Extended notes for

# Twisted homological stability for configuration spaces

Martin Palmer // 22 December 2017

**Abstract**

This is a collection of some remarks, explanations and examples that complement the main text of the paper [Pal13a], but which have not been included (or which have been abridged) in that paper in order to keep it reasonably compact. We will henceforth write [Pal13a] = [P].

## Contents

(NB: The top-level sections correspond to those of [P])



## 1. Introduction

**1.a. Summary of related results.** This is a more detailed version of Remark 1.5 of [P], summarising twisted homological stability results related to the main result of that paper.

(a) *Symmetric groups.* First, there is the stability result [Bet02, Theorem 4.3] for the symmetric groups, of which [P] is a generalisation, corresponding to setting $M = \mathbb{R}^\infty$ and $X = *$.

(b) *Braid groups.* Another example is [CF13, Corollary 4.4], which concerns the braid groups $\beta_n = \pi_1(C_n(\mathbb{R}^2))$. The local coefficient systems in this case are the rational $\beta_n$-representations $V_{\lambda[n]}$. Here, $\lambda$ is a fixed Young diagram with $|\lambda|$ boxes, $\lambda[n]$ is the Young diagram obtained by adding a new row of length $n - |\lambda|$ to the top of $\lambda$ and $V_{\lambda[n]}$ is the irreducible $\Sigma_n$-representation corresponding to $\lambda[n]$, viewed as a $\beta_n$-representation via the projection $\beta_n \to \Sigma_n$. A more general example concerning the braid groups is [RW17, Theorem D], which proves homological stability for $\beta_n$ with coefficients in any finite-degree functor from a certain category $\boldsymbol{\mathcal{U}\beta}$ to the category of abelian groups. In particular, the Burau representations $\beta_n \to \mathrm{Aut}(\mathbb{Z}[t^{\pm 1}]^n)$ (see Example 4.3 and Corollary F of [RW17]) fit into their setup (cf. also §4.b below). Note that in this example the local coefficient systems do not factor through the projection $\beta_n \to \Sigma_n$. In fact, the homology of $\beta_n$ with coefficients in the *reduced complex* Burau representations $\beta_n \to \mathrm{Aut}(\mathbb{C}[t^{\pm 1}]^{n-1})$ is explicitly computed in [Che17] (*cf.* §1.c below), and one can directly read off a stable range from the computation. Interestingly, the stable range obtained in [Che17] for the reduced *complex* Burau representations has slope 1, whereas the stable range obtained in [RW17] for the *integral*

2010 *Mathematics Subject Classification*: Primary 55R80; secondary 57N65.
*Key words and phrases*: Configuration spaces, homological stability, polynomial twisted coefficients.
*Address*: Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
*Email address*: palmer@math.uni-bonn.de

Burau representations has slope $\frac{1}{2}$. This parallels the improvements to the range of Theorem A of [P] discussed in Remark 1.4 of [P]. The Burau representations fit into a much larger family of representations of the braid groups, called the *Lawrence representations*, which are discussed in §4.b below.

(c) *Noetherian categories.* There is a general theorem, due to Putman and Sam [PS14, Theorem 4.2], which proves twisted homological stability for the automorphism groups of a so-called *complemented category* $A$ (this is a special case of the *homogeneous categories* of [RW17]) with coefficients in any finitely-generated functor $A \to R$-mod, as long as the automorphism groups are homologically stable with *untwisted* $R$-coefficients and the category $\mathsf{Fun}(A, R\text{-mod})$ is *Noetherian*. An example of a complemented category is $\mathrm{FI}_G$ for any group $G$, whose automorphism groups are the wreath products $G \wr \Sigma_n$. In [SS17], Sam and Snowden develop very general methods for proving that $\mathsf{Fun}(A, R\text{-mod})$ is Noetherian for certain kinds of "combinatorial categories" $A$, and in [SS14] they use these to prove (see Corollary 1.2.2) that $\mathsf{Fun}(\mathrm{FI}_G, R\text{-mod})$ is Noetherian as long as $R$ is a left-Noetherian ring and $G$ is polycyclic-by-finite. The sequence of groups $G \wr \Sigma_n$ is homologically stable with constant integral coefficients, so by the above it is also homologically stable with any finitely-generated twisted coefficients, as long as $G$ is polycyclic-by-finite.

In fact, this is true for any group $G$ — this is part of Theorem D of [RW17]. We note that, if the dimension of $M$ is at least 3, there is an isomorphism $\mathcal{B}_{\mathsf{f}}(M,X) \cong \mathrm{FI}_{\pi_1(M \times X)}$ (see §3.1 of [P] for the definition of $\mathcal{B}_{\mathsf{f}}(M,X)$), in particular $\pi_1(C_n(M,X)) \cong \pi_1(M \times X) \wr \Sigma_n$, so, setting $G = \pi_1(M \times X)$, we have twisted homological stability for the sequence $\pi_1(C_n(M,X))$ of fundamental groups of configuration spaces, with coefficients in any finitely generated functor $\mathcal{B}_{\mathsf{f}}(M,X) \to \mathsf{Ab}$. This statement is different to that of Theorem A of [P], since the configuration spaces $C_n(M,X)$ are not aspherical when $\dim(M) \geqslant 3$.

(d) *$E_2$-algebras.* The result of [RW17] for the braid groups mentioned above is part of a much more general framework for proving twisted homological stability results for sequences of groups. Since it only concerns discrete groups, it cannot apply to configuration spaces on manifolds of dimension at least 3, since these have non-trivial higher homotopy groups. However, in very recent work, Krannich [Kra17] extends the framework of Randal-Williams and Wahl to a topological setting: the input is an $\mathbb{N}$-graded $E_1$-module $\mathcal{M}$ over an $E_2$ algebra (this is related to the notion of module over a braided monoidal category via the fundamental groupoid and classifying space functors), and the output is a semi-simplicial space augmented over $\mathcal{M}$ such that, if its graded pieces have diverging connectivity, then the graded pieces of $\mathcal{M}$ are homologically stable with certain finite-degree twisted coefficients. Considering the $E_2$-algebra $C(\mathbb{R}^2) = \sqcup_n C_n(\mathbb{R}^2)$ as a module over itself, this recovers the theorem of Randal-Williams and Wahl that $C_n(\mathbb{R}^2) = B\beta_n$ is homologically stable with finite-degree coefficients $\boldsymbol{\mathcal{U}\beta} \to \mathsf{Ab}$. It may also be applied to the $E_1$-module $C(M,\pi)$ over the $E_2$-algebra $C(\mathbb{R}^n, \pi_{\mathrm{tr}})$ (for $n \geqslant 2$) for any open, connected manifold $M$ and fibre bundle $\pi\colon E \to M$ with path-connected fibres, where $\pi_{\mathrm{tr}}$ denotes the trivial bundle over $\mathbb{R}^n$ with the same fibres as $\pi$, with the module structure determined by a choice of $n$-ball embedded "near infinity" in $M$ and a trivialisation of $\pi$ over this ball. This then implies twisted homological stability for $C_n(M,\pi)$ with coefficients in any finite-degree functor defined on an analogue of the category $\boldsymbol{\mathcal{U}\beta}$.

(e) *Alternating coefficients.* There is a sequence of $\pi_1(C_n(M,X))$-modules that does not fit into the framework of [P] (it does not form a twisted coefficient system at all, let alone a finite-degree one), but which nevertheless does exhibit homological stability. Every loop in $C_n(M,X)$ induces a permutation of its base configuration, so there is a natural projection map $\pi_1(C_n(M,X)) \to \Sigma_n$, which we can compose with the sign homomorphism to obtain a map $\pi_1(C_n(M,X)) \to \mathbb{Z}/2$. This makes $\mathbb{Z}[\mathbb{Z}/2]$ into a $\pi_1(C_n(M,X))$-module, and its kernel corresponds to a double cover $C_n^+(M,X) \to C_n(M,X)$. The space $C_n^+(M,X)$ is the "oriented configuration space" where each configuration is equipped with an ordering of its points up to even permutations. One can easily see that

$$H_*(C_n^+(M,X);\mathbb{Z}) \cong H_*(C_n(M,X);\mathbb{Z}[\mathbb{Z}/2]). \tag{1.1}$$

In [Pal13b] the author proved that the sequence of spaces $C_n^+(M,X)$, with analogous stabilisation maps, is homologically stable as $n \to \infty$, in the range $* \leqslant \frac{n-5}{3}$. Via the identification (1.1) this is twisted homological stability for $C_n(M,X)$ with respect to the local coefficient system $\mathbb{Z}[\mathbb{Z}/2]$.

This is an example of an *abelian* local coefficient system, so this result is a special case of *abelian homological stability* for $C_n(M,X)$, which is proved in Theorem D(i) of [Kra17] (see Corollary E for the special case). In the case when $M = S$ is a surface and $X = BG$ is aspherical, the configuration spaces $C_n(M,X)$ are classifying spaces of $G \wr \beta_n^S$, where $\beta_n^S$ denotes the surface braid group on $S$. Abelian homological stability in this case (if $S$ is the interior of a compact, connected surface with exactly one boundary-component) also follows from the main result of [RW17].

**1.b. A conjecture that is now known.** An earlier draft of [P] contained a conjecture that one should be able to extend its Theorem A to more general twisted coefficient systems, namely those defined only on a certain subcategory $\mathcal{B}_{\mathsf{f}}(M,X)$ of $\mathcal{B}(M,X)$. Recall that a twisted coefficient system for the sequence $\{C_n(M,X)\}$ is a functor from a category $\mathcal{B}(M,X)$ (called the *partial braid category* on $M$) to the category of abelian groups. This has a subcategory $\mathcal{B}_{\mathsf{f}}(M,X)$ (called the *injective braid category* on $M$), which is related to $\mathcal{B}(M,X)$ in the same way that the category FI of finite sets and injections is related to the larger category FI$\sharp$ of finite sets and partially-defined injections. See §2.3 and §3.1 of [P] for the precise definitions of these categories. See also §3.1 of [P] and §3.13 of [Pal17] for remarks about extending the notions of *degree* resp. *height* of functors defined on $\mathcal{B}(M,X)$ to functors defined only on $\mathcal{B}_{\mathsf{f}}(M,X)$.

This conjecture has recently been confirmed by Krannich [Kra17, Theorem D and §5.2], who in fact proves twisted homological stability for $C_n(M,X)$ for any finite-degree twisted coefficient system defined on a certain category $\mathcal{C}^X(M)$. There is a functor $f\colon \mathcal{C}^X(M) \to \mathcal{B}_{\mathsf{f}}(M,X)$ such that $-\circ f$ preserves degree (see diagram (15) of [Kra17]; *cf.* also Lemma 4.2 of [Pal17]), so his result applies also to functors defined on $\mathcal{B}_{\mathsf{f}}(M,X)$. See also §1.a(d) above.

**1.c. Stable twisted homology.** Once we know homological stability for a sequence of spaces, the natural next step is to compute the *stable homology*, i.e., the homology in the stable range. When the coefficients are untwisted, the answer is given by [Seg73] and [McD75] in terms of the fibrewise one-point compactification $\dot{T}M$ of the tangent bundle $TM$ of $M$, namely $\lim_{n\to\infty} H_*(C_n(M,X)) \cong H_*(\Gamma_\circ(\dot{T}M \wedge X_+))$, where $\dot{T}M \wedge X_+$ is the fibrewise smash product over $M$ and $\Gamma_\circ(-)$ denotes the space of degree-zero compactly-supported sections of a bundle. The stable twisted homology is also known for certain non-constant twisted coefficient systems, including the reduced and unreduced Coxeter representations [Vas92, §I.5] (which is also recovered and extended in work in progress of Arthur Soulié) and the reduced Burau representations $V_n$ over $\mathbb{C}$ [Che17] (this was mentioned in §1.a(b)). In the latter case it is shown, by directly calculating the twisted homology in all degrees, that $\lim_{n\to\infty} H_*(C_n(\mathbb{R}^2);V_n) \cong \mathbb{C}$ in positive degrees and 0 in degree $*=0$.

In the articles [DV10] and [DV15], Djament and Vespa introduce a general method, using functor homology, for computing stable twisted homology of families of groups, and carry out this programme in the case of orthogonal and symplectic groups, and automorphism group of free groups. Their methods may be adaptable to the case of configuration spaces on surfaces (equivalently, surface braid groups); however, they are unlikely to be directly adaptable to configuration spaces on higher-dimensional manifolds, since these are not aspherical spaces.

Stable twisted homology of $\mathrm{Aut}(F_n)$, as well as that of $\mathrm{Out}(F_n)$, has been calculated also by Randal-Williams [Ran16]. His technique is more topological, in that he uses explicit models for the classifying spaces of the groups under investigation, so it appears to be more adaptable to the case of configuration spaces in general. The central idea is to introduce an extra parameter $Y$, which is a topological space, specialise it to be an Eilenberg-MacLane space $K(V,\ell)$ for a rational vector space $V$, and then compute the (untwisted) homology of these spaces in the stable range, as $GL(V)$-modules, in two different ways, and compare these two calculations. For $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ Randal-Williams uses spaces of graphs in $\mathbb{R}^\infty$ labelled by $Y$ for this auxiliary construction. Moreover, he explains in an appendix how to adapt his methods to the mapping class groups of (closed, unpunctured) surfaces, for which the auxiliary spaces are spaces of embedded subsurfaces in $\mathbb{R}^\infty$ equipped with continuous maps to $Y$, introduced by Cohen and Madsen in [CM09]. For the configuration spaces $C_n(M,X)$, a natural candidate to play an analogous role is $C_n(M, X\times Y)$.

We note that both approaches (Djament-Vespa and Randal-Williams) require one to know, in advance, the *untwisted* stable homology of the sequence that one is investigating.[1] As another

[1] More precisely, the Randal-Williams approach requires one to know the untwisted stable homology of the sequence, after introducing the parameter $Y$, as a functor of $Y$.

remark, we note that both Randal-Williams and Djament-Vespa have calculations for stable twisted homology of $\mathrm{Aut}(F_n)$ with respect to twisted coefficient systems defined either on the category **gr** of finitely-generated free groups or on its opposite $\mathbf{gr}^{\mathrm{op}}$. See Theorem A(ii) of [Ran16] and Theorem 4 of [Ves15] for such "contravariant" calculations (the latter result is a consequence of combining the main theorems of both [Ves15] and [Dja15]).

**1.d. Representation stability.** Let $F_n(M,X)$ denote the configuration space of $n$ distinct, ordered points in $M$ labelled by $X$, which is an $(n!)$-sheeted covering space of $C_n(M,X)$. In the notation of Remark 1.7 of [P] it may also be written as $C_{(1,\ldots,1)}(M,X)$, where the partition contains $n$ instances of the number 1. The sequence of graded $\mathbb{Q}[\Sigma_n]$-modules $\{H^*(F_n(M,X);\mathbb{Q})\}$ satisfies *representation stability*, a notion introduced in [CF13] and proved in this case by [Chu12]. Roughly, this says that for each fixed degree $*$ and Young diagram $\lambda$, the number of copies of the irreducible $\Sigma_n$-representation $V_{\lambda[n]}$ in the $n$th term $H^*(F_n(M,X);\mathbb{Q})$ of the sequence is eventually independent of $n$.[2] See §1.a(b) for an explanation of this notation. Moreover, the stability in this case is *uniform*: the bound on "eventually" depends only on $*$ and not on $\lambda$.

The rational homology of $F_n(M,X)$ is related to the groups appearing in Corollary C of [P] as follows:

$$H_*(F_n(M,X);\mathbb{Q}) \otimes_{\mathbb{Q}[\Sigma_n]} \mathbb{Q}[\Sigma_n/\Sigma_{\lambda[n]}] \;\cong\; H_*(C_n(M,X);\mathbb{Q}[\Sigma_n/\Sigma_{\lambda[n]}]). \tag{1.2}$$

This follows from the collapse of the Künneth spectral sequence for the singular chain complex $C_*(F_n(M,X);\mathbb{Q})$ and the module $\mathbb{Q}[\Sigma_n/\Sigma_{\lambda[n]}]$ over the ring $\mathbb{Q}[\Sigma_n]$. By Corollary C of [P], this sequence of graded groups is stable in the range $* \leqslant n-|\lambda|$ (or the range $* < n-|\lambda|$ if $M$ is an orientable surface). There is an argument due to Søren Galatius (personal communication), involving only the representation theory of symmetric groups, that takes stability of the left-hand side of (1.2) as input and proves representation stability for $\{H^*(F_n(M,X);\mathbb{Q})\}$. This therefore reveals a link between twisted homological stability and representation stability.

More quantitatively, the argument of Galatius proves representation stability in the range $n \geqslant |\lambda| + \max\{*+o, |\lambda|+1\}$ (where we set $o=1$ for orientable surfaces and $o=0$ otherwise). For comparison, the range obtained in [Chu12] is $n \geqslant 2*$ for manifolds of dimension at least three and $n \geqslant 4*$ for surfaces. So the range obtained by Galatius' argument improves the range of [Chu12] for surfaces when $|\lambda| \leqslant 2*-1$ (and also works equally well for non-orientable manifolds). If we define the *complexity* of a Young diagram $\mu$ to be the number of boxes below the first row, $\kappa(\mu) = |\mu| - \mu_1$, then we can say that the range is improved for Young diagrams with low complexity, since $\kappa(\lambda[n]) = |\lambda|$. However, it does not recover *uniform* representation stability, as the range depends on $\lambda$ as well as on $*$.

In fact, representation stability for $\{H^*(F_n(M,X);\mathbb{Q})\}$ may be deduced more directly from twisted homological stability for $C_n(M,X)$, as long as one knows this for more general coefficient systems. The representations $V_{\lambda[n]}$ extend to a finitely-generated FI-module by Proposition 3.4.1 of [CEF15], which is therefore a twisted coefficient system on FI of finite degree (in the sense of [RW17, Kra17]) by Proposition 3.4.2 of [SS14]. This may be pulled back to a twisted coefficient system on $\mathcal{B}_{\mathsf{f}}(M,X)$ (see §3.1 of [P] for the definition of this category) along the canonical functor $\mathcal{B}_{\mathsf{f}}(M,X) \to \mathrm{FI}$, which preserves degree (*cf.* Lemma 4.2 of [Pal17]). Thus, by the main result of [Kra17],[3] the sequence of rational vector spaces

$$H_*(F_n(M,X);\mathbb{Q}) \otimes_{\mathbb{Q}[\Sigma_n]} V_{\lambda[n]} \;\cong\; H_*(C_n(M,X);V_{\lambda[n]})$$

is stable as $n \to \infty$, which immediately implies representation stability, since the dimension of the left-hand side is the multiplicity of the irreducible $V_{\lambda[n]}$ in $H^*(F_n(M,X);\mathbb{Q})$. See also Example 5.13(ii) and Corollary 5.17 of [Kra17]. We note that this second approach would not work using

[2] This statement is stronger than it may appear at first: it is true for all $n$, including values of $n$ for which $\lambda[n]$ is not a valid Young diagram (it is only a valid Young diagram if $n \geqslant |\lambda| + \lambda_1$). By definition, the number of copies of $V_{\lambda[n]}$ in a $\Sigma_n$-representation, when $\lambda[n]$ is not a valid Young diagram, is zero. Thus, if the range of stability (for a fixed homological degree $*$ and Young diagram $\lambda$) includes values of $n$ for which $\lambda[n]$ is not a valid Young diagram, then the multiplicity of $V_{\lambda[n]}$ in $H^*(F_n(M,X);\mathbb{Q})$ must in fact be zero for all values of $n$. (See also the second paragraph after the statement of Theorem 1 in [Chu12].)

[3] Or by the main result of [RW17] in the case when $M$ is the interior of a compact, connected surface with exactly one boundary component and $X = BG$.

Theorem A of [P], since the representations $V_{\lambda[n]}$ assemble into a twisted coefficient system on FI, but not on the larger category $\mathrm{FI}\sharp = \Sigma$. The approach described further above, using the representations $\mathbb{Q}[\Sigma_n/\Sigma_{\lambda[n]}]$ instead of $V_{\lambda[n]}$, does work with Theorem A of [P], since these do assemble to form an $\mathrm{FI}\sharp$-module (see Example 4.6 of [P]).

## 2. Twisted coefficient systems

**2.a. Coefficient systems on braid categories of very low degree.** In this section we freely use the notation of [P] – see that paper for any unexplained notation or terminology.

In Remark 3.4 of [P] it is shown that all functors $\mathcal{B}(M,X) \to \mathsf{Ab}$ or $\mathcal{B}_{\mathsf{f}}(M,X) \to \mathsf{Ab}$ of degree at most zero are isomorphic to a constant functor. In this section, we give an alternative argument for this statement, and generalise it to the setting where we restrict the domain category to its full subcategory on objects $\geqslant \kappa$, for a fixed non-negative integer $\kappa$. (Recall that both $\mathcal{B}(M,X)$ and $\mathcal{B}_{\mathsf{f}}(M,X)$ have all non-negative integers as objects.)

Denote these full subcategories by $\mathcal{B}(M,X)_{\geqslant\kappa}$ and $\mathcal{B}_{\mathsf{f}}(M,X)_{\geqslant\kappa}$ respectively, and let $\mathcal{D}$ denote either of them (for any $\kappa$). Since the stabilisation endofunctor $S$ and its associated natural transformation $\mathrm{id} \Rightarrow S$ restrict to $\mathcal{D}$, we have a notion of degree for functors $\mathcal{D} \to \mathsf{Ab}$. We will show in this section that:

**Proposition 2.1** *If $T\colon \mathcal{D} \to \mathsf{Ab}$ is of degree at most zero, it is isomorphic to a constant functor.*

This will follow easily from the following lemma.

**Lemma 2.2** *The classifying space $B\mathcal{D}$ is simply-connected.*

*Proof of Proposition 2.1.* It is easy to see that $T$ has degree at most zero if and only if its image is contained in $\mathsf{Ab}^{\sim}$, the underlying groupoid of $\mathsf{Ab}$.

For a given category $\mathcal{C}$, there is a universal functor from $\mathcal{C}$ to a groupoid, whose target is $\mathcal{G}(\mathcal{C})$, the *Grothendieck groupoid* of $\mathcal{C}$. The functor $\mathcal{G}$ is the left adjoint of the inclusion $\mathsf{Groupoid} \hookrightarrow \mathsf{Cat}$ whereas the underlying groupoid functor $(\cdot)^{\sim}$ mentioned above is the right adjoint. Alternatively, we may think of $\mathcal{G}(\mathcal{C})$ as the localisation $\mathcal{C}[\mathcal{C}^{-1}]$ of $\mathcal{C}$ at all of its morphisms, or as $\Pi_1(B\mathcal{C})$, the fundamental groupoid of the classifying space, or nerve, of $\mathcal{C}$.

By the universal property, if $T\colon \mathcal{C}_1 \to \mathcal{C}_2$ is a functor, its image will be contained in $\mathcal{C}_2^{\sim}$ if and only if it factors through $\mathcal{C}_1 \to \mathcal{G}(\mathcal{C}_1)$. In our setting, we therefore know that $T$ factors through $\mathcal{G}(\mathcal{D})$. In addition, we know by Lemma 2.2 that $\mathcal{G}(\mathcal{D}) \simeq *$. Thus $T$ factors up to isomorphism through the trivial category on one object. □

It therefore remains to prove Lemma 2.2. If $\kappa = 0$ then $\mathcal{D}$ has an initial object, and so $B\mathcal{D}$ is contractible, in particular simply-connected. If $\mathcal{D} = \mathcal{B}(M,X)_{\geqslant\kappa}$ (for any value of $\kappa$), then $\mathcal{D}$ is a *filtered* category, which again implies that its classifying space is (weakly) contractible. The remaining case is therefore $\mathcal{D} = \mathcal{B}_{\mathsf{f}}(M,X)_{\geqslant\kappa}$ for a positive integer $\kappa$, which has no initial or terminal object and is not filtered.

Let's say that a category $\mathcal{C}$ satisfies *property* $(P)$ if for any functor $T\colon \mathcal{C} \to \mathcal{G}$ to a groupoid, if $f$ and $g$ are parallel morphisms in $\mathcal{C}$ then $Tf = Tg$. In other words, $T$ sends every diagram to a commutative diagram. A category $\mathcal{C}$ has *diameter one* if, for each pair of objects $x, y \in \mathcal{C}$, there exists either a morphism $x \to y$ or $y \to x$.

Note that, if $B\mathcal{C}$ is simply-connected, then $\mathcal{G}(\mathcal{C}) \simeq *$ and thus any functor $T\colon \mathcal{C} \to \mathcal{G}$ to a groupoid is isomorphic to a constant functor. But if $T$ is isomorphic to a constant functor, it is easy to check that it sends parallel morphisms in $\mathcal{C}$ to the same morphism in $\mathcal{G}$. Thus any simply-connected category satisfies property $(P)$. Conversely:

**Lemma 2.3** *Any category $\mathcal{C}$ of diameter one that satisfies property $(P)$ is simply-connected.*

*Proof.* The functor $\mathrm{ob}\colon \mathsf{Cat} \to \mathsf{Set}$ that forgets the morphisms of a category has a left adjoint taking a set $S$ to the discrete category on $S$ and a right adjoint taking $S$ to the *indiscrete* category $\mathrm{ind}(S)$ on $S$, i.e., the category whose objects are the elements of $S$ and which has a unique morphism between any pair of objects. We will show that any functor $\mathcal{C} \to \mathcal{G}$ to a groupoid factors through the unit $\eta_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{ind}(\mathrm{ob}(\mathcal{C}))$ of this adjunction. Thus $\mathrm{ind}(\mathrm{ob}(\mathcal{C}))$ satisfies the universal property of $\mathcal{G}(\mathcal{C})$. But $\mathrm{ind}(\mathrm{ob}(\mathcal{C}))$ is equivalent to the trivial category $*$, so we have $\mathcal{G}(\mathcal{C}) \cong \mathrm{ind}(\mathrm{ob}(\mathcal{C})) \simeq *$. Thus $B\mathcal{C}$ is simply-connected.

Let $T\colon \mathcal{C} \to \mathcal{G}$ be a functor to a groupoid. We need to find $T'\colon \mathrm{ind}(\mathrm{ob}(\mathcal{C})) \to \mathcal{G}$ so that

$$T' \circ \eta_{\mathcal{C}} \;=\; T. \tag{2.1}$$

Clearly we must define $T'(c) = T(c)$ on objects of $\mathcal{C}$. Given objects $c, d \in \mathcal{C}$, write $\langle c, d\rangle$ for the unique morphism $c \to d$ in $\mathrm{ind}(\mathrm{ob}(\mathcal{C}))$. Since $\mathcal{C}$ has diameter one there exists either a morphism $f\colon c \to d$ or $g\colon d \to c$. Define $T'(\langle c, d\rangle)$ to be either $Tf$ or $(Tg)^{-1}$. This will satisfy (2.1) as long as $T'$ is a well-defined functor. But any two morphisms $c \to d$ are sent to the same morphism in $\mathcal{G}$ by property $(P)$, and similarly for any two morphisms $d \to c$. Thus to see that $T'$ is well-defined it remains to consider the case of morphisms $f\colon c \to d$ and $g\colon d \to c$ and show that $Tf = (Tg)^{-1}$. But this follows since $gf$ is parallel to $\mathrm{id}_c$. It is similarly trivial to check that $T'$ preserves composition and identities. Hence $T$ factors through $\eta_{\mathcal{C}}$, as desired. ∎

Clearly $\mathcal{D}$ has diameter one. Thus the next lemma, together with Lemma 2.3, implies that $B\mathcal{D}$ is simply-connected, completing the proof of Lemma 2.2 and therefore of Proposition 2.1.

**Lemma 2.4** *The category $\mathcal{D}$ satisfies property $(P)$.*

*Proof.* If $\mathcal{D} = \mathcal{B}(M, X)_{\geqslant \kappa}$ then it is filtered and thus $B\mathcal{D}$ is contractible, as observed above. Then the observation before Lemma 2.3 tells us that $\mathcal{D}$ satisfies property $(P)$. For the rest of this proof we therefore assume that $\mathcal{D} = \mathcal{B}_{\mathsf{f}}(M, X)_{\geqslant \kappa}$.

Let $T\colon \mathcal{D} \to \mathcal{G}$ be a functor to a groupoid. Write $B_n = \mathrm{End}_{\mathcal{D}}(n)$ for the endomorphism monoid of the object $n \geqslant \kappa$ in $\mathcal{D}$. Since $\mathcal{D}$ is an EI-category, this endomorphism monoid is in fact a group, which acts on $T_n = T(n)$.

*Claim.* The action of $B_n$ on $T_n$ is trivial for all $n \geqslant \kappa$.

Now let $n \geqslant m \geqslant \kappa$ be two objects of $\mathcal{D}$ and write $\iota_m^n = \iota_{n-1} \circ \cdots \circ \iota_m \colon m \to n$. Any morphism $f\colon m \to n$ may be written as $b_f \circ \iota_m^n$ for some (non-unique) element $b_f \in B_n$. If $g$ is parallel to $f$ we need to show that $Tf = Tg$. But this follows from the claim above, which tells us that $Tb_f = \mathrm{id}_{T_n} = Tb_g$.

It just remains to prove the claim. There is a homomorphism

$$s_n\colon B_n \longrightarrow B_{n+1}$$

taking a loop $\gamma$ in $C_n(M, X)$ to the loop $s_n \circ \gamma$ in $C_{n+1}(M, X)$. Write $s_m^n = s_{n-1} \circ \cdots \circ s_m$. It is easy to check that $T\iota_n\colon T_n \to T_{n+1}$ is $s_n$-equivariant, and so $T\iota_m^n$ is $s_m^n$-equivariant. Recall that $B_n$ is the fundamental group of $C_n(M, X)$ based at the configuration $\{a_1, \ldots, a_n\} \subset M$ with all labels equal to $x_0$. Each loop induces a permutation of the points in this configuration, so there is a homomorphism $B_n \to \Sigma_n$ to the symmetric group on $n$ letters. Choose any element $\Delta_n \in B_{2n}$ whose induced permutation of $\{a_1, \ldots, a_{2n}\}$ swaps the subsets $\{a_1, \ldots, a_n\}$ and $\{a_{n+1}, \ldots, a_{2n}\}$. It follows that for any $b \in B_n$ we have:

$$\iota_n^{2n} = \Delta_n \circ s_n^{2n}(b) \circ \Delta_n^{-1} \circ \iota_n^{2n}. \tag{2.2}$$

Now fix $n \geqslant \kappa$, an element $b \in B_n$ and an element $x \in T_n$.[4] Our task is to show that $b \cdot x = x$. The equation above implies that $\Delta_n \circ s_n^{2n}(b) \circ \Delta_n^{-1}$ acts trivially on the image of the morphism

$$T\iota_n^{2n}\colon T_n \longrightarrow T_{2n}.$$

But this is an isomorphism, since $\mathcal{G}$ is a groupoid, and hence surjective. So $\Delta_n \circ s_n^{2n}(b) \circ \Delta_n^{-1}$ acts trivially on $T_{2n}$, and therefore so does $s_n^{2n}(b)$. Hence

$$T\iota_n^{2n}(x) = s_n^{2n}(b) \cdot T\iota_n^{2n}(x) = T\iota_n^{2n}(b \cdot x),$$

where the second step follows from $s_n^{2n}$-equivariance of $T\iota_n^{2n}$. But $T\iota_n^{2n}$ is injective, so $x = b \cdot x$. ∎

[4] Since $T_n$ is an object of an arbitrary groupoid $\mathcal{G}$, we cannot strictly speak about elements of its objects. To make the argument correct in general, the elements should be interpreted as *generalised elements*. The justification for writing the argument in terms of actual elements is that, in practice, we are interested in the fact that all functors $\mathcal{D} \to \mathsf{Ab}^{\sim}$ are isomorphic to a constant functor, and $\mathsf{Ab}^{\sim}$ is a concrete groupoid.

## 4. Examples of twisted coefficient systems

**4.a. A generalisation of Example 4.6.** Example 4.6 of [P] may be generalised as follows. Fix $\ell > 0$ and write

$$\Lambda_\ell = \{\lambda \mid \lambda \in \mathbb{N}^k \text{ for some } k > 0 \text{ and } \lambda \vdash \ell' \text{ for some } \ell' \leqslant \ell\}.$$

This set has a partial order where $(\lambda_1, \ldots, \lambda_k) \leqslant (\mu_1, \ldots, \mu_l)$ if and only if there exists an injection $\alpha \colon \{1, \ldots, k\} \hookrightarrow \{1, \ldots, l\}$ such that $\lambda_i \leqslant \mu_{\alpha(i)}$ for each $i \in \{1, \ldots, k\}$. For $\lambda \leqslant \mu$, write $[\lambda, \mu]$ for the interval $\{\nu \in \Lambda_\ell \mid \lambda \leqslant \nu \leqslant \mu\}$. We may then define a functor $P_{[\lambda,\mu]} \colon \Sigma \to \mathsf{Se}^{\mathsf{fin}}$ as follows. On objects it is defined by

$$P_{[\lambda,\mu]}(n) = \begin{cases} \{\text{ordered decompositions of } \underline{n} \text{ with type} \in [\lambda, \mu]\} & n \geqslant |\lambda| \\ \varnothing & n < |\lambda|. \end{cases}$$

Given a partially-defined injection $j \colon \{1, \ldots, m\} \dashrightarrow \{1, \ldots, n\}$, we define $P_{[\lambda,\mu]}(j)$ to be the empty function if either $m < |\lambda|$ or $n < |\lambda|$; otherwise it takes $(S_1, \ldots, S_k)$ to $(j(S_1), \ldots, j(S_k))$ if this is an ordered decomposition with type $\in [\lambda, \mu]$, and is undefined on $(S_1, \ldots, S_k)$ if not.

As in Remark 4.8 of [P], we may compose this with the functor $R(-) \colon \mathsf{Se}^{\mathsf{fin}} \to \mathsf{Ab}$ for any ring $R$ to obtain a twisted coefficient system $RP_{[\lambda,\mu]} \colon \Sigma \to \mathsf{Ab}$, which has degree $|\mu|$ by a similar argument to Lemma 4.7 and Remark 4.8 of [P].

**4.b. Examples of twisted coefficient systems: Lawrence representations.** In this section we briefly introduce a viewpoint on the *Lawrence representations* of the braid groups (a family of representations of the braid groups including the Burau representation and the Lawrence-Krammer-Bigelow representation). We will investigate these further in future work.

**The construction.** To describe the Lawrence representations we will use twisted homology as described in §5.1 of [P] (see in particular Remark 5.2) as a continuous functor $\mathsf{Top}_R \to \text{gr-}R\text{-mod}$, where an object of $\mathsf{Top}_R$ is a locally path-connected and semi-locally simply-connected space equipped with a bundle $R$-modules, and morphisms are continuous maps covered by bundle maps (restricting to an $R$-linear isomorphism on each fibre). For each $k \geqslant 0$ we can pick out the $k$-graded piece of a graded $R$-module, so we get a continuous functor

$$H_k \colon \mathsf{Top}_R \longrightarrow R\text{-mod}.$$

For a group $G$, we may define $\mathsf{Top}_G$ exactly like $\mathsf{Top}_R$, using bundles of $G$-sets instead of $R$-modules. If $S$ is a $G$-set, the free $R$-module $R\langle S \rangle$ with basis $S$ is an $R[G]$-module. Applying this construction to each fibre of a bundle defines a continuous functor

$$R\langle\cdot\rangle \colon \mathsf{Top}_G \longrightarrow \mathsf{Top}_{R[G]}.$$

Now fix a positive integer $m$ and let $G = \mathbb{Z}$ if $m = 1$ and $G = \mathbb{Z}^2$ if $m \geqslant 2$. Let $\beta_n$ be the classical $n$-th braid group, in other words $\beta_n = \pi_1(C_n(\mathbb{D}^2))$. Another description of $\beta_n$ is as a mapping class group: let $B_n$ be the topological group of self-diffeomorphisms of $\mathbb{D}^2$ restricting to the identity on the boundary and sending a chosen subset $\{p_1, \ldots, p_n\} \subset \text{int}(\mathbb{D}^2)$ to itself. Then $\beta_n = \pi_0(B_n)$, the group of path-components of $B_n$. Let $\boldsymbol{\beta}$ be the groupoid with objects $0, 1, 2, \ldots$ and automorphisms $\text{Aut}_{\boldsymbol{\beta}}(n) = \beta_n$ and with no morphisms between distinct objects (*cf.* [RW17, Sou17]). Similarly let $\boldsymbol{B}$ be the topological groupoid with the same objects and with automorphisms $\text{Aut}_{\boldsymbol{B}}(n) = B_n$ and with no morphisms between distinct objects. Our viewpoint on the Lawrence construction is that it is a continuous functor (defined in a moment)

$$L_m \colon \boldsymbol{B} \longrightarrow \mathsf{Top}_G,$$

which we then compose with $\mathbb{Z}\langle\cdot\rangle$ and $H_m$ to obtain a functor $\boldsymbol{B} \to \mathbb{Z}[G]\text{-mod}$. Since the target is a discrete category, this continuous functor descends to a well-defined functor

$$\mathcal{L}_m \colon \boldsymbol{\beta} = \pi_0(\boldsymbol{B}) \longrightarrow \mathbb{Z}[G]\text{-mod}$$

which is the *Lawrence representation at level m*. We now define the continuous functor $L_m$. The space $L_m(n)$ is defined to be the configuration space $C_m(\mathbb{D}_n)$, where $\mathbb{D}_n$ denotes the closed 2-disc with $n$ interior points $p_1, \ldots, p_n$ removed. Choose a configuration in the boundary of the disc as a basepoint. There is a surjective homomorphism $\pi_1(C_m(\mathbb{D}_n)) \to G$ defined as follows. When $m = 1$ the group $G$ is $\mathbb{Z}$, and we send a loop in $C_1(\mathbb{D}_n) = \mathbb{D}_n$ to the sum of the winding numbers of the loop around each of the punctures $p_1, \ldots, p_n$. When $m = 2$ the group $G$ is $\mathbb{Z}^2$, and we send a loop $\gamma$ of configurations to the pair $(t(\gamma), w(\gamma))$, where $w(\gamma)$ is the total winding number of all points of the configuration around all punctures,[5] and $t(\gamma)$ is the total winding number of all points of the configuration around each other — more precisely, the latter means that we ignore the punctures, view $\gamma$ as an element of $\beta_m$ and take its image under the abelianisation map $\beta_m \to \mathbb{Z}$. See also [Bud05, §2] for a description of this quotient homomorphism. The covering space $\widetilde{C}_m(\mathbb{D}_n) \to C_m(\mathbb{D}_n)$ corresponding to the kernel of this quotient is a principal $G$-bundle (since it is a regular covering space with deck transformation group $G$), and therefore in particular a bundle of $G$-sets. This defines the object $L_m(n)$ of $\mathsf{Top}_G$. It remains to define the functor $L_m$ on automorphisms (recall that the only morphisms of $\boldsymbol{B}$ are automorphisms). There is a continuous action of $B_n$ on $C_m(\mathbb{D}_n)$ by applying a diffeomorphism to each point of a configuration. One can check that the quotient $\pi_1(C_m(\mathbb{D}_n)) \to G$ is invariant under the induced action of $B_n$ on $\pi_1(C_m(\mathbb{D}_n))$. This implies that the action of $B_n$ on the space $C_m(\mathbb{D}_n)$ lifts (uniquely) to an action on the object $L_m(n)$ of $\mathsf{Top}_G$ — in other words, it acts by automorphisms of bundles of $G$-sets. This defines the functor $L_m$ on automorphisms. Note that all actions of $B_n$ that we have considered are continuous actions, so this is indeed a *continuous* functor, as stated above.

**Special cases.** When $m = 1$ the twisted homology $H_1(\mathbb{D}_n; \mathbb{Z}[\mathbb{Z}])$, equivalently, the integral homology of the covering space $\widetilde{\mathbb{D}}_n$, is a free $\mathbb{Z}[\mathbb{Z}]$-module of rank $n-1$, so we obtain representations

$$\beta_n \longrightarrow GL_{n-1}(\mathbb{Z}[\mathbb{Z}]).$$

These are the reduced Burau representations of the braid groups. When $m = 2$ the twisted homology $H_2(C_2(\mathbb{D}_n); \mathbb{Z}[\mathbb{Z}^2])$, equivalently, the integral homology of the covering space $\widetilde{C}_2(\mathbb{D}_n)$, is a free $\mathbb{Z}[\mathbb{Z}^2]$-module of rank $\binom{n}{2}$ (see Proposition 3.6 of [PP02] or Theorem 4.1 of [Big03]), so we obtain representations

$$\beta_n \longrightarrow GL_{\binom{n}{2}}(\mathbb{Z}[\mathbb{Z}^2]).$$

These are the *Lawrence-Krammer-Bigelow* representations of the braid groups, which were shown by Bigelow [Big01] and Krammer [Kra02] to be faithful.

**Variants.** One may also define variants of the Lawrence construction, for example by taking homology relative to some subspace of the configuration space, and/or by taking Borel-Moore homology in place of ordinary homology.

If we modify the above construction using instead *reduced* (twisted) homology, in other words, homology relative to a point on the boundary of the configuration space, we denote the resulting representation of $\boldsymbol{\beta}$ by $\mathcal{L}_m^{\mathsf{r}}$. If we instead use the *Borel-Moore* homology (this requires us to restrict $\mathsf{Top}_G$ and $\mathsf{Top}_R$ to their subcategories of of proper maps, or alternatively use the opposite of their subcategories of open embeddings), we denote the resulting representation of $\boldsymbol{\beta}$ by $\mathcal{L}_m^{\mathsf{bm}}$. We note that the $\mathbb{Z}[\mathbb{Z}]$-module $\mathcal{L}_1^{\mathsf{r}}(n)$ is free of rank $n$, and is the *unreduced* Burau representation of $\beta_n$ (in contrast to $\mathcal{L}_1(n)$, which, as noted above, is the *reduced* Burau representation of $\beta_n$).

**Twisted coefficient systems extending the Lawrence representation.** There are functors

$$\boldsymbol{\beta} \hookrightarrow \boldsymbol{\mathcal{U}\beta} \longrightarrow \mathcal{B}_{\mathsf{f}}(\mathbb{R}^2) \hookrightarrow \mathcal{B}(\mathbb{R}^2),$$

where the first and last are faithful, and the composite functor $\boldsymbol{\beta} \to \mathcal{B}_{\mathsf{f}}(\mathbb{R}^2)$ is also faithful (although the middle one is not). The notation $\boldsymbol{\mathcal{U}}(-)$ denotes a construction of Quillen, studied in [RW17,

[5] We are talking about unordered configurations, so not every point of the configuration returns to where it started under the loop $\gamma$. Nevertheless, we obtain $m$ paths in $\mathbb{D}_n$, which concatenate to form $\ell \leqslant m$ loops, and we take $w(\gamma)$ to be the sum of the winding numbers of each of these loops around each of the punctures $p_1, \ldots, p_n$. Note that if we ignore the punctures, the loop $\gamma$ determines a braid in $\beta_m$, which induces a permutation in $\Sigma_m$. The number $\ell$ of loops in $\mathbb{D}_n$ is the number of cycles in the cycle decomposition of this permutation.

§1.1], and the category $\boldsymbol{\mathcal{U}\beta}$ is studied in [Sou17]. The *partial braid category* $\mathcal{B}(\mathbb{R}^2)$ is defined in §2.3 of [P] and its subcategory $\mathcal{B}_{\mathsf{f}}(\mathbb{R}^2)$, the *injective braid category*, is defined in §3.1 of [P]. All three categories have $\boldsymbol{\beta}$ as their underlying groupoid, and all of the above functors are the identity on the object set (which is the set of non-negative integers in each case).

The Lawrence representation $\mathcal{L}_m \colon \boldsymbol{\beta} \to \mathbb{Z}[G]$-mod and its variants $\mathcal{L}_m^{\mathsf{r}}$ and $\mathcal{L}_m^{\mathsf{bm}}$ extend to $\boldsymbol{\mathcal{U}\beta}$ (see Example 4.15 of [RW17] or Example 2.27 of [Sou17] for $\mathcal{L}_1^{\mathsf{r}}$, see Example 2.28 of [Sou17] for $\mathcal{L}_1$ and see Example 2.37 of [Sou17] for $\mathcal{L}_2$).[6] However, they do not in general extend to $\mathcal{B}_{\mathsf{f}}(\mathbb{R}^2)$. In work to appear separately, we will explain these facts geometrically, using our viewpoint of the Lawrence construction as a functor $\boldsymbol{B} \to \mathsf{Top}_G$, considering the problem of extending it to certain topological categories containing $\boldsymbol{B}$. We will also explore specialisations of $\mathcal{L}_m$ and its variants, and investigate when they extend to $\mathcal{B}_{\mathsf{f}}(\mathbb{R}^2)$ and when they extend to $\mathcal{B}(\mathbb{R}^2)$. By a *specialisation* of $\mathcal{L}_m$, we mean $\mathrm{Ind}_\theta \circ \mathcal{L}_m$, where $\theta \colon \mathbb{Z}[G] \to R$ is a ring homomorphism and $\mathrm{Ind}_\theta$ is the functor $\mathbb{Z}[G]$-mod $\to R$-mod given by induction along $\theta$. We conjecture that the extensions of $\mathcal{L}_m$, $\mathcal{L}_m^{\mathsf{r}}$ and $\mathcal{L}_m^{\mathsf{bm}}$ to $\boldsymbol{\mathcal{U}\beta}$ have degree $m$, with two anomalous exceptions that $\mathcal{L}_1$ and $\mathcal{L}_1^{\mathsf{bm}}$ have degree 2.[7] This would imply, using the main theorem of [RW17], that the Lawrence representations, at each level $m$, are homologically stable. We note that for $\mathcal{L}_1^{\mathsf{r}}$ this is explicitly proved in [RW17] as Corollary F, and the representations $\mathcal{L}_1$ and $\mathcal{L}_2$ are shown to have degree 2 in Corollary 2.36 and Proposition 2.40 of [Sou17].[6] We also intend to explore other variants of the Lawrence representations, varying the construction $\boldsymbol{B} \to \mathsf{Top}_G$ (and the group $G$) by using different kinds of configuration spaces, and more generally studying surface braid groups. It would also be interesting to investigate the consequences of homological stability for such representations, since the construction of Lawrence is closely related to the Jones polynomial [Law93, Big02] and more generally the quantum $\mathfrak{sl}(n)$ polynomials [Law96, Big07].

## References


[Bet02] S. Betley. *Twisted homology of symmetric groups*. *Proc. Amer. Math. Soc.* 130.12 (2002), 3439–3445 (electronic) (↑ p. 1).

[Big01] S. J. Bigelow. *Braid groups are linear*. *J. Amer. Math. Soc.* 14 (2001), pp. 471–486. {arxiv:math/0005038} (↑ pp. 8, 9).

[Big02] S. Bigelow. *A homological definition of the Jones polynomial*. *Geom. Topol. Monogr.* 4 (2002), pp. 29–41. {arxiv:math/0201221} (↑ p. 9).

[Big03] S. Bigelow. *The Lawrence-Krammer representation*. *Topology and Geometry of Manifolds*. Ed. by G. Matić and C. McCrory. Vol. 71. Proceedings of Symposia in Pure Mathematics. 2003, pp. 51–68. {arxiv:math/0204057} (↑ p. 8).

[Big07] S. Bigelow. *A homological definition of the HOMFLY polynomial*. *Algebr. Geom. Topol.* 7 (2007), pp. 1409–1440. {arxiv:math/0608527} (↑ p. 9).

[Bud05] R. D. Budney. *On the image of the Lawrence-Krammer representation*. *J. Knot Theory Ramifications* 14.6 (2005), pp. 773–789. {arxiv:math/0202246} (↑ p. 8).

[CEF15] T. Church, J. S. Ellenberg and B. Farb. *FI-modules and stability for representations of symmetric groups*. *Duke Math. J.* 164.9 (2015), pp. 1833–1910. {arxiv:1204.4533} (↑ p. 4).

[CF13] T. Church and B. Farb. *Representation theory and homological stability*. *Adv. Math.* 245 (2013), pp. 250–314. {arxiv:1008.1368} (↑ pp. 1, 4).

[Che17] W. Chen. *Homology of braid groups, the Burau representation, and points on superelliptic curves over finite fields*. *Israel J. Math.* 220.2 (2017), pp. 739–762. {arxiv:1506.02189} (↑ pp. 1, 3).

[Chu12] T. Church. *Homological stability for configuration spaces of manifolds*. *Invent. Math.* 188.2 (2012), pp. 465–504. {arxiv:1103.2441} (↑ p. 4).


[6] To be pedantic, we note that the version $\mathfrak{LK}$ of the Lawrence-Krammer-Bigelow representation studied by Soulié is not quite the same as $\mathcal{L}_2$. The representation $\mathfrak{LK}$ is defined over $\mathbb{C}[\mathbb{Z}^2]$ by certain formulas, which are also valid over $\mathbb{Z}[\mathbb{Z}^2]$, so they also define an integral version $\mathfrak{LK}_{\mathbb{Z}}$ related to $\mathfrak{LK}$ by $\mathfrak{LK} \cong \mathfrak{LK}_{\mathbb{Z}} \otimes \mathbb{C}[\mathbb{Z}^2]$, where $\otimes$ is the tensor product over $\mathbb{Z}[\mathbb{Z}^2]$. By Theorem 1.2 of [PP02], the representations $\mathfrak{LK}_{\mathbb{Z}}$ and $\mathcal{L}_2 \colon \boldsymbol{\beta} \to \mathbb{Z}[\mathbb{Z}^2]$-mod are not isomorphic, but they become isomorphic after composing with the functor $\mathbb{Z}[\mathbb{Z}^2]$-mod $\to \mathbb{Q}(x,y)$-mod given by induction along the inclusion of $\mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ into its field of fractions. See also Theorem 4.1 of [Big01], which says that $\mathfrak{LK}_{\mathbb{Z}}$ and $\mathcal{L}_2$ become isomorphic after inducing along any embedding of $\mathbb{Z}[\mathbb{Z}^2]$ into $\mathbb{R}$ given by a pair of algebraically independent real numbers. (Of course, any such embedding factors through the field of fractions, so this also follows from the previous statement.)

[7] On the other hand, it is not hard to see that the *weak* degree (defined by Djament and Vespa [DV13, Définition 1.22], see also Definition 2.1 of [Pal17]) of $\mathcal{L}_1$ is 1, rather than 2 (this observation was pointed out to the author by Arthur Soulié) which suggests that it may be useful to study the *weak* degree of Lawrence representations.

[CM09] R. L. Cohen and I. Madsen. *Surfaces in a background space and the homology of mapping class groups*. *Algebraic geometry—Seattle 2005. Part 1*. Vol. 80. Proc. Sympos. Pure Math. Providence, RI: Amer. Math. Soc., 2009, pp. 43–76. {arxiv:math/0601750} (↑ p. 3).

[Dja15] A. Djament. *Décomposition de Hodge pour l'homologie stable des groupes d'automorphismes des groupes libres*. ArXiv:1510.03546v4. v4: 2016, v1: 2015 (↑ p. 4).

[DV10] A. Djament and C. Vespa. *Sur l'homologie des groupes orthogonaux et symplectiques à coefficients tordus*. *Ann. Sci. Éc. Norm. Supér. (4)* 43.3 (2010), pp. 395–459. {arxiv:0808.4035} (↑ p. 3).

[DV13] A. Djament and C. Vespa. *Foncteurs faiblement polynomiaux*. ArXiv:1308.4106v5. v5: 2017, v1: 2013. To appear in IMRN. (↑ p. 9).

[DV15] A. Djament and C. Vespa. *Sur l'homologie des groupes d'automorphismes des groupes libres à coefficients polynomiaux*. *Comment. Math. Helv.* 90.1 (2015), pp. 33–58. {arxiv:1210.4030} (↑ p. 3).

[Kra02] D. Krammer. *Braid groups are linear*. *Ann. of Math. (2)* 155.1 (2002), pp. 131–156. {arxiv:math/0405198} (↑ p. 8).

[Kra17] M. Krannich. *Homological stability of topological moduli spaces*. ArXiv:1710.08484v1. 2017 (↑ pp. 2–4).

[Law93] R. J. Lawrence. *A functorial approach to the one-variable Jones polynomial*. *J. Differential Geom.* 37.3 (1993), pp. 689–710 (↑ p. 9).

[Law96] R. J. Lawrence. *Braid group representations associated with $\mathfrak{sl}_m$*. *J. Knot Theory Ramifications* 5 (1996), pp. 637–660 (↑ p. 9).

[McD75] D. McDuff. *Configuration spaces of positive and negative particles*. *Topology* 14 (1975), pp. 91–107 (↑ p. 3).

[Pal13a] M. Palmer. *Twisted homological stability for configuration spaces*. ArXiv:1308.4397v3. v3: 2017, v1: 2013 (↑ p. 1).

[Pal13b] M. Palmer. *Homological stability for oriented configuration spaces*. *Trans. Amer. Math. Soc.* 365.7 (2013), pp. 3675–3711. {arxiv:1106.4540} (↑ p. 2).

[Pal17] M. Palmer. *A comparison of twisted coefficient systems*. ArXiv:1712.06310v1. 2017 (↑ pp. 3, 4, 9).

[PP02] L. Paoluzzi and L. Paris. *A note on the Lawrence-Krammer-Bigelow representation*. *Algebr. Geom. Topol.* 2 (2002), pp. 499–518. {arxiv:math/0111186} (↑ pp. 8, 9).

[PS14] A. Putman and S. V. Sam. *Representation stability and finite linear groups*. ArXiv:1408.3694v3. v3: 2017, v1: 2014. To appear in Duke Mathematical Journal. (↑ p. 2).

[Ran16] O. Randal-Williams. *Cohomology of automorphism groups of free groups with twisted coefficients*. ArXiv:1604.01701v2. v2: 2017, v1: 2016. To appear in Selecta Mathematica. (↑ pp. 3, 4).

[RW17] O. Randal-Williams and N. Wahl. *Homological stability for automorphism groups*. *Adv. Math.* 318 (2017), pp. 534–626. {arxiv:1409.3541} (↑ pp. 1–4, 7–9).

[Seg73] G. Segal. *Configuration-spaces and iterated loop-spaces*. *Invent. Math.* 21 (1973), pp. 213–221 (↑ p. 3).

[Sou17] A. Soulié. *The Long-Moody construction and polynomial functors*. ArXiv:1702.08279v3. 2017 (↑ pp. 7, 9).

[SS14] S. V. Sam and A. Snowden. *Representations of categories of G-maps*. ArXiv:1410.6054v4. v4: 2016, v1: 2014. To appear in J. Reine Angew. Math. (↑ pp. 2, 4).

[SS17] S. V. Sam and A. Snowden. *Gröbner methods for representations of combinatorial categories*. *J. Amer. Math. Soc.* 30.1 (2017), pp. 159–203. {arxiv:1409.1670} (↑ p. 2).

[Vas92] V. A. Vassiliev. *Complements of discriminants of smooth maps: topology and applications*. Vol. 98. Translations of Mathematical Monographs. Translated from the Russian by B. Goldfarb. American Mathematical Society, Providence, RI, 1992, pp. vi+208 (↑ p. 3).

[Ves15] C. Vespa. *Extensions between functors from free groups*. ArXiv:1511.03098v3. v3: 2016, v1: 2015 (↑ p. 4).