May 20, 2021 ①

Line arrangements, same combinatorial perspectives

Def <u>arrangement of hyperplanes</u> $\mathcal{A}$:

a finite set of hyperplanes (codim 1 subspaces) $\mathcal{A} = \{H_1, .., H_n\}$ in an l-dimensional affine / projective vector space $V/\mathbb{K}$.

- our context: $V = \mathbb{C}^3$ or $\mathbb{CP}^2$ $(l = 3)$ and $\mathbb{K} = \mathbb{C}$.
- $\mathcal{A}$ is called <u>central</u> if $\underbrace{\bigcap_{H \in \mathcal{A}} H}_{\text{center of } \mathcal{A}} \neq \emptyset$.
- we will work from now on with <u>central</u> arrangements
- $\{x_1, \dots, x_l\}$ basis for V^*, $S = \mathbb{K}[x_1, .., x_l]$

  $H \in \mathcal{A}$ is $\ker \alpha_H$, $\alpha_H \in V^*$

  $$f_{\mathcal{A}} = \prod_{H \in \mathcal{A}} \alpha_H$$

  defining polynomial of $\mathcal{A}$
- $\mathcal{A}$ is called <u>essential</u> if $\bigcap_{H \in \mathcal{A}} H = \{0\}$.

one can allways restrict to this situation by taking a quotient of the ambient vector space by the center of the arrangement

Def The <u>intersection lattice</u> of $\mathcal{A}$:

$$L(\mathcal{A}) = \{ \bigcap_{H \in \mathcal{B}} \mid \mathcal{B} \subset \mathcal{A} \}$$

$\|$

(a geometric lattice) the partial ordered set of all intersections of various subsets of $\mathcal{A}$, ordered by reverse inclusion.

- to make sense of the title: a property/invariant associated to $\mathcal{A}$ is called <u>combinatorial</u> (or <u>combinatorially determined</u>) if it is completely determined by $L(\mathcal{A})$ [regardless of it's nature, which can be top., geom., or alg.]

- a topological property/invariant usually involves the complement of $\mathcal{A}$:

$$M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H.$$

$\swarrow$

smooth manifold of real dimension 2ℓ, CW complex of finite type, as homotopy type

- we consider the combinatorial nature of some <u>algebraic</u> properties

Def: A *derivation* is a $\mathbb{K}$-linear map

$$\theta : S \to S$$

$$\theta(f \cdot g) = f\,\theta(g) + g\,\theta(f), \quad \forall f, g \in S$$

$\mathrm{Der}_{\mathbb{K}} S$
↳ the S-module of derivations of S.

- a canonical basis for $\mathrm{Der}_{\mathbb{K}} S$:

$$\{\partial x_i\}_{i=\overline{1,\ell}}$$

→ partial derivatives with respect to x_i

Def: The *module of $\mathcal{A}$-derivations* $D(\mathcal{A})$

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}\, S \;/\; \theta(f^{\mathcal{A}}) \in f^{\mathcal{A}} S\}$$

- $\mathcal{A}$ is called *free* if $D(\mathcal{A})$ is a free S-module

Examples:
- the boolean arrangement

$$f^{\mathcal{A}} = x_1 \cdots x_\ell$$

$$D(\mathcal{A}) = \langle x_1 \partial x_1, \ldots, x_\ell \partial x_\ell \rangle.$$

- the braid arrangement

$$f^{\mathcal{A}} = \prod_{1 \le i < j \le \ell} (x_i - x_j)$$

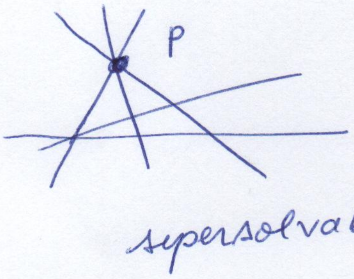
$\Downarrow$

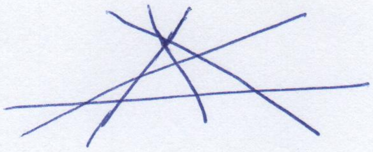
- fiber-type arrangements / supersolvable arrangements
(Falk - Randell) (Stanley)

it's complement is the total space of a sequence of fibrations

• supersolvability for line arrangements:

∃ a multiple point P in $L(A)$ such that all the other multiple points in $L(A)$ sit on a line of A that passes through P.





• reflection arrangements are free (Terao)

Conjecture (Terao) If A is free then so is any other arrangement in it's lattice isomorphism class.

open, even for $l=3$.

there is no lattice so that it's moduli space contains a free and a non-free arrangement

(Yuzvinsky) · free arrangements form an open subset in the moduli space associated to a given lattice, $V(L(A))$.

usually either very big or 0-dimensional; when 0-dim., it has finitely many components, Galois conjugates

rigid arrangements

here Terao's conj. holds!

Exponents of a free arrangement

$\mathcal{A}$ free, $D(\mathcal{A}) = \langle \theta_1, \dots, \theta_\ell \rangle$

homogeneous basis.

$$\theta_i = \sum_{j=1}^{\ell} a_j^i \partial x_j \,, \quad \deg (a_j^i) = d_i \,, \ \forall j$$

$\deg(\theta_i) := d_i$

→ not necessarily ordered

$$\exp(\mathcal{A}) := (d_1, \dots, d_\ell)$$

Rem: $\mathcal{A} \neq \emptyset$, then $\theta_E = \sum_i x_i \partial x_i \in D(\mathcal{A})$.

- we may assume $d_1 = 1$

Examples: • $\ell = 2 \Rightarrow \mathcal{A}$ free and $\exp(\mathcal{A}) = (1, |\mathcal{A}|-1)$

Say $f^{\mathcal{A}} = x_1 \cdot f_0^{\mathcal{A}}$, $f_0^{\mathcal{A}} \in \mathbb{K}[x_1, x_2]$

$$D(\mathcal{A}) = \langle \underset{\substack{\parallel \\ x_1 \partial x_1 + x_2 \partial x_2}}{\theta_E} , f_0^{\mathcal{A}} \cdot \partial x_2 \rangle$$

• $\mathcal{A}$ boolean $\subset V^{(\ell)}$, $\exp(\mathcal{A}) = \underbrace{(1, \dots, 1)}_{\ell \text{ components}}$

• $\mathcal{A}$ braid arr. $\subset V^{(\ell)}$, $\exp(\mathcal{A}) = (1, 2, \dots, \ell-1)$

characteristic polynomial of an arrangement ⑥

Möbius function : $\mu : L(\mathcal{A}) \to \mathbb{Z}$

- $\mu(V) = 1$
- $\mu(X) = -\sum\limits_{X \subsetneq Y,\ Y \in L(\mathcal{A})} \mu(Y)$, for $\underset{L(\mathcal{A})}{X \in} \neq V$

$\Rightarrow \mu(H) = -1$, $\forall\ H \in \mathcal{A}$

$$\mu(X) = \#\{H \in \mathcal{A} \mid X \subset H\} - 1$$
$$= |\mathcal{A}_X| - 1 \text{ , for any } X \text{ with } \operatorname{codim} X = 2$$

char poly of $\mathcal{A}$:

$$\underbrace{\chi(\mathcal{A}, t)}_{\text{combinatorial}} = \sum_{X \in L(\mathcal{A})} \mu(X)\, t^{\dim X}$$

For $\mathcal{A}$ central $\chi(\mathcal{A}, t) \ \vdots\ (t-1)$

$$\chi_0(\mathcal{A}, t) := \chi(\mathcal{A}; t) \big/ (t-1)$$

<u>Terao's factorization theorem</u> $\mathcal{A}$ free

with $\exp(\mathcal{A}) = (d_1, \dots, d_l)$ then

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{l} (t - d_i)$$

<u>Deletion - Restriction</u> $X \in \mathcal{L}(\mathcal{A})$

$$\mathcal{A}_X = \{ H \in \mathcal{A} \mid H \supset X \} \subset \mathcal{A} \subset V$$

$\downarrow$

localization of $\mathcal{A}$ at X

$$\mathcal{A}^X = \{ H \cap X \mid X \in \mathcal{A} \setminus \mathcal{A}_X \} \subset X$$

$\downarrow$

restriction of $\mathcal{A}$ onto X

- let $X = H \in \mathcal{A}$.

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A} \setminus \{H\}, t) - \chi(\mathcal{A}^H, t)$$

- $\mathcal{A}$ free $\rightarrow$ $\mathcal{A}_X$ free, $\forall X \in \mathcal{A}$

$\rightarrow$ restrictions of free arr. are not necessarily free

$\swarrow$

(Orlik - Solomon - Terao) $- \mathcal{A}^H$ free for $\forall H \in \mathcal{A}$, $\mathcal{A}$ Coxeter arr.

(Hoge - Röhrle) $\rightarrow$ for all restrictions of reflection arrangements

<u>Thm (Terao)</u> 1980 Let $H \in \mathcal{A}$, $\mathcal{A}' = \mathcal{A} \setminus \{H\}$, $\mathcal{A}'' = \mathcal{A}^H$.
Then any of the following two imply the third:

1) $\mathcal{A}$ free with $\exp(\mathcal{A}) = (d_1, d_2, .., d_{\ell-1}, d_\ell)$
2) $\mathcal{A}'$ free with $\exp(\mathcal{A}') = (d_1, .., d_{\ell-1}, d_\ell - 1)$
3) $\mathcal{A}''$ free with $\exp(\mathcal{A}'') = (d_1, .., d_{\ell-1})$

- a generalization of the previous result:

<u>Division Theorem</u> (Abe, 2016, Inv. Math)

$\ell = 3$: $\mathcal{A}$ is free if $\mathcal{A}''$ is free and $\chi(\mathcal{A}'',t) \mid \chi(\mathcal{A},t)$.

<u>Thm</u> (Abe)

Let $\chi_0(\mathcal{A},t) = (t-a)(t-b)$, $a, b \in \mathbb{R}$, $a \le b$. Then $\mathcal{A}$ is free iff there is a line L (not necessarily in $\mathcal{A}$) such that $n_L = a+1$ or $n_L = b+1$, where

$$n_L := \#\{H \cap L \neq \emptyset \mid H \in \mathcal{A},\ H \neq L\}.$$

In any case, $n_L \le a+1$ or $n_L \ge b+1$.

- we identify central arr in $\mathbb{C}^3$ to line arr in $\mathbb{P}\mathbb{C}^2$ and sometimes omit the exponent $1 \to$ from θ_E

<u>Thm</u> (Terao) <u>Addition - deletion theorem</u>.

Let $\mathcal{A}$ be an arrangement in $\mathbb{P}^2_{\mathbb{C}}$ (central in $\mathbb{C}^3$), $\exp(\mathcal{A}) = (a,b)$

1. Let $H \notin \mathcal{A}$. Then $\mathcal{B} = \mathcal{A} \cup \{H\}$ is free with $\exp(\mathcal{B}) = (a, b+1)$ iff $|\mathcal{A} \cap H| = a+1$

2. Let $H \in \mathcal{A}$. Then $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is free with $\exp(\mathcal{A}') = (a, b-1)$ iff $|\mathcal{A}' \cap H| = a+1$

Def A is called <u>inductively free</u> if it can be obtained from the empty arr. using only addition, and <u>recursively free</u> if it can be constructed from the empty arrangement using both addition and deletion techniques.

- free, not recursively free : example by Hoge - Cuntz (27 lines), <u>Abe - Cuntz - Kawanoue - Nozawa</u> (<u>13 lines</u>) → minimal $|A|$.
(ACKN)

Thm (ACKN)
Free arrangements in $\mathbb{C}P^2$ with at most 12 lines are recursively free.

- research in this direction (small $|A|$) is motivated by the search of counterexamples.
  - Wakefield - Yuzvinsky → $|A| \leq 11$
  - Faenzi - Vallès → $|A| < 12$
  ACKN
  - Dimca - Ibadula - M. → $|A| < 13$.
  - Barakat - Behrends - Jefferson - Kühne - Leuner
  ↳ a database of all rank 3 matroids with at most <u>14</u> atoms and integrally splitting char. poly.

Some combinatorial sufficient conditions for freeness:

- (Abe) If $(a,b) = \exp(A)$ and $a \leq 5$, then the conjecture holds for A.

fre

Terao

- (Dimca-Sticlaru) A fre with $\exp(A) = (a,b)$ and $a \leq m(A)$ then the Terao conj. holds for A. (i.e. in the lattice iso. class of A)

$\| $ def

maximal multiplicity of the multiple points of A.

<u>Weaker notion of freeness</u>:

<u>Def</u> (Dimca-Sticlaru) $A \subseteq \mathbb{CP}^2$ is called <u>nearly-free</u> if $D(A)$ has a minimal generating system

$$\theta_E, \theta, \phi_1, \phi_2 \text{ such that}$$

$$\deg \theta = a \leq b = \deg \phi_1 = \deg \phi_2$$

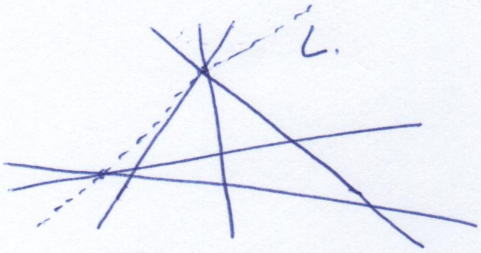
with the unique relation

$$h_0 \theta_E + h\theta + \beta_1\phi_1 + \beta_2\phi_2 = 0,$$

$$h_0, h, \beta_1, \beta_2 \in S, \ \deg \beta_1 = \deg \beta_2 = 1$$

We call the pair <u>(a,b)</u> the exponents of the nearly-free arrangement A.

- (Dimca–Sticlaru) $\chi(\mathcal{A},t) = (t-a)(t-b+1)+1$



$\mathcal{B}$.

$|\mathcal{A}| = 6$

$\exp(\mathcal{A}) = (2,4)$

$|\mathcal{B}| = 7$

$\exp(\mathcal{B}) = (2,4)$

$\mathcal{A}$ nearly free ; $\mathcal{A} \cup \{L\}$ free

Free – nearly-free interplay

Thm (Abe–Dimca) $\mathcal{A} \subseteq \mathbb{CP}^2$, $H \in \mathcal{A}$, $\mathcal{B} = \mathcal{A} \setminus \{H\}$, $a \le b$ non-negative integers. Then any two of the following imply the third:

1. $\mathcal{A}$ free with $\exp(\mathcal{A}) = (a,b)$
2. $\mathcal{B}$ is nearly free with $\exp(\mathcal{B}) = (a,b)$
3. $|\mathcal{A}^H| = a$

- it makes sense to consider a nearly free version of the Terao conjecture:

? is near-freeness combinatorial ?

Thm (DIM) The 'near Terao' conjecture holds for $\mathcal{A} \subseteq \mathbb{CP}^2$ with $|\mathcal{A}| \le 12$.

Thm (Abe–Ibadula–M) $\mathcal{A} \subseteq \mathbb{CP}^2$ nearly-free with $\exp(\mathcal{A}) = (a,b)$ and $a \le 5$ then the 'near Terao' conjecture holds in the lattice isomorphism class of $\mathcal{A}$.

Def (Abe) $\mathcal{A} \subseteq \mathbb{CP}^2$ is called plus-one-generated (POG) with $\exp(\mathcal{A}) = (a, b)$ and level d if $D(\mathcal{A})$ has a minimal generating system of homogeneous generators

$$\theta_E, \theta_1, \theta_2, \varphi \text{ such that}$$

$$\deg \theta_1 = a, \deg \theta_2 = b, \deg \varphi = d \text{ and}$$

$$\beta_0 \theta_E + \beta_1 \theta_1 + \beta_2 \theta_2 + \alpha \varphi = 0,$$

$$\beta_0, \beta_1, \beta_2, \alpha \in S, \deg \alpha = 1.$$

$\Downarrow$ $d \geq b$.

Rem: • POG with $d = b \iff$ nearly fr.

• POG arrangements are 'next-to' fr ones:

Thm (Abe) $\mathcal{A} \subseteq \mathbb{CP}^2$

The following are equivalent:

1. $\mathcal{A}$ fr with $\exp(\mathcal{A}) = (a, b)$
2. For any $H \in \mathcal{A}$, $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is either fr with $\exp(\mathcal{A}) = (a, b-1)$ and $|\mathcal{A}'| - |\mathcal{A}^H| = b-1$ or POG with $\exp(\mathcal{A}') = (a, b)$ and level $|\mathcal{A}'| - |\mathcal{A}^H|$.
3. For some . . . .
4. For any $L \notin \mathcal{A}$, $\mathcal{B} = \mathcal{A} \cup \{L\}$ is either free with $\exp(\mathcal{B}) = (a, b+1)$ and $|\mathcal{B}^H| = a+1$ or POG with $\exp(\mathcal{B}) = (a+1, b+1)$ and level $|\mathcal{B}^H| - 1$.
5. For some . . . .