1\}$. The objects are the non-negative integers $\{0, 1, 2, \ldots\}$ and the sources and targets of the generating morphisms are as follows:

$$\sigma_{i,n}^\epsilon\colon n \to n \qquad \iota_n\colon n \to n+1 \qquad \pi_{n+1}\colon n+1 \to n.$$

The relations are:

(i) $\sigma_{i,n}^{-1}$ is an inverse for $\sigma_{i,n} := \sigma_{i,n}^{+1}$

(b) the usual braid relations on $\{\sigma_{i,n}\}_{i=1}^{n-1}$ for each n

(c) the stabilisation and forgetful maps "commute" with the σ maps:

$$\sigma_{i+1,n+1}^\epsilon \iota_n = \iota_n \sigma_{i,n}^\epsilon \qquad \text{and} \qquad \sigma_{i,n}^\epsilon \pi_{n+1} = \pi_{n+1} \sigma_{i+1,n+1}^\epsilon$$

(e) edge effects:

$$\pi_{n+1}(\sigma_{1,n+1})^k \iota_n = \begin{cases} \mathrm{id}_n & \text{for } k \text{ even} \\ \iota_{n-1}\pi_n & \text{for } k \text{ odd.} \end{cases} \tag{4.8}$$

Write $\mathcal{B}_0(\mathbb{R}^2)$ for the subcategory generated by $\{\sigma_{i,n}^\epsilon\}$ and $\mathcal{B}_1(\mathbb{R}^2)$ for the subcategory generated by these together with $\{\iota_n\}$. Note that $\mathcal{B}_0(\mathbb{R}^2)$ is the category with objects $\mathbb{N}$, where every morphism is an automorphism and with $\mathrm{Aut}(n) = \beta_n$, the classical braid group on n strands.

The Burau representation $\beta_n \to \mathrm{Aut}(\mathbb{Z}[t^{\pm 1}]^n) = GL_n(\mathbb{Z}[t^{\pm 1}])$ is defined by sending the generator $\sigma_{i,n}$ of β_n to the matrix $I_{i-1} \oplus \left(\begin{smallmatrix} 1-t & t \\ 1 & 0 \end{smallmatrix}\right) \oplus I_{n-i-1}$. This defines a functor $\mathcal{B}_0(\mathbb{R}^2) \to \mathsf{Ab}$, which easily extends to a functor $\mathbf{bu}\colon \mathcal{B}_1(\mathbb{R}^2) \to \mathsf{Ab}$ by sending ι_n to the inclusion $\mathbb{Z}[t^{\pm 1}]^n \hookrightarrow \mathbb{Z}[t^{\pm 1}]^{n+1}$ that takes $(f_1, \ldots, f_n)$ to $(0, f_1, \ldots, f_n)$. In order to define a twisted coefficient system for $\{C_n(\mathbb{R}^2)\}$ we would need to extend this further to the morphisms $\{\pi_{n+1}\}$. However, $\mathbf{bu}$ does not naturally extend in this way. We can attempt to define $\mathbf{bu}(\pi_{n+1})$ to be the projection $\mathbb{Z}[t^{\pm 1}]^{n+1} \to \mathbb{Z}[t^{\pm 1}]^n$ that takes $(f_1, \ldots, f_{n+1})$ to $(f_2, \ldots, f_{n+1})$; this satisfies the commutation relations (c), but not the relations (e). Instead, it turns out that the right-hand side of (4.8) is I_n or $(0) \oplus I_{n-1}$ depending on whether k is even or odd, and the left-hand side is equal to

$$\left(\tfrac{(-t)^k + t}{t+1}\right) \oplus I_{n-1}.$$

Note that these are equal when $t = 1$, but for no other values of $t \in \mathbb{C}$ (consider $k = 2$). So this only defines a twisted coefficient system when t is evaluated at 1, in which case the Burau representation is simply the projection $\beta_n \to \Sigma_n$ followed by the permutation representation of Σ_n on $\mathbb{Z}[t^{\pm 1}]$. This twisted coefficient system has degree 1. It seems likely that $\mathbf{bu}$ cannot be extended to $\mathcal{B}(\mathbb{R}^2)$ at all, even if we allow ourselves to restrict to $\mathbf{bu}|_{\mathcal{B}_0(\mathbb{R}^2)}$ and redefine it on $\{\iota_n\}$.

5. A twisted Serre spectral sequence

To prove Theorem 1.3 we will need a generalisation of the basic Serre spectral sequence, allowing the base space to be equipped with a local coefficient system. It is a special case of (the homology version of) an *equivariant* generalisation of the Serre spectral sequence constructed by Moerdijk and Svensson in [MS93]. This section gives a brief description of their spectral sequence and deduces the particular case that we will need.

We start by describing an alternative basepoint-independent viewpoint on (co)homology with local coefficients (in the non-equivariant setting).

Definition 5.1 For a space Y let $\Delta(Y)$ be the category whose objects are all singular simplices in Y, and whose morphisms are simplicial operations (generated by face and degeneracy maps). Denote the fundamental groupoid of Y by $\pi(Y)$, and the standard n-simplex by Δ^n. There is a canonical functor $\upsilon_Y\colon \Delta(Y) \to \pi(Y)$ which takes a singular simplex $\Delta^n \to Y$ to the image of its barycentre b_n. A morphism $\Delta^k \xrightarrow{\alpha} \Delta^n \to Y$ is taken to the image of the straight-line path in Δ^n from $\alpha(b_k)$ to b_n.

A covariant (resp. contravariant) functor $\Delta(Y) \to \mathsf{Ab}$ is a *coefficient system* for homology (resp. cohomology); it is a *local coefficient system* if it factors up to natural isomorphism through υ_Y.

The functor $\upsilon_Y\colon \Delta(Y) \to \pi(Y)$ encapsulates most of the combinatorics needed to define (co)homology with local coefficients. The definition makes sense for any (not necessarily local) coefficient system, but it is only homotopy-invariant for local coefficient systems.

Definition 5.2 (Homology) Given a space Y and coefficient system $M\colon \Delta(Y) \to \mathsf{Ab}$, the homology $H_*(Y;M)$ is the homology of the chain complex $C_*(\Delta(Y);M)$:

$$\xrightarrow{\partial_{n+1}} \bigoplus_{\sigma\in N_n\Delta(Y)} M(\sigma_0) \xrightarrow{\partial_n} \bigoplus_{\tau\in N_{n-1}\Delta(Y)} M(\tau_0) \xrightarrow{\partial_{n-1}}$$

where $N_\bullet\Delta(Y)$ denotes the nerve of the category $\Delta(Y)$, and for a chain of singular simplices $\sigma = (\Delta^{k_0} \to \Delta^{k_1} \to \cdots \to \Delta^{k_n} \to Y)$ of $N_n\Delta(Y)$, the 0th one $\Delta^{k_0} \to Y$ is denoted by σ_0. The map ∂_n is the alternating sum of maps ∂_n^i which are defined using the ith face map of $N_\bullet\Delta(Y)$.[5]

Definition 5.3 (Cohomology) Given a space Y and coefficient system $M\colon \Delta(Y)^{\mathrm{op}} \to \mathsf{Ab}$, the cohomology $H^*(Y;M)$ is the homology of the cochain complex $C^*(\Delta(Y);M)$:

$$\xrightarrow{\delta_{n-1}} \prod_{\sigma\in N_n\Delta(Y)} M(\sigma_0) \xrightarrow{\delta_n} \prod_{\tau\in N_{n+1}\Delta(Y)} M(\tau_0) \xrightarrow{\delta_{n+1}}$$

where the map δ_n is the alternating sum of maps δ_n^i which are defined using the ith face map of $N_\bullet\Delta(Y)$.[6]

This reduces to ordinary (untwisted) homology and cohomology when M is constant. (Although it does not reduce to the usual singular (co)chain complex, one can show that it does compute the same homology as it; cf. [MS93, Theorem 2.2].)

In [MS93] the above is generalised to the equivariant setting: they define $\upsilon_Y\colon \Delta_G(Y) \to \pi_G(Y)$ for a G-space Y, and equivariant twisted cohomology $H^*_G(Y;M)$ for any coefficient system $\Delta_G(Y)^{\mathrm{op}} \to \mathsf{Ab}$. Again a coefficient system is *local* if it factors up to natural isomorphism through υ_Y. Cohomology with respect to local coefficient systems is G-homotopy invariant [MS93, Theorem 2.3]. Their main theorem is the existence of a twisted equivariant Serre spectral sequence:

Theorem 5.4 ([MS93, Theorem 3.2]) *For any G-fibration $f\colon Y \to X$ (i.e. $Y^H \to X^H$ is a fibration for all $H \leqslant G$) and any local coefficient system M on Y, there is a local coefficient system $H^q_G(f;M)$ on X for each $q \geqslant 0$ and a spectral sequence*

$$E_2^{p,q} = H^p_G\big(X; H^q_G(f;M)\big) \;\Rightarrow\; H^*_G(Y;M) \tag{5.1}$$

[5] For $\sigma \in N_n\Delta(Y)$, let τ be its ith face. There is a canonical map $\sigma_0 \to \tau_0$ (which is the identity except when $i=0$) inducing a map $M(\sigma_0) \to M(\tau_0)$. The direct sum of these maps is ∂_n^i.

[6] Given an element $\{g_\sigma \in M(\sigma_0) \mid \sigma \in N_n\Delta(Y)\}$, we need to choose an element of $M(\tau_0)$ for each $\tau \in N_{n+1}\Delta(Y)$. Let σ be the ith face of τ, which has a canonical map $\tau_0 \to \sigma_0$ (which is the identity except when $i=0$). Apply M to get a map $M(\sigma_0) \to M(\tau_0)$ and take the image of g_σ under this map.

with the usual cohomological grading.

Remark 5.5 We will describe the local coefficient system $H^q(f;M)$ in the non-equivariant case. As a functor $\Delta(X)^{\mathrm{op}} \to \mathsf{Ab}$ it does the following. A singular simplex $\Delta^k \xrightarrow{\sigma} X$ is taken to the cohomology $H^q(\sigma^*(Y);M)$, where $\sigma^*(Y)$ is the pullback of σ and f, and we denote any pullback of the coefficients M also by M. A morphism $\Delta^l \xrightarrow{\alpha} \Delta^k \xrightarrow{\sigma} X$ induces a map of pullbacks $(\sigma \circ \alpha)^*(Y) \to \sigma^*(Y)$ and hence a map on cohomology.

It is a *local* coefficient system since it factors up to natural isomorphism through v_X by the following functor $\pi(X)^{\mathrm{op}} \to \mathsf{Ab}$. A point $x \in X$ is taken to $H^q(f^{-1}(x);M)$. Given a homotopy class $[I \xrightarrow{p} X]$ of paths from x to y, there are induced maps of pullbacks $f^{-1}(x) \hookrightarrow p^*(Y) \hookleftarrow f^{-1}(y)$. These induce maps on cohomology, and since they are *isomorphisms*[7] the first one can be inverted to get a composite map $H^q(f^{-1}(x);M) \to H^q(f^{-1}(y);M)$. One can check that this map is independent of the choice of representing path p.

In [MS93] the authors point out that there is an analogous version of the spectral sequence (5.1) for homology. We will only need the non-equivariant (but twisted) version, which is:[8]

Theorem 5.6 *For any fibration $f\colon Y \to X$ and any local coefficient system M on Y, there is a local coefficient system $H_q(f;M)$ on X for each $q \geqslant 0$ and a spectral sequence*

$$E^2_{p,q} = H_p\big(X; H_q(f;M)\big) \;\Rightarrow\; H_*(Y;M) \tag{5.2}$$

with the usual homological grading.

The description of the local coefficient systems $H_q(f;M)$ is the same as above, replacing cohomology with homology. When the local coefficient system M on Y is pulled back from the base X, they are built out of the *untwisted* homology of each fibre.

We now return to the viewpoint of local coefficient systems as an action of the fundamental group on an abelian group. In the special case where the local coefficient system on Y is a pullback of one on X the above can be rephrased as:

Corollary 5.7 *For any fibration $f\colon Y \to X$ with fibre F over the basepoint $x_0 \in X$, and any $\pi_1(X)$-module M, there is a spectral sequence*

$$E^2_{p,q} = H_p\big(X; H_q(F;M)\big) \;\Rightarrow\; H_*(Y;M) \tag{5.3}$$

with the usual homological grading. Here the action of $\pi_1(Y)$ on M is pulled back from that of $\pi_1(X)$ via f_* and the action of $\pi_1(F)$ on M is trivial. The action of $\pi_1(X)$ on $H_q(F;M)$ is induced by its diagonal action on the chain complex $S_*(X) \otimes_{\mathbb{Z}} M$.

This is natural for maps of fibrations in the obvious way:

Proposition 5.8 *Suppose we have a map of fibrations (the vertical maps are fibrations, and the square commutes on the nose):*

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

and a $\pi_1(X')$-module M. Denote the fibres over the basepoints by F and F' respectively. Then there is a map of spectral sequences (5.3) where:

- *The map $F \to F'$ induces a map of untwisted homology $H_q(F;M) \to H_q(F';M)$, which is equivariant w.r.t. the homomorphism $\pi_1(X) \to \pi_1(X')$, so it induces a map of twisted homology $H_p(X;H_q(F;M)) \to H_p(X';H_q(F';M))$. This is the map on the E^2 pages.*
- *The action of $\pi_1(Y)$ on M is the pullback of the action of $\pi_1(Y')$ on M, so the map $Y \to Y'$ induces a map of twisted homology $H_*(Y;M) \to H_*(Y';M)$. This is the map in the limit.*

---

[7] The inclusion $\{0\} \hookrightarrow [0,1]$ is an acyclic cofibration, so its pullback along the fibration f is again an acyclic cofibration, in particular a weak equivalence.

[8] This was also stated (referencing [MS93]) as Theorem 4.1 of [Han09a].

6. Proof of twisted homological stability

We now use the twisted Serre spectral sequence of the previous section to prove Theorem 1.3. We first record another fact we will use:

Lemma 6.1 (Shapiro for covering spaces) *Suppose we have a based space X which is locally nice enough to have a universal cover, a subgroup H of $\pi_1(X)$ and an H-module A. Let $\hat{X}$ be the (based) covering space corresponding to H. Then*

$$H_*(\hat{X};A) \cong H_*(X;\mathbb{Z}\pi_1(X)\otimes_{\mathbb{Z}H} A). \tag{6.1}$$

Moreover, given a map of the above data, namely a (based) map $f\colon X\to X'$ such that $f_*(H)\subseteq H'$ (so that there is a unique based lift $\hat{f}\colon \hat{X}\to\hat{X}'$) and a map $\phi\colon A\to A'$ which is equivariant w.r.t. f_*, the identification (6.1) is natural in the sense that

$$\begin{array}{ccc} H_*(\hat{X};A) & \longrightarrow & H_*(\hat{X}';A') \\ \wr\| & & \wr\| \\ H_*(X;\mathbb{Z}\pi_1(X)\otimes_{\mathbb{Z}H} A) & \longrightarrow & H_*(X';\mathbb{Z}\pi_1(X')\otimes_{\mathbb{Z}H'} A') \end{array} \tag{6.2}$$

commutes.

Proof. Denote the singular chain complex functor by $S_*(\ )$ and the universal cover of X by $\widetilde{X}$. Then we have an isomorphism of chain complexes

$$S_*(\widetilde{X})\otimes_{\mathbb{Z}H} A \longrightarrow S_*(\widetilde{X})\otimes_{\mathbb{Z}\pi_1(X)}\mathbb{Z}\pi_1(X)\otimes_{\mathbb{Z}H} A$$

given by $\sigma\otimes a\mapsto\sigma\otimes[c_x]\otimes a$, where c_x is the constant loop at the basepoint x of X. Taking homology gives the identification (6.1). Let $\widetilde{f}$ denote the unique (based) lift of f to $\widetilde{X}\to\widetilde{X}'$. The diagram (6.2) is induced by

$$\begin{array}{ccc} S_*(\widetilde{X})\otimes_{\mathbb{Z}H} A & \longrightarrow & S_*(\widetilde{X}')\otimes_{\mathbb{Z}H'} A' \\ \cong\downarrow & & \downarrow\cong \\ S_*(\widetilde{X})\otimes_{\mathbb{Z}\pi_1(X)}\mathbb{Z}\pi_1(X)\otimes_{\mathbb{Z}H} A & \longrightarrow & S_*(\widetilde{X}')\otimes_{\mathbb{Z}\pi_1(X')}\mathbb{Z}\pi_1(X')\otimes_{\mathbb{Z}H'} A' \end{array}$$

and one can check that both routes around the square send $\sigma\otimes a$ to $\widetilde{f}_\sharp(\sigma)\otimes[c_{x'}]\otimes\phi(a)$. □

This will be applied to the following covering spaces of configuration spaces:

Definition 6.2 The configuration space $C_{(k,n-k)}(M,X)$ of k red and $n-k$ green points in M with labels in X is defined to be

$$(\mathrm{Emb}(n,M)\times X^n)/(\Sigma_{n-k}\times\Sigma_k)$$

(cf. Remark 1.10), and we give it the basepoint $\{(a_1,x_0),\ldots,(a_n,x_0)\}$ with the points $a_1,\ldots,a_{n-k}$ coloured green and the points $a_{n-k+1},\ldots,a_n$ coloured red. There is also a stabilisation map $s_n^k\colon C_{(k,n-k)}(M,X)\to C_{(k,n-k+1)}(M,X)$, which is defined exactly as in §2.2, and adds a new green point to the configuration.

Definition 6.3 Let $f\colon C_{(k,n-k)}(M,X)\to C_k(M,X)$ be the map which forgets the green points. We will also need the following two maps for technical reasons: Define $p\colon C_k(M,X)\to C_k(M,X)$ to be the self-homotopy-equivalence induced by the self-embedding $e|_M\colon M\hookrightarrow M$ (see §2.1). Choose a self-diffeomorphism of M which is isotopic to the identity and which takes a_i to $a_{i+n-k+1}$ for $i=1,\ldots,k$. Denote by ϕ the self-homeomorphism $C_k(M,X)\to C_k(M,X)$ induced by this.

The forgetful maps f are locally trivial fibre bundles, so we have a map of fibrations:

$$\begin{array}{ccc} C_{(k,n-k)}(M,X) & \xrightarrow{s_n^k} & C_{(k,n-k+1)}(M,X) \\ f\downarrow & & \downarrow\phi^{-1}\circ f \\ C_k(M,X) & \xrightarrow[\phi^{-1}\circ p]{} & C_k(M,X) \end{array} \tag{6.3}$$

The p is there to ensure that it commutes on the nose, and the ϕ^{-1} is there to deal with basepoints: on the bottom-left we have to give $C_k(M,X)$ the basepoint $\{(a_{n-k+1},x_0),\dots,(a_n,x_0)\}$, but on the bottom-right we can give it its usual basepoint of $\{(a_1,x_0),\dots,(a_k,x_0)\}$.

The map s_n^k restricted to the fibres over the basepoints is a map

$$C_{n-k}(M\smallsetminus\{a_{n-k+1},\dots,a_n\},X)\to C_{n-k+1}(M\smallsetminus\{a_{n-k+2},\dots,a_{n+1}\},X),$$

but this can be identified, up to homeomorphism, with the stabilisation map $s_{n-k}\colon C_{n-k}(M_k,X)\to C_{n-k+1}(M_k,X)$, where M_k is M with a subset of $M\smallsetminus U$ of size k removed (see §2.1 for notation).

Finally, before beginning the proof proper, we mention how a certain local coefficient system pulls back along the maps in (6.3). The covering space $C_{(k,n-k)}(M,X)\to C_n(M,X)$ corresponds to the subgroup $G_n^k\leqslant G_n=\pi_1C_n(M,X)$. Recall from Proposition 3.3 that T_n^k is a G_n^k-module (it is a sub-G_n^k-module of T_n), so it is a local coefficient system for $C_{(k,n-k)}(M,X)$.

Lemma 6.4 *The local coefficient system T_k^k on the right-hand base space pulls back to the local coefficient systems T_n^k and T_{n+1}^k on the total spaces of (6.3).*

Proof. By Lemma 3.9, the left-inverse π_k^n of $\iota_k^n\colon T_k\to T_n$ restricts to a bijection $T_n^k\to T_k^k$. So this is an isomorphism of abelian groups, and it is enough to check that it is equivariant w.r.t. the map on π_1 induced by the composite $\phi^{-1}\circ p\circ f$ in (6.3). This is true because both $e|_M\colon M\hookrightarrow M$ (which induces p) and the diffeomorphism which induces ϕ are isotopic to the identity. Exactly the same argument works for the right-hand side. □

Proof of Theorem 1.3 (except the split-injectivity claim). We need to show that the map

$$H_*(C_n(M,X);T_n)\longrightarrow H_*(C_{n+1}(M,X);T_{n+1}) \tag{6.4}$$

induced by s_n and ι_n is an isomorphism in the range $*\leqslant\frac{n-d}{2}$. By the decomposition (3.2) of Proposition 3.3, and the fact that T has degree d, this is the same as the map

$$\bigoplus_{k=0}^{d}H_*(C_n(M,X);\mathbb{Z}G_n\otimes_{\mathbb{Z}G_n^k}T_n^k)\longrightarrow\bigoplus_{k=0}^{d}H_*(C_{n+1}(M,X);\mathbb{Z}G_{n+1}\otimes_{\mathbb{Z}G_{n+1}^k}T_{n+1}^k) \tag{6.5}$$

induced by s_n, ι_n and $(s_n)_*$. By Shapiro's Lemma for covering spaces (Lemma 6.1) this is isomorphic to the map

$$\bigoplus_{k=0}^{d}H_*(C_{(k,n-k)}(M,X);T_n^k)\longrightarrow\bigoplus_{k=0}^{d}H_*(C_{(k,n-k+1)}(M,X);T_{n+1}^k) \tag{6.6}$$

induced by s_n^k and ι_n. The map of fibrations (6.3) gives the following map of twisted Serre spectral sequences (Corollary 5.7, Proposition 5.8 and Lemma 6.4):

$$\begin{array}{ccc} E^2_{p,q}=H_p(C_k(M,X);H_q(C_{n-k}(M_k,X);T_k^k)) & \Rightarrow & H_*(C_{(k,n-k)}(M,X);T_n^k)\\ \downarrow & & \downarrow\\ E^2_{p,q}=H_p(C_k(M,X);H_q(C_{n-k+1}(M_k,X);T_k^k)) & \Rightarrow & H_*(C_{(k,n-k+1)}(M,X);T_{n+1}^k). \end{array} \tag{6.7}$$

The map in the limit is the kth summand of (6.6), and the map on E^2 pages is induced by the stabilisation map s_{n-k} on the fibres and the homotopy-equivalence $\phi^{-1}\circ p$ on the base. Note that T_k^k is a *constant* coefficient system once it has been pulled back to the fibres $C_{n-k}(M_k,X)$ and $C_{n-k+1}(M_k,X)$, since it was originally pulled back from the base.

Hence, by *untwisted* homological stability for configuration spaces (Theorem 1.2) and the universal coefficient theorem, the map on E^2 pages is an isomorphism for $q\leqslant\frac{n-k}{2}$ (and all $p\geqslant 0$). By the Zeeman comparison theorem[9] it is therefore an isomorphism in the limit for $*\leqslant\frac{n-k}{2}$. So in the range $*\leqslant\frac{n-d}{2}$ each summand in (6.6) is an isomorphism, so (6.4) is an isomorphism. □

[9] The required implication is contained in the proof of Theorem 1 of [Zee57], although stronger hypotheses are stated there. An explicit statement of the comparison theorem which applies to our case is Theorem 1.2 of [Iva93]. It is also written in Remarque 2.10 of [CDG13].

Remark 6.5 When M is at least 3-dimensional, the stabilisation map $C_n(M,X) \to C_{n+1}(M,X)$ is an isomorphism on homology with coefficients in $\mathbb{Z}[\frac{1}{2}]$ in the larger range $* \leqslant n$, by [KM14b]. Since $\mathbb{Z}[\frac{1}{2}]$ is a PID, this implies, via the universal coefficient theorem, the same for homology with coefficients in any $\mathbb{Z}[\frac{1}{2}]$-module. If $T\colon \mathcal{B}(M,X) \to \mathbb{Z}[\frac{1}{2}]\text{-}\mathsf{mod} < \mathsf{Ab}$ is a twisted coefficient system of degree d of $\mathbb{Z}[\frac{1}{2}]$-modules, then the constant coefficients T_k^k appearing in (6.7) above are all $\mathbb{Z}[\frac{1}{2}]$-modules, and the same proof tells us that the map

$$H_*(C_n(M,X);T_n) \to H_*(C_{n+1}(M,X);T_{n+1}) \tag{6.8}$$

is an isomorphism in the larger range $* \leqslant n-d$ (rather than just $* \leqslant \frac{n-d}{2}$). When M is a surface, there is a similar improvement to the range for rational coefficients. In this case, the stabilisation map is an isomorphism on homology with rational coefficients in the range $* \leqslant n$ in the non-orientable case and in the range $* < n$ in the orientable case, by [Chu12, Corollary 3] and [Knu14, Theorem 1.3].[10] Thus if $T\colon \mathcal{B}(M,X) \to \mathsf{Vect}_{\mathbb{Q}} < \mathsf{Ab}$ is a *rational* twisted coefficient system of degree d, then the map (6.8) is an isomorphism in either the range $* \leqslant n-d$ (for non-orientable surfaces) or the range $* < n-d$ (for orientable surfaces).

7. Split-injectivity

To prove the split-injectivity part of Theorem 1.3 we will use the following lemma which was used implicitly by Nakaoka in [Nak60] and later written down explicitly by Dold in [Dol62]:

Lemma 7.1 ([Dol62, Lemma 2]) *Given a sequence $0 \to A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots$ of abelian groups and homomorphisms, the following is sufficient to imply that each of the maps ϕ_i is split-injective: There exist maps $\tau_{k,n}\colon A_n \to A_k$ for $1 \leqslant k \leqslant n$ with $\tau_{n,n} = \mathrm{id}$ such that*

$$\mathrm{im}(\tau_{k,n} - \tau_{k,n+1} \circ \phi_n) \leqslant \mathrm{im}(\phi_{k-1}). \tag{7.1}$$

Let $U_n(M,X)$ be the universal cover of $C_n(M,X)$. One can think of its elements as n-strand "open-ended braids" in $M \times [0,1]$ (n pairwise disjoint paths in $M \times [0,1]$ which are the identity in the second coordinate and start at $\{(a_1,0),\ldots,(a_n,0)\}$, up to endpoint-preserving homotopy) with each strand labelled by the based path space PX. Let $\widetilde{s}_n\colon U_n(M,X) \to U_{n+1}(M,X)$ be the lift of the stabilisation map which applies $e|_M \times \mathrm{id}_{[0,1]}$ to the braid and adds a vertical strand at a_1 labelled by the constant path c_{x_0}.

As before, denote $\pi_1 C_n(M,X)$ by G_n, and denote the singular chain complex of a space by $S_*(\ \ )$. Let $T\colon \mathcal{B}(M,X) \to \mathsf{Ab}$ be any twisted coefficient system (we do not assume finite-degree in this section). Then the map

$$(s_n;\iota_n)_*\colon H_*(C_n(M,X);T_n) \longrightarrow H_*(C_{n+1}(M,X);T_{n+1}). \tag{7.2}$$

is induced by the map of chain complexes

$$(\widetilde{s}_n)_\sharp \otimes \iota_n\colon S_*(U_n(M,X)) \otimes_{\mathbb{Z}G_n} T_n \longrightarrow S_*(U_{n+1}(M,X)) \otimes_{\mathbb{Z}G_{n+1}} T_{n+1}.$$

Proof of Theorem 1.3 (split-injectivity claim). We want to prove that (7.2) is split-injective for all $*$ and n. By Dold's Lemma 7.1, it is sufficient to construct chain maps

$$t_{k,n}\colon S_*(U_n(M,X)) \otimes_{\mathbb{Z}G_n} T_n \longrightarrow S_*(U_k(M,X)) \otimes_{\mathbb{Z}G_k} T_k$$

for $1 \leqslant k \leqslant n$ such that $t_{n,n} = \mathrm{id}$ and

$$t_{k,n} \simeq t_{k,n+1} \circ ((\widetilde{s}_n)_\sharp \otimes \iota_n) - ((\widetilde{s}_{k-1})_\sharp \otimes \iota_{k-1}) \circ t_{k-1,n}. \tag{7.3}$$

Let $S \subseteq \{1,\ldots,n\}$. There is a unique partially-defined injection $\{1,\ldots,n\} \dashrightarrow \{1,\ldots,|S|\}$ which is order-preserving and is defined precisely on S. This is a morphism $n \to |S|$ in the category

[10] The maps used in these two references to induce isomorphisms between configuration spaces are not the stabilisation maps. However, we may reduce to the case where the manifolds are of finite type, so that the rational homology of the configuration spaces is a finite-dimensional vector space in each degree. Moreover, the stabilisation maps are always split-injective in all degrees (see Theorem 1.2). So the fact that $H_*(C_n(M,X);\mathbb{Q})$ and $H_*(C_{n+1}(M,X);\mathbb{Q})$ are abstractly isomorphic in a range implies that the stabilisation map is an isomorphism in this range.

Σ. Let $\pi_{S,n}$ be the lift along $\mathcal{B}(M,X) \to \Sigma$ to a morphism $(x_0)^n \to (x_0)^{|S|}$ given by travelling along the paths p_i (see §2.1) and keeping the labels constant. By our standard abuse of notation we will denote its image under T also by $\pi_{S,n}\colon T_n \to T_{|S|}$.

We also define a map $p_{S,n}\colon U_n(M,X) \to U_{|S|}(M,X)$ as follows. Given an open-ended braid in $U_n(M,X)$, forget the strands which start at $(a_i,0)$ for $i \in \{1,\dots,n\} \smallsetminus S$, and then concatenate this with the reverse of $\pi_{S,n}\colon (x_0)^n \to (x_0)^{|S|}$ to get an open-ended braid in $U_{|S|}(M,X)$.

Directly from these definitions one can check (where the notation $(S-1)$ means $\{s-1 \mid s \in S\}$):

(a) If $1 \notin S$ then $\pi_{S,n+1} \circ \iota_n = \pi_{(S-1),n}$ and $p_{S,n+1} \circ \widetilde{s}_n \simeq p_{(S-1),n}$.

(b) If $1 \in S$ then $\pi_{S,n+1} \circ \iota_n = \iota_{|S|-1} \circ \pi_{(S\smallsetminus\{1\}-1),n}$ and $p_{S,n+1} \circ \widetilde{s}_n = \widetilde{s}_{|S|-1} \circ p_{(S\smallsetminus\{1\}-1),n}$.

We now define $t_{k,n}$ to be the following chain map:

$$\sigma \otimes x \;\mapsto\; \sum_{S\subseteq\{1,\dots,n\},\,|S|=k} (p_{S,n})_\sharp(\sigma) \otimes \pi_{S,n}(x).$$

Clearly $t_{n,n} = \mathrm{id}$, so we just need to check the identity (7.3). The right-hand side of this is:

$$\begin{aligned} \sigma \otimes x \quad\mapsto\quad & \sum_{S\subseteq\{1,\dots,n+1\},\,|S|=k} \big((p_{S,n+1})_\sharp \circ (\widetilde{s}_n)_\sharp(\sigma)\big) \otimes \big(\pi_{S,n+1} \circ \iota_n(x)\big) \\ - & \sum_{R\subseteq\{1,\dots,n\},\,|R|=k-1} \big((\widetilde{s}_{k-1})_\sharp \circ (p_{R,n})_\sharp(\sigma)\big) \otimes \big(\iota_{k-1} \circ \pi_{R,n}(x)\big). \end{aligned} \tag{7.4}$$

Using (a) and (b) above, we see that the top line of this decomposition is chain-homotopic to:

$$\begin{aligned} \sigma \otimes x \quad\mapsto\quad & \sum_{S\subseteq\{1,\dots,n+1\},\,|S|=k,\,1\in S} \big((\widetilde{s}_{k-1})_\sharp \circ (p_{(S\smallsetminus\{1\}-1),n})_\sharp(\sigma)\big) \otimes \big(\iota_{k-1} \circ \pi_{(S\smallsetminus\{1\}-1),n}(x)\big) \\ + & \sum_{S\subseteq\{1,\dots,n+1\},\,|S|=k,\,1\notin S} (p_{(S-1),n})_\sharp(\sigma) \otimes \pi_{(S-1),n}(x). \end{aligned} \tag{7.5}$$

The first line of (7.5) cancels with the second line of (7.4), leaving just the second line of (7.5), which is precisely $t_{k,n}$, as required. ☐

References

[Bet02] S. Betley. *Twisted homology of symmetric groups*. *Proc. Amer. Math. Soc.* 130.12 (2002), 3439–3445 (electronic) (cited on pp. 1, 2, 12).

[Bol12] S. K. Boldsen. *Improved homological stability for the mapping class group with integral or twisted coefficients*. *Math. Z.* 270.1-2 (2012), pp. 297–329. {arxiv:0904.3269} (cited on pp. 2, 12).

[CDG13] G. Collinet, A. Djament and J. T. Griffin. *Stabilité homologique pour les groupes d'automorphismes des produits libres*. *Int. Math. Res. Not. IMRN* 19 (2013), pp. 4451–4476. {arxiv:1109.2686} (cited on pp. 7–9, 20).

[CEF12] T. Church, J. Ellenberg and B. Farb. *FI-modules: a new approach to stability for S_n-representations*. ArXiv:1204.4533v2. 2012 (cited on p. 7).

[CF13] T. Church and B. Farb. *Representation theory and homological stability*. *Adv. Math.* 245 (2013), pp. 250–314. {arxiv:1008.1368} (cited on pp. 2, 4).

[Che14] W. Chen. *Homology of the braid group with coefficients in the reduced Burau representation*. ArXiv:1409.7422v1. 2014 (cited on p. 3).

[Chu12] T. Church. *Homological stability for configuration spaces of manifolds*. *Invent. Math.* 188.2 (2012), pp. 465–504. {arxiv:1103.2441} (cited on pp. 2, 4, 21).

[CM09] R. L. Cohen and I. Madsen. *Surfaces in a background space and the homology of mapping class groups*. *Algebraic geometry—Seattle 2005. Part 1.* Vol. 80. Proc. Sympos. Pure Math. Providence, RI: Amer. Math. Soc., 2009, pp. 43–76. {arxiv:math/0601750} (cited on pp. 2, 12).

[CP14] F. Cantero and M. Palmer. *On homological stability for configuration spaces on closed background manifolds*. ArXiv:1406.4916v2. 2014 (cited on p. 4).

[Dol62] A. Dold. *Decomposition theorems for $S(n)$-complexes*. *Ann. of Math. (2)* 75 (1962), pp. 8–16 (cited on p. 21).

[DV13] A. Djament and C. Vespa. *De la structure des foncteurs polynomiaux sur les espaces hermitiens*. ArXiv:1308.4106v2. 2013 (v2: 2014) (cited on pp. 5, 8).

[Dwy80] W. G. Dwyer. *Twisted homological stability for general linear groups*. *Ann. of Math. (2)* 111.2 (1980), pp. 239–251 (cited on pp. 2, 12).

[EML54] S. Eilenberg and S. Mac Lane. *On the groups $H(\Pi, n)$. II. Methods of computation*. *Ann. of Math. (2)* 60 (1954), pp. 49–139 (cited on p. 8).

[Han09a] E. Hanbury. *An open-closed cobordism category with background space*. *Algebr. Geom. Topol.* 9.2 (2009), pp. 833–863. {arxiv:0902.0705} (cited on p. 18).

[Han09b] E. Hanbury. *Homological stability of non-orientable mapping class groups with marked points*. *Proc. Amer. Math. Soc.* 137.1 (2009), pp. 385–392. {arxiv:0806.1082} (cited on p. 14).

[Heu09] C. Heunen. *Categorical quantum models and logics*. PhD thesis. Radboud University Nijmegen, 2009 (cited on p. 7).

[HPV12] M. Hartl, T. Pirashvili and C. Vespa. *Polynomial functors from Algebras over a set-operad and non-linear Mackey functors*. ArXiv:1209.1607v2. 2012. To appear in IMRN (cited on p. 8).

[HV11] M. Hartl and C. Vespa. *Quadratic functors on pointed categories*. *Adv. Math.* 226.5 (2011), pp. 3927–4010. {arxiv:0810.4502} (cited on p. 8).

[Iva93] N. V. Ivanov. *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*. *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*. Vol. 150. Contemp. Math. Providence, RI: Amer. Math. Soc., 1993, pp. 149–194 (cited on pp. 2, 12, 20).

[KM14a] A. Kupers and J. Miller. *E_n-cell attachments and a local-to-global principle for homological stability*. ArXiv:1405.7087v1. 2014 (cited on p. 4).

[KM14b] A. Kupers and J. Miller. *Improved homological stability for configuration spaces after inverting* 2. ArXiv:1405.4441v3. 2014. To appear in Homology, Homotopy and Applications (cited on pp. 2, 21).

[Knu14] B. Knudsen. *Betti numbers and stability for configuration spaces via factorization homology*. ArXiv:1405.6696v4. 2014 (cited on pp. 2, 21).

[McD75] D. McDuff. *Configuration spaces of positive and negative particles*. *Topology* 14 (1975), pp. 91–107 (cited on p. 2).

[MS93] I. Moerdijk and J.-A. Svensson. *The equivariant Serre spectral sequence*. *Proc. Amer. Math. Soc.* 118.1 (1993), pp. 263–278 (cited on pp. 17, 18).

[Nak60] M. Nakaoka. *Decomposition theorem for homology groups of symmetric groups*. *Ann. of Math. (2)* 71 (1960), pp. 16–42 (cited on p. 21).

[Pal] M. Palmer. *Homological stability for spaces of disconnected submanifolds*. In preparation (cited on p. 7).

[Pal13] M. Palmer. *Homological stability for oriented configuration spaces*. *Trans. Amer. Math. Soc.* 365.7 (2013), pp. 3675–3711. {arxiv:1106.4540} (cited on p. 5).

[RW13] O. Randal-Williams. *Homological stability for unordered configuration spaces*. *Q. J. Math.* 64.1 (2013), pp. 303–326. {arxiv:1105.5257} (cited on pp. 2, 4).

[Seg73] G. Segal. *Configuration-spaces and iterated loop-spaces*. *Invent. Math.* 21 (1973), pp. 213–221 (cited on p. 2).

[Seg79] G. Segal. *The topology of spaces of rational functions*. *Acta Math.* 143.1-2 (1979), pp. 39–72 (cited on p. 2).

[Wah14] N. Wahl. *Homological stability for automorphism groups*. ArXiv:1409.3541v1. 2014 (cited on pp. 2, 3, 5, 12).

[Zee57] E. C. Zeeman. *A proof of the comparison theorem for spectral sequences*. *Proc. Cambridge Philos. Soc.* 53 (1957), pp. 57–62 (cited on p. 20).