Appendix to part I.

Saito's criterion (1980) Let $\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$, $\mathcal{A} \subset \mathbb{P}^{\ell-1}_{\mathbb{C}}$.

Let $M(\theta_1, \dots, \theta_\ell)$ be a matrix with (i,j)-th entry $\theta_j(x_i)$.

TFAE: 1. $\det M(\theta_1, \dots, \theta_\ell) = f^{\mathcal{A}}$, up to a scalar

2. $\theta_1, \dots, \theta_\ell$ is a basis for $D(\mathcal{A})$ over S $\overset{}{=} \mathbb{C}[x_1, \dots, x_\ell]$.

Examples:

- $\mathcal{A} \subset V = \mathbb{C}^2$, $f^{\mathcal{A}} = x_1 \cdot f_0^{\mathcal{A}}$ \ $D(\mathcal{A})$ refl. module of rank 2 $\Rightarrow$ free.

central

$S = \mathbb{C}[x_1, x_2]$ $\qquad \theta_E = x_1 \partial x_1 + x_2 \partial x_2$

$$\theta = f_0^{\mathcal{A}} \cdot \partial x_2$$

$$\det M(\theta_E, \theta) = \begin{vmatrix} x_1 & 0 \\ x_2 & f_0^{\mathcal{A}} \end{vmatrix} = f_{\mathcal{A}} \Rightarrow \mathcal{A} \text{ free}$$

(of exp $(1, |\mathcal{A}|-1)$)

- $\mathcal{A}$: $f_{\mathcal{A}} = x_1 \cdots x_\ell$ (boolean arrangement)

$$D(\mathcal{A}) = \langle x_1 \partial x_1, \dots, x_\ell \partial x_\ell \rangle$$

$$\det M(x_1 \partial x_1, \dots, x_\ell \partial x_\ell) = \begin{vmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots 0 & x_\ell \end{vmatrix} = f^{\mathcal{A}}.$$

$\theta \in D(\mathcal{A})$; $\theta(f^{\mathcal{A}}) = \sum_i g_i x_1 \cdots \hat{x_i} \cdots x_\ell = f_{\mathcal{A}} \cdot \sum_i \frac{g_i}{x_i} \rightsquigarrow$
$\overset{\shortparallel}{\sum_i g_i \partial x_i}$

take $g_i = x_i \partial x_i \Rightarrow \theta \in D(\mathcal{A})$

• $\mathcal{A}_\ell$: $f_{\mathcal{A}} = \prod\limits_{1\le i<j\le \ell} (x_i - x_j) \subset \mathbb{C}^\ell$

for $1\le k\le \ell$: $\theta_k = \sum\limits_{i=1}^{k} x_i^{k-1}\partial x_i$; $\theta_k(x_i-x_j) = x_i^{k-1} - x_j^{k-1}$

$$\det M(\theta_1,\dots,\theta_\ell) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{\ell-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_\ell & x_\ell^\ell & & x_\ell^{\ell-1} \end{vmatrix} = \prod_{1\le i<j\le \ell} (x_i - x_j) = f_{\mathcal{A}}$$

? $\exp(\mathcal{A}^\ell) = (1, 2, \dots, \ell-1)$

$$M(\mathcal{A}_\ell) \xrightarrow{\pi_{\ell-1}} M(\mathcal{A}_{\ell-1}) \xrightarrow{\pi_{\ell-2}} \cdots \xrightarrow{\pi_2} M(\mathcal{A}_2) \longrightarrow M(\mathcal{A}_1) = \mathbb{C}^*$$

π_k fibrations with fiber $\mathbb{C}\setminus\{k \text{ points}\} =: F_k$

Thm (Falk–Randell) $H^*(M(\mathcal{A})) \cong H^*(F_1)\otimes\cdots\otimes H^*(F_\ell)$

fiber-type

iso of graded k-modules

$\Downarrow$

$$\mathrm{Poin}(M(\mathcal{A}),t) = \prod_{k=1}^{\ell} (1+d_k t)$$

$$\chi(\mathcal{A},t) = \prod_{i=1}^{k} (t-d_i) \overset{\text{Terao}}{\Longleftarrow} (d_1,\dots,d_\ell) = \exp(\mathcal{A})$$

• In general $\chi(\mathcal{A},t) = t^\ell\, \mathrm{Poin}(M(\mathcal{A}), -t^{-1})$ (Orlik–Solomon 1980)

Line arrangements, some combinatorial perspectives, II.

Multiarrangements

$\subset V^{(H)}$

Def A multiarrangement is a pair $(\mathcal{A}, m)$, $\mathcal{A}$ arrangement and $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ a map, called multiplicity.

- $\ell = \alpha \rightsquigarrow (\mathcal{A}, m)$ α-multiarrangement
- $\mathcal{A}$ arrangement $\Leftrightarrow (\mathcal{A}, m)$ with $m \equiv 1$

$(\mathcal{A}, m) \rightsquigarrow$ associated derivation module

$$D(\mathcal{A}, m) := \{ \theta \in \mathrm{Der}\, S \,/\, \theta(\alpha_H) \in S\alpha_H^{m(H)} \ \forall H \in \mathcal{A},\ H = \mathrm{Ker}\, \alpha_H \}$$

$(\mathcal{A}, m)$ free $\overset{\mathrm{def}}{\Longleftrightarrow}$ $D(\mathcal{A}, m)$ free S-module

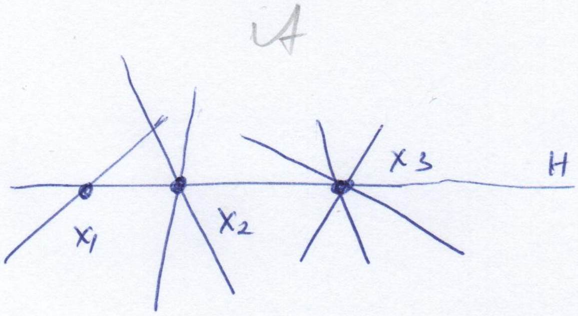
$\Downarrow$

$\exp(\mathcal{A}, m)$ = degrees of the elements in a homogeneous basis for $D(\mathcal{A}, m)$.

Ziegler restrictions : $\mathcal{A} \ni H$, the Ziegler restriction of $\mathcal{A}$ onto H is the multiarrangement $(\mathcal{A}^H, m^H)$ with

$$m^H(X) = \# \{ L \in \mathcal{A} \setminus \{H\} \,/\, L \cap H = X \}, \ X \in \mathcal{A}^H$$

Exp



$m^H(x_1)=1$

$m^H(x_2)=2$

$m^H(x_3)=3$.

$$\Delta(\mathcal{A}^H, m^H) = \{ \theta \in \mathrm{Der}\,\bar{S} \;/\; \theta(\alpha_K) \in \bar{S}(\alpha_K)^{m^H(K)}, \ \forall K \in \mathcal{A}^H \}$$

$$\bar{S} = S/(\alpha_H)$$

- Any $(\mathcal{A}, m)$ 2-multiarrangement is free and it's exponents

$\exp(\mathcal{A}, m) = (a,b)$ are such that

difficult to compute generally

$$a+b = |m| := \sum_{H \in \mathcal{A}} m(H)$$

$\Rightarrow$ $\mathcal{A} \subseteq \mathbb{CP}^2$ $(\mathbb{C}^3)$ $\Rightarrow$ $\Delta(\mathcal{A}^H, m^H)$ free, of rank 2, $\forall H \in \mathcal{A}$.

Prop (Wakefield-Yuzvinsky) Let $\mathcal{A} \subseteq \mathbb{CP}^2$ $(\mathbb{C}^3)$ and $(\mathcal{A}^H, m^H)$ it's Ziegler restriction onto H, for some $H \in \mathcal{A}$.

1. If $n_H := |\mathcal{A}^H| \geq (|\mathcal{A}|+1)/2$, then $\exp(\mathcal{A}^H, m^H) = (|\mathcal{A}| - n_H, n_H - 1)$

2. If $\exists X \in \mathcal{A}^H$ with $m = m^H(X) \geq (|\mathcal{A}|-1)/2$, then

$$\exp(\mathcal{A}^H, m^H) = (m, |\mathcal{A}|-1-m)$$

$D_H(\mathcal{A}) := \{ \theta \in D(\mathcal{A}) \mid \theta(\alpha_H) = 0 \}$, $H \in \mathcal{A}$

$\cap$ $\qquad\qquad$ $H = \ker \alpha_H$

$D(\mathcal{A})$

$$D(\mathcal{A}) \cong S\theta_E \oplus D_H(\mathcal{A})$$

Rco of S-alg.

$\pi_H : D_H(\mathcal{A}) \to D(\mathcal{A}^H, m^H)$ (defined by taking modulo α_H)

the Ziegler restriction map

Thm (Ziegler, 1989) Let $\mathcal{A}$ be free with $\exp(\mathcal{A}) = (a, b)$.

Then, for any $H \in \mathcal{A}$, the Ziegler restriction $(\mathcal{A}^H, m^H)$ is also free of exponents (a, b) and π_H is surjective.

motivation for our focus on multiarrangements: Yoshinaga connected 2-multiarrangements to the freeness of 3-arrangements ($\mathcal{A} \in \mathbb{C}^3$).

Yoshinaga's criterion (2005) $H \in \mathcal{A}$, $\chi_0(\mathcal{A},t) = t^2 - (|\mathcal{A}|-1)t + b_2^\circ(\mathcal{A})$

and $\exp(\mathcal{A}^H, m^H) = (a,b)$. Then $\dim_{\mathbb{C}} \operatorname{Coker} \pi_H = b_2^\circ(\mathcal{A}) - ab$

and $\mathcal{A}$ free iff $b_2^\circ(\mathcal{A}) = ab$, in which case $\exp(\mathcal{A}) = (a,b)$.

Prop Let $\chi_0(\mathcal{A},t) = (t-n)(t-n-r)+1$, $n, r \geq 0$. If $r \neq 2$

then $\nexists H$ such that $n+1 < n_H < n+r+1$. If $r=2$,

then the same conclusion holds, except when

$\mathcal{A}$ is free with $\exp(\mathcal{A}) = (n+1, n+1)$. In this case, if $n_H > n+1$,

for some $H \in \mathcal{A}$, then $n_H = n+2$.

Sketch of proof: We may assume $r \geq 2$. Say $\exists H$ s.th

$$n+1 \leq n_H \leq n+r+1 \quad (*)$$

$$(a,b)_\leq = \exp(\mathcal{A}^H, m^H) \Rightarrow a+b = 2n+r \quad (|\mathcal{A}^H|)$$

We will show that $n+1 \leq a \leq b \leq n+r+1$ (**)

Enough to show $a \geq n+1$.

Case I: $n_H \leq (|\mathcal{A}|+1)/2 \xrightarrow{\text{Abe}} a \geq \underbrace{n_H - 1}_{> n}$.

Case II : $n_H > (|A|+1)/2 \overset{W-Y}{\rightsquigarrow} a \in \{ n_H - 1, |A| - n_H \}$

Since ⊙ holds $\Rightarrow$ $n_H - 1 > n$
$|A| - n_H > n$
$\Rightarrow a > n.$

$2n+r = a+b$ and ⊙⊙ $\Rightarrow ab \geq (n+1)(n+r-1)$

$$0 \leq b_2^0(A) - ab \leq n(n+r) + 1 - (n+1)(n+r-1) = \underbrace{2-r}_{\leq 0}$$

Contradiction to Yoshinaga's criterion,

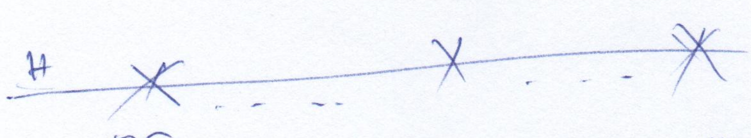
if $r \neq 2$.

If $r = 2 \Rightarrow b_2^0(A) - ab = 0 \Rightarrow A$ free with $\exp(A) = (n+1, n+1)$.

So, if $n_H > n+1 \overset{T.Th}{\Longrightarrow} n_H = n+2$.

Def.: An arrangement $A \subseteq \mathbb{CP}^2$ is called balanced if each of its Ziegler restrictions are balanced, i.e.

$\forall H \in A$
$\forall K \in A^H$
$\Big| \; m^H(K) \leq (|A|-1)/2$



$3 = m^H(x)$

$$\sum_{x \in A^H} m^H(x) = |A| - 1$$

Prop. $\mathcal{A}$ balanced and $\exists H \in \mathcal{A}$ such that

$r = n_H - 2$ and $\chi_0(\mathcal{A}, t) = (t-n)(t-n-r)+1$, $n, r \geq 0$.

If $r \neq 2$, then $\mathcal{A}$ is nearly free. If $r = 2$, then

$\mathcal{A}$ is either nearly free, or free with exp (n_H, n_H).

- If $\mathcal{A}$ (free) is not balanced, then the Terao conjecture holds for $\mathcal{A}$ (Wakefield-Yuzvinsky).

- If $\mathcal{A}$ is nearly free and non-balanced, then all the other arrangements in the realisation/moduli space of $L(\mathcal{A})$ are also nearly free. (Abe-Dimca)

Thm Let $\mathcal{A} \subseteq \mathbb{P}^2$, $\chi_0(\mathcal{A}, t) = (t-n)(t-n-r)+1$, $n, r \geq 0$.

If $r \geq n > 0$, then whether $\mathcal{A}$ is neither free nor nearly free depends only on $L(\mathcal{A})$.

Cor: $\mathcal{A}$ nearly free of exponents $(a, b)_\leq$ such that $a \leq \frac{|\mathcal{A}|}{3}$.

Then the "near Terao" conjecture holds in $L(\mathcal{A})$.

Prop. $\mathcal{A}$ nearly free with exponents $(a,b)_{\leq}$.

Then $m_H \leq b+1$, $\forall H \in \mathcal{A}$.

↙ a Yoshinaga-type criterion for nearly free arrangements

Thm (Abe–Dimca).

Let $\mathcal{A} \subseteq \mathbb{P}^2$ with $\chi_0(\mathcal{A}, t) = t^2 - (|\mathcal{A}|-1)t + b_2^0(\mathcal{A})$

Then $\mathcal{A}$ is nearly free if $\exists\, H \in \mathcal{A}$ such that

$b_2^0(\mathcal{A}) - ab = 1$, where $(a,b) = \exp(\mathcal{A}^H, m^H)$.

Sketch of proof for Thm A

- enough to assume $\mathcal{A}$ is balanced, and $r \neq 2$.

If $r = 2 \Rightarrow n \leq 2$ and $|\mathcal{A}| = 2n + 2 + r \leq 7$

↳ in this range both freeness and near freeness are combinatorial

- key Prop ⊛ $m_H \leq n+1$ or $m_H \geq n+r+1$

  if equality in either case $\xrightarrow{\text{Abe–Dimca}}$ $\mathcal{A}$ nearly free

- assume $m_H \leq n \leq r$ or $m_H \geq n+r+2$, $\forall H \in \mathcal{A}$.

If $\exists\, H$ such that $\lnot$, let $(a,b)_{\leq} = \exp(\mathcal{A}^H, m^H)$

Thm, $b-a \leq n_H - 2 \leq r-2$

but $a+b = 2n+r$ $\Big| \Rightarrow ab \geq (n+1)(n+r-1)$

$0 \leq b_2^0(A) - ab \leq n(n+r)+1 - (n+1)(n+r-1) = 2-r.$

$\Downarrow$

$r=1 \Rightarrow A$ nearly free

. If $\nexists H$ such that $n_H \leq n$, then $n_H \geq n+r+2$, $\forall H \in A$.

but if $n_H \geq n+r+3 \Rightarrow A$ neither free nor nearly free

$\Rightarrow n_H = n+r+2$, $\forall H \in A$.

. fix H, take $A' = A \setminus \{H\}$.

we will apply Yoshinaga criterion to A'

. take $L \in A'$ such that $A_{H \cap L} = \{H, L\}$: this is possible

since $|A| = 2n+r+1$ and $|A^H| = n+r+2$.

$|(A')^L| = n+r+1 \xrightarrow{W-Y} exp((A')^L; m^L) = (n-1, n+r)$

Y. crit $\Rightarrow A'$ free with $exp(A') = (n-1, n+r)$

$|A^H| = n+r+2$ $\Big| \overset{A.D.}{\Rightarrow} A$ nearly-free with $exp(A) = (n, n+r+1)$

$A \subseteq \mathbb{CP}^2 \rightsquigarrow$ assoc. sheaf (actually vector bundle)

rep of Ziegler restrictions onto $H \in A$ $\longmapsto$ splitting type of the restriction to H of the vector bundle

graded S-module of Jacobian syzygies

$$AR(f^A) := \{(a,b,c) \in S^3 \,/\, a\frac{\partial f^A}{\partial x} + b\frac{\partial f^A}{\partial y} + c\frac{\partial f^A}{\partial z} = 0\}$$

$$(a,b,c) \mapsto (a\partial x + b\partial y + c\partial z) \qquad \cong$$

$$D_0(A) := \{\theta \in \mathrm{Der}\, S \,/\, \theta(f^A) = 0\}.$$

Moreover $D_0(A) \cong D_H(A)$

$$\theta \longmapsto \theta - \left(\theta(\alpha_H)/\alpha_H\right)\theta_E$$

- A free $\Leftrightarrow D_0(A)$ free S-module $\Leftrightarrow AR(f^A)$ free S-mod.

$AR(f^A) \rightsquigarrow \widetilde{AR(f^A)}$ (sheaf associated to the S-module $AR(f^A)$)

(Nullstellensatz-Schenck)

$\widetilde{AR(f^A)}$ locally free $\xLeftarrow{\text{for } l=3}$ locally free iff A_x free for all $x \in L(A)_{H_0}$

Equiv of categories:

locally free sheaves $\longleftrightarrow$ vector bundles

from graded S-modules to sheaves:

$S = \mathbb{C}[x_1, x_2, x_3]$ $\qquad \mathbb{P}^2_{\mathbb{C}} = U_{x_1} \cup U_{x_2} \cup U_{x_3}$ $\qquad U_{x_i} : \{x_i \neq 0\}$

M - graded S-module

$\rightsquigarrow$

localization by x_i

$\tilde{M}$ sheaf on $\mathbb{P}^2_{\mathbb{C}}$ with sections $\Gamma(U_i, \tilde{M}) = (M \otimes_S S[\frac{1}{x_i}])_0$

degree zero component

$\mathcal{O} := \tilde{S}$; $\mathcal{O}(k) := \widetilde{S[k]}$

$S[k]_d = S_{k+d}$ $\qquad \Gamma_*(\tilde{M}) := \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathbb{P}^2_{\mathbb{C}}, \tilde{M} \otimes \mathcal{O}(d))$

degrees

$\exists$ a natural homomorphism

$$M \rightsquigarrow \Gamma_*(\tilde{M})$$

(Abe-Yoshinaga)

For $M = AR(f^A)$ this is an isomorphism, so we may assume an identification $AR(f^A) \longleftrightarrow \Gamma_*(\widetilde{AR(f^A)})$

$\mathcal{D}_0(A)$

- $\mathcal{E}$ rank 2 vector bundle on $\mathbb{P}^2_{\mathbb{C}}$ and $L \subset \mathbb{P}^2_{\mathbb{C}}$ line.
- $\mathcal{E}|_L$ vector bundle on $\mathbb{P}^1_{\mathbb{C}}$

$\rightarrow$ this allways splits, by Grothendieck's splitting thm.

$$\mathcal{E}|_L = \mathcal{O}_L(-e_1) \oplus \mathcal{O}_L(-e_2)$$

$(e_1, e_2) \overset{\text{def}}{=}$ the splitting type of $\mathcal{E}$ along L

Thm (Yoshinaga) Let $\mathcal{E}$ be a rank 2 vector bundle over $\mathbb{P}^2_{\mathbb{C}}$ and $L \subset \mathbb{P}^2_{\mathbb{C}}$ a line.

Let $\mathcal{E}|_L = \mathcal{O}_L(-e_1) \oplus \mathcal{O}_L(-e_2)$. Then $c_2(\mathcal{E}) \geq e_1 e_2$,

furthermore

$$c_2(\mathcal{E}) - e_1 e_2 = \dim_{\mathbb{C}} \operatorname{coker}\left(\Gamma_*(\mathcal{E}) \xrightarrow{\pi_L} \Gamma_*(\mathcal{E}|_L)\right)$$

where $c_2(\mathcal{E})$ is the second Chern number of $\mathcal{E}$ and π_L is the morphism of graded modules induced by the restriction of $\mathcal{E}$ to L.

Moreover $\mathcal{E}$ is splitting iff $c_2(\mathcal{E}) = e_1 e_2$.

Rem: $c_2(\mathcal{E})$, when $\mathcal{E} = \widetilde{AR(f^{\mathcal{A}})}$ is computed using the characteristic poly $\chi(\mathcal{A}, t)$ for locally free $\mathcal{A}$.

Yoshinaga: the splitting type along $H \in \mathcal{A} \Longleftrightarrow$ the exponents for the Ziegler restriction $(\mathcal{A}^H, m^H)$

(14) II

Thm Let A be POG with $\exp(A) = (a,b)$ and level d and L arbitrary line in $\mathbb{P}^2$.
Let (e_1^L, e_2^L) be the splitting type of $\widetilde{AR(f^A)}$ over L. Then $(e_1^L, e_2^L) \in \{(a-1,b), (a,b-1), \underbrace{(a+b-d-1,d)}\}$

at most one line $L \in A$.

Proof (Sketch): ~~consideration~~.

(Ahn–Min cc). $a-(d-b+1) \leq e_1^L \leq a$

$$e_1^L + e_2^L = |A| - 1 = a+b-1.$$

. assume $e_1^L < a-1$ [consider $L \in A$]

$$\Downarrow$$

$$b+1 \leq e_2^L \leq d$$

the Ziegler restriction map

$$\begin{array}{ccc} \Delta_L(A) & \xrightarrow{\pi_L} & \Delta(A^L, m^L) \\ | & & | \\ \langle \theta_2, \theta_3, \phi \rangle & & \langle \theta_1^L, \theta_2^L \rangle \\ \text{degs: } \downarrow \ \downarrow \ \downarrow & & \text{degs: } \downarrow \quad \downarrow \\ a \quad b \quad d & & e_1^L \quad e_2^L \end{array}$$

$\Rightarrow \pi_L(\theta_2), \pi_L(\theta_3) \in \langle \theta_1^L \rangle$

$\pi_L(\phi) = r\theta_1^L + s\theta_2^L$, $r, s \in S/(\alpha_L)$, $s \neq 0$

(Yoshinaga's criterion) $\dim_{\mathbb{K}} \mathrm{Coker}(\pi_L) < \infty$; even for $s \neq 0$, if $\deg \theta_2^L < d$ we can construct an infinite series in $\Delta(A^L, m^L)$ not in $\mathrm{Im}\, \pi_L$: $f\theta_1^L + \theta_2^L$, f arb. monomial

Corollary: $\mathcal{A}$ POG of $\exp(a,b)$ and level d,

then, for any $H \in \mathcal{A} \Rightarrow m_H \leq d+1$.

Prop. $\mathcal{A}$ POG of $\exp(a,b)$ and level d, $H \in \mathcal{A}$.

If $m_H \geq a \Rightarrow m_H \in \{a, a+1, b, b+1, d+1\}$.

A Yoshinaga-type criterion (for POG)

$\mathcal{A} \subseteq \mathbb{CP}^2$, $H \in \mathcal{A}$, $D(\mathcal{A}^H, m^H) = \langle \theta_1^H, \theta_2^H \rangle$.

$\boxed{P}$: Let $\mathcal{A}$ be such that $\exists\, H \in \mathcal{A}$ such that one of the conditions holds:

1. $\exists\, \alpha \in S$, $0 \neq \alpha$, $\deg \alpha = 1$ s.th. $\{\theta_1^H, \theta_2^H\} \neq \operatorname{Im} \pi \supset \{\theta_1^H, \alpha\theta_2^H\}$

2. $\exists\, \beta \in S$, $0 \neq \beta$, $\deg \beta = 1$ s.th. $\{\theta_1^H, \theta_2^H\} \neq \operatorname{Im} \pi \supset \{\theta_2^H, \beta\theta_1^H\}$

where π is the Ziegler restriction map onto H.

Thm. $\mathcal{A}$ satisfies $\boxed{P} \Leftrightarrow \mathcal{A}$ POG.

$\boxed{P} \rightarrow$ construct a basis for $D_H(\mathcal{A})$

$\mathcal{A}$ POG $\rightarrow$ take $H \in \mathcal{A}$ with $\exp(\mathcal{A}^H, m^H) \in \{(a, b-1), (a-1, b)\}$
$|\mathcal{A}| \geq 2$

and use π and a basis of $D_H(\mathcal{A})$ to construct a good basis for $D(\mathcal{A}^H, m^H)$ and get α, β.