Moduli spaces and homological stability phenomena

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Moduli & Friends seminar
IMAR
26 May 2021

Plan

I — "classical" stability results (< 2000)

II — Moduli spaces of Riemann surfaces (Madsen-Weiss theorem, etc.)

III — Higher-dim. analogues of II (Galatius-Randal-Williams, etc.)

Outline for today:

Whitney & Bott (40s + 50s)

Configuration spaces (70s)

$GL_n(\mathbb{R})$ & pseudoisotopy & alg. K-theory (80s)

Monopoles (80s + 90s)

General idea

- family of moduli spaces $\mathcal{M}_n$ $\quad n \in \mathbb{N}$ or $\mathbb{Z}$

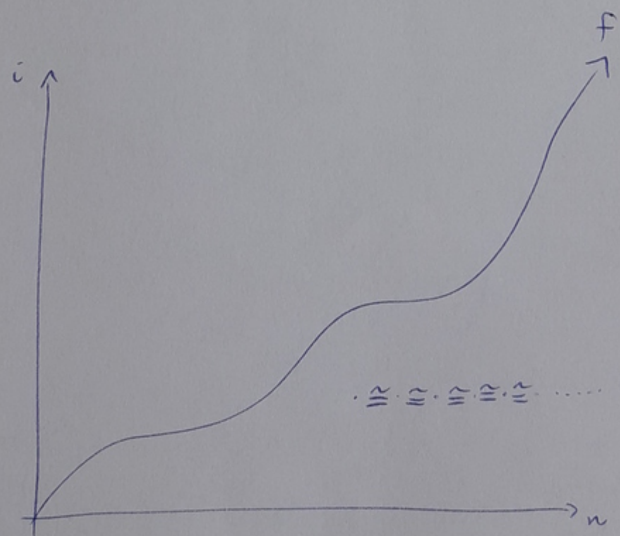
  (configuration spaces of n points
  classifying spaces of $O(n)$
  $\qquad GL_n(R)$
  $\{$Riemann surfaces of genus $g = n\}$ $\quad \dots$)

Q_1: Is $H_i(\mathcal{M}_n) \cong H_i(\mathcal{M}_{n+1})$ for all $i \leq f(n)$

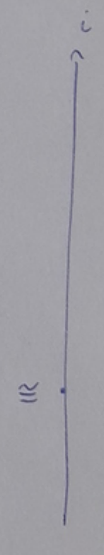
for some <u>unbounded</u> function f ?

Q_2: Is there a natural "$\mathcal{M}_\infty$", and can $H_i(\mathcal{M}_\infty)$ be calculated?

[Or the same questions with π_i instead of H_i.]



$H_i(\mathcal{M}_n)$



$H_i(\mathcal{M}_\infty)$

A_2: Typically of the form $\mathcal{M}_\infty := \mathrm{hocolim}\,(\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to \cdots)$

and $H_i(\mathcal{M}_\infty) \cong \underbrace{H_i(*)}_{}$ computable via standard techniques

Whitney & Bott (40s + 50s)

M smooth d-dim. manifold

$\mathrm{Emb}(M, \mathbb{R}^n) :=$ space of smooth embeddings $M \hookrightarrow \mathbb{R}^n$

$\mathcal{M}_n(M) := \mathrm{Emb}(M, \mathbb{R}^n) / \mathrm{Diff}(M)$ = moduli space of smooth submanifolds of $\mathbb{R}^n$ that are diffeomorphic to M

Fact: The projection
$$\begin{array}{c} \mathrm{Emb}(M, \mathbb{R}^n) \\ \downarrow \\ \mathcal{M}_n(M) \end{array}$$
is a fibre bundle with fibre $\mathrm{Diff}(M)$.

Whitney: When $n \geq 2d$ $\quad \mathrm{Emb}(M, \mathbb{R}^n) \neq \emptyset$.

When $n \geq 2(d+k)$ $\quad \pi_i(\mathrm{Emb}(M, \mathbb{R}^n)) = 0 \quad \forall i \leq k-1$

Corollary (Stability Q_1):
$$\pi_i(\mathcal{M}_n(M)) \cong \pi_i(\mathcal{M}_{n+1}(M))$$
for all $i \leq f(n) = \frac{1}{2}(n-2d) - 1$

proof:
$$\begin{array}{ccc} \mathrm{Diff}(M) & \xrightarrow{id} & \mathrm{Diff}(M) \\ \downarrow & & \downarrow \\ \mathrm{Emb}(M, \mathbb{R}^n) & \longrightarrow & \mathrm{Emb}(M, \mathbb{R}^{n+1}) \\ \downarrow & & \downarrow \\ \mathcal{M}_n(M) & \longrightarrow & \mathcal{M}_{n+1}(M) \end{array}$$

$\rightsquigarrow$ map of LES on π_*

$\rightsquigarrow$ 5-lemma.

For stability Q_2:

G topological group

Def $BG :=$ any space that admits a (numerable) universal principal G-bundle

bundles with fibre = structure group $= G$

it classifies all principal G-bundles over X via $[X, BG]$ $= \mathrm{Map}(X, BG)/\text{homotopy}$

Lemma This is unique up to htpy equivalence.

Theorem (Milnor) This exists for any G.

Proposition A principal G-bundle is universal $\iff$ its total space is contractible.

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Corollary (Stability Q_2). $\mathcal{M}_\infty(M) \simeq B\mathrm{Diff}(M)$

proof: Letting $n \to \infty$ we get a principal $\mathrm{Diff}(M)$-bundle

$$\begin{array}{c} \mathrm{Emb}(M, \mathbb{R}^\infty) \\ \big\downarrow \\ \mathcal{M}_\infty(M) \end{array}$$

NB: This can also be proved directly, without Whitney's embedding theorem

Whitney's theorem (+ Whitehead theorem) $\Rightarrow$ $\mathrm{Emb}(M, \mathbb{R}^\infty) \simeq *$. //

Similarly,

$$\begin{tikzcd} Gr_n(\mathbb{R}^\infty) \arrow[d, twoheadrightarrow] \\ Gr_n(\mathbb{R}^\infty)/O(n) \end{tikzcd}$$

Grassmannian of n-planes in $\mathbb{R}^\infty$

- is a principal $O(n)$ - bundle
- $Gr_n(\mathbb{R}^\infty) \simeq *$

So its base is $BO(n)$.

The map $Gr_n(\mathbb{R}^\infty) \longrightarrow Gr_{n+1}(\mathbb{R}^\infty)$

$$V \longmapsto V \oplus \mathbb{R} \subset \mathbb{R}^\infty \oplus \mathbb{R} \cong \mathbb{R}^\infty$$

descends to $BO(n) \longrightarrow BO(n+1)$.

<u>Proposition</u> (Stability Q_1) The map $BO(n) \longrightarrow BO(n+1)$ induces $\cong$ on π_i for all $i \leq f(n) = n-1$.

<u>proof</u> The map $Gr_n(\mathbb{R}^\infty) \longrightarrow Gr_{n+1}(\mathbb{R}^\infty)$ is $\simeq$ to a fibre bundle with fibre $= S^n$.

$$\begin{tikzcd} & O(n) \arrow[r] \arrow[d] & O(n+1) \arrow[r] \arrow[d] & S^n \\ S^n \arrow[r] & Gr_n(\mathbb{R}^\infty) \arrow[r] \arrow[d, twoheadrightarrow] & Gr_{n+1}(\mathbb{R}^\infty) \arrow[d, twoheadrightarrow] & \\ & BO(n) \arrow[r] & BO(n+1) & \end{tikzcd}$$

(rows / columns are fibre sequences)

$\rightsquigarrow$ map of LES on π_*

$\rightsquigarrow$ 5-lemma.

//.

Theorem (Bott) (Stability Q_2)

$$BO := \operatorname*{hocolim}_{n \to \infty} \left(BO(n) \right)$$

$$\pi_i(BO) \cong \begin{cases} \mathbb{Z}/2 & i \equiv 1 \pmod 8 \\ \mathbb{Z}/2 & 2 \\ 0 & 3 \\ \mathbb{Z} & 4 \\ 0 & 5 \\ 0 & 6 \\ 0 & 7 \\ \mathbb{Z} & 8 \end{cases} \qquad \text{for } i \geq 1$$

[+ Similar story for unitary groups $U(n)$.]

Configuration spaces (70s)

M smooth manifold

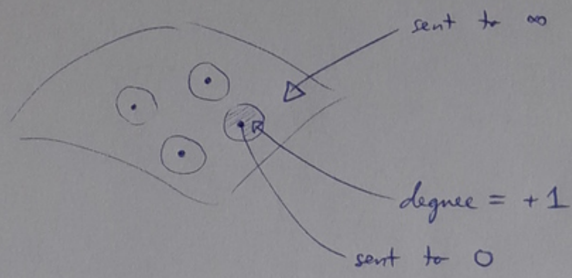
$$F_n(M) = \{ (x_1 \dots x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j \}$$

$$C_n(M) = F_n(M) / \Sigma_n$$

$$\Gamma_n(M) = \{ \text{sections of } \underbrace{T^\infty M \to M}_{\substack{\text{fibrewise 1-pt} \\ \text{compact}^\text{n} \text{ of} \\ TM \to M}} \text{ of } \underbrace{\text{degree} = k}_{\substack{\text{alg. int. \# with the} \\ \text{section at } 0}} \}$$

compactly-supported: outside of a compact $K \subset M$, equal to the section at ∞

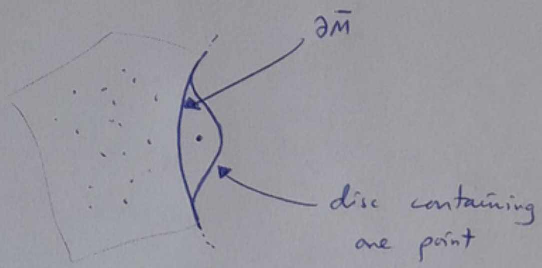
Maps: $\mathrm{scan}_n : C_n(M) \longrightarrow \Gamma_n(M)$



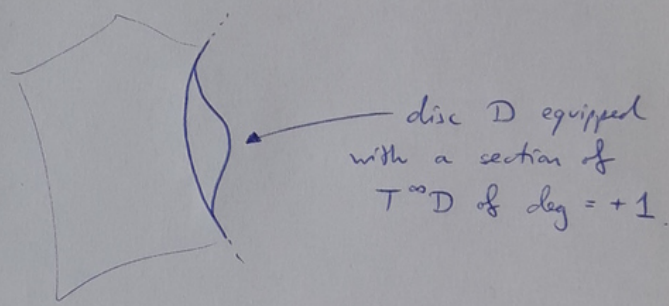
(defined precisely using the exponential map)

if $M = \mathrm{int}(\bar{M})$, $\partial\bar{M} \neq \phi$:

$C_n(M) \longrightarrow C_{n+1}(M)$



$\Gamma_n(M) \longrightarrow \Gamma_{n+1}(M)$



Obs $\Gamma_n(M) \to \Gamma_{n+1}(M)$ has a homotopy inverse.

We have a diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & C_n(M) \arrow[r] \arrow[d, "\mathrm{scan}_n"] & C_{n+1}(M) \arrow[r] \arrow[d, "\mathrm{scan}_{n+1}"] & \cdots & C_\infty(M) \arrow[d, "\mathrm{scan}_\infty"'] \\
\cdots \arrow[r, "\simeq"'] & \Gamma_n(M) \arrow[r, "\simeq"'] & \Gamma_{n+1}(M) \arrow[r, "\simeq"'] & \cdots & \Gamma_0(M)
\end{tikzcd}
$$

<u>Theorem</u> (McDuff '75) $\mathrm{scan}_n : C_n(M) \longrightarrow \Gamma_n(M)$

induces $\cong$ on H_i for all $i \leq f(n)$

$f(n)$ = non-explicit unbounded non-decreasing function.

<u>Theorem</u> (Segal '79) when $M = \mathrm{int}(\bar{M})$, $\partial\bar{M} \neq \phi$

we may take $f(n) = n/2$ in the above statement.

<u>Corollary</u> $(Q_1 + Q_2)$: if $M = \mathrm{int}(\bar{M})$, $\partial\bar{M} \neq \phi$, then

(1) $C_n(M) \longrightarrow C_{n+1}(M)$ induces $\cong$ on H_i for $i \leq n/2$

(2) $H_i(C_\infty(M)) \cong H_i(\Gamma_0(M))$

Special case — <u>braid groups</u>

- $B_n = \pi_1 C_n(\mathbb{R}^2)$
- [Fadell-Neuwirth] $\pi_i C_n(\mathbb{R}^2) = 0$ for $i \geq 2$
- Hence $\widetilde{C_n(\mathbb{R}^2)} \twoheadrightarrow C_n(\mathbb{R}^2)$ is a principal B_n-bundle with contractible total space
- So $C_n(\mathbb{R}^2) \simeq BB_n$

(1) $H_i(B_n) \cong H_i(B_{n+1})$ for $i \leq n/2$ [Arnol'd '69]

(2) $H_i(B_\infty) \cong H_i(\mathrm{Map}^c_0(\mathbb{R}^2, S^2))$

$\cong H_i(\Omega^2_0 S^2)$ ⟶ will see again later

More recently:

$$C_n^+(M) = F_n(M)/A_n$$

(double cover of $C_n(M)$)

<u>Thm</u> (P. '13) if $M = \mathrm{int}(\bar{M})$, $\partial\bar{M} \neq \phi$

then $$H_i(C_n^+(M)) \cong H_i(C_{n+1}^+(M)) \quad \text{for } i \leq n/3$$

(and the "slope" of $1/3$ is optimal for $\mathbb{Z}$ coefficients)

<u>Thm</u> (Miller - P. '15) $$H_i(C_\infty^+(M)) \cong H_i(\Gamma_0^+(M))$$

for a certain double cover $\Gamma_0^+(M) \longrightarrow \Gamma_0(M)$.

$GL_n(R)$ & algebraic K-theory & pseudoisotopy (80s)

$GL_n(R)$ (as discrete group)

R ring.

Thm (Charney) (Stability Q_1) If R is a Dedekind domain (e.g. PID)

$$H_i(GL_n(R)) \cong H_i(GL_{n+1}(R)) \quad \text{for} \quad i \leq f(n) = \frac{n-5}{4}$$

Stability Q_2 $\longleftrightarrow$ $\boxed{\text{alg. K-theory of rings}}$

Def (Quillen) $X \longmapsto X^+$: {spaces} $\longrightarrow$ {spaces}

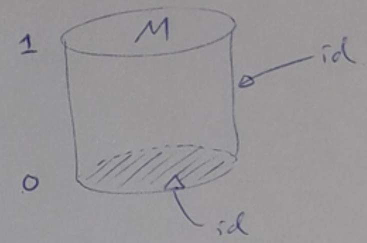
characterised by $H_i(X^+) \cong H_i(X)$ (with local coeffs)

$$\pi_1(X^+) = \pi_1(X) / \text{maximal perfect subgroup}$$

$$K_i(R) := \pi_i(BGL_\infty(R)^+)$$

M manifold

$$Ps(M) = \mathrm{Diff}\left(M \times I \,,\, (\partial M \times I) \cup (M \times \{0\})\right)$$



NB: $F \in Ps(M)$ preserves levels $M \times \{t\}$
$\iff$ F is an isotopy to the identity (rel. ∂M)

Def $f, g \in \mathrm{Diff}(M)$ are pseudoisotopic iff $\exists\, F \in Ps(M)$: $f = g \circ F_1$.

Note isotopic $\Rightarrow$ pseudoisotopic

Thm (Cerf '70) if $\pi_1(M) = 0$ & $\dim(M) \geq 6$ & $\partial M = \emptyset$
then $Ps(M)$ is path-connected

hence: pseudoisotopic $\Rightarrow$ isotopic.

$$Ps(M) \longrightarrow Ps(M \times I) \qquad F \longmapsto F \times id_I$$

Thm (Igusa '88) (Stability Q_1) This induces $\cong$ on π_i for $i \leq f(n) = \min\left(\frac{n-8}{2}, \frac{n-5}{3}\right)$

$n = \dim(M)$.

Stability Q_2 $\longleftrightarrow$ $\boxed{\text{alg. K-theory of spaces}}$

[Waldhausen] $X \longmapsto A(X)$ space

Thm (Waldhausen '78) $Ps(M \times I^\infty) \simeq \Omega^2 Wh(M)$

$$A(M) \simeq Wh(M) \times \Omega^\infty \Sigma^\infty (M_+)$$

Monopoles (80s + 90s)

$\mathcal{M}_n$ = magnetic monopoles on $\mathbb{R}^3$ of total charge $= n$.

$= (A, \Phi)$ satisfying the Bogomolny equations

$$D_A \Phi = \underbrace{* F_A}_{\text{curvature}}$$

A: connection on $SU(2) \times \mathbb{R}^3 \to \mathbb{R}^3$

Φ: Higgs field

Theorem (Donaldson '84)

$$\mathcal{M}_n \cong \underbrace{\left\{ f : \mathbb{CP}^1 \to \mathbb{CP}^1 \;\middle|\; \begin{array}{l} f \text{ holomorphic} \\ f(\infty) = 0 \\ \text{degree}(f) = n \end{array} \right\}}_{=: R_n}$$

(topologised as a subspace of $\Omega^2 S^2 = \mathrm{Map}_*(S^2, S^2)$)

Map_*: based continuous maps.

Note: $R_n \ni f \longleftrightarrow$ n zeros in $\mathbb{R}^2$ (may collide)
n poles in $\mathbb{R}^2$ (— " —)

poles may not collide with zeros.

$\boxed{\text{"add a new pole \& zero far away"}} : R_n \longrightarrow R_{n+1}$

We have:

$$\begin{array}{ccccccc} \cdots \to & R_n & \longrightarrow & R_{n+1} & \to \cdots \\ & \big\downarrow & & \big\downarrow & \\ \cdots \to & \Omega^2_n S^2 & \xrightarrow{\simeq} & \Omega^2_{n+1} S^2 & \to \cdots \end{array}$$

<u>Thm</u> (Segal '79) The inclusion $R_n \hookrightarrow \Omega^2_n S^2$ induces $\cong$ on π_i for $i \leq f(n) = n$.

<u>Corollary</u> (Stability $Q_1 + Q_2$)

(1) $H_i(\mathcal{M}_n) \cong H_i(\mathcal{M}_{n+1})$ for $i \leq n$.

(2) $H_i(\mathcal{M}_\infty) \cong \underline{H_i(\Omega^2_0 S^2)}$

This was also the stable homology of the braid groups!

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<u>Thm</u> (Cohen-Cohen-Mann-Milgram '91)

In fact $\mathcal{M}_n$ is <u>stably htpy equivalent</u> to BB_{2n} !

In particular $\boxed{H_i(\mathcal{M}_n) \cong H_i(B_{2n})}$.

<u>Next time</u>: Moduli spaces of Riemann surfaces ......