Moduli spaces and homological stability phenomena

II

Moduli and Friends seminar
IMAR
9 June 2021

Plan

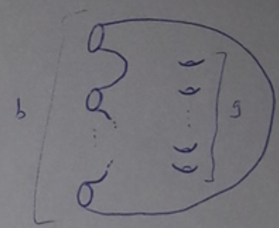
1. Moduli spaces of Riemann surfaces
2. Stabilisation maps & homological stability
3. Stable (co)homology — MMM classes
   Mumford conjecture
   scanning maps
   Madsen-Weiss theorem
4. Embedded surfaces
5. Higher dimensions
6. Disconnected submanifolds

(1)

Moduli spaces of Riemann surfaces

Fix $g \geq 2$
$b \geq 0$

$\Sigma_{g,b}$ — smooth, connected, compact, oriented surface of genus g with b ∂-components



$$\mathrm{Riem}(M) := \left\{ \begin{array}{c} \mathrm{Hom}(TM \oplus TM, \underline{\mathbb{R}}) \\ s \curvearrowright \downarrow \\ M \end{array} \;\middle|\; \begin{array}{l} s \text{ smooth} \\ s(p) : T_pM \oplus T_pM \longrightarrow \mathbb{R} \text{ is a} \\ \text{pos. definite inner product, } \forall p \in M \end{array} \right\}$$

with the compact-open topology

$$\mathrm{Hyp}(M) := \left\{ s \in \mathrm{Riem}(M) \;\middle|\; \sec_{(M,s)}(V) = -1 \quad \text{for all } \begin{array}{l} p \in M \\ V \subseteq T_pM \\ \text{2-dim} \end{array} \right\}$$

We can pull back Riemannian metrics along diffeo's $M \xrightarrow{\varphi} M$, and φ^* preserves the property that $\sec \equiv -1$, so:

$$\mathrm{Hyp}(M) \curvearrowleft \mathrm{Diff}(M)$$

In particular:

$$\boxed{\mathcal{H}_{g,b} := \mathrm{Hyp}(\Sigma_{g,b}) \curvearrowleft \mathrm{Diff}^+_\partial(\Sigma_{g,b})}$$

Smallprint: in $\mathcal{H}_{g,b}$, also require that ∂-components are geodesics & have length 1.

Def (Mapping class group)

$$\Gamma_{g,b} := \pi_0\left(\mathrm{Diff}^+_\partial(\Sigma_{g,b})\right) \cong \mathrm{Diff}^+_\partial(\Sigma_{g,b}) \Big/ \underbrace{\mathrm{Diff}^+_\partial(\Sigma_{g,b})_1}_{\text{path-component containing the identity.}}$$

Def (Teichmüller space & Moduli space)

$$\begin{tikzcd}
\mathcal{H}_{g,b} \arrow[d, "-/\mathrm{Diff}^+_\partial(\Sigma_{g,b})"'] \arrow[r, bend left, "-/\mathrm{Diff}^+_\partial(\Sigma_{g,b})_1"] & \mathcal{T}_{g,b} \arrow[dl, bend left, "-/\Gamma_{g,b}"] \\
\mathcal{M}_{g,b} &
\end{tikzcd}$$

$\mathcal{T}_{g,b}$ — Teichmüller space

$\mathcal{M}_{g,b}$ — Moduli space

Fact

The action $\Gamma_{g,b} \curvearrowright \mathcal{T}_{g,b}$ is properly discontinuous and
$\begin{cases} \text{free} & b \geq 1 \\ \text{has } \underline{\text{finite}} \text{ point-stabilisers} & b = 0 \end{cases}$

Thm [Teichmüller] $\mathcal{T}_{g,b} \cong \mathbb{R}^{6g-6+2b}$ ← contractible!

Coro

$$\begin{cases} \mathcal{M}_{g,b} \simeq B\Gamma_{g,b} & b \geq 1 \\ H_*(\mathcal{M}_{g,b}; \mathbb{Q}) \cong H_*(B\Gamma_{g,b}; \mathbb{Q}) & b = 0 \end{cases}$$

Thm [Earle–Eells '69 $b=0$; Earle–Schatz '70 $b\geq 1$, Gramain '73]

$$\mathrm{Diff}^+_\partial(\Sigma_{g,b})_1 \text{ is } \underline{\text{contractible}}.$$

Cor $$B\mathrm{Diff}^+_\partial(\Sigma_{g,b}) \xrightarrow{\simeq} B\Gamma_{g,b}$$

Summary: $$\mathcal{M}_{g,b} \simeq B\Gamma_{g,b} \simeq B\mathrm{Diff}^+_\partial(\Sigma_{g,b})$$

(rationally, if $b=0$)

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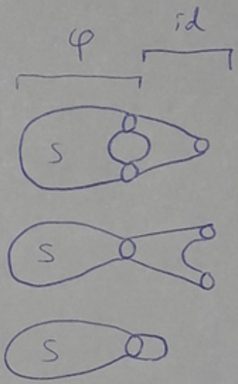
② Stabilisation maps

$(b\geq 2)$ $\quad \Gamma_{g,b} \xrightarrow{\alpha} \Gamma_{g+1,b-1}$

$(b\geq 1)$ $\quad \Gamma_{g,b} \xrightarrow{\beta} \Gamma_{g,b+1}$

$(b\geq 1)$ $\quad \Gamma_{g,b} \xrightarrow{\delta} \Gamma_{g,b-1}$



We are especially interested in the sequence:

$$\begin{tikzcd}
\cdots \arrow[r, "\alpha"] & \Gamma_{g,1} \arrow[r, "\beta"] \arrow[d, "\delta"] \arrow[rr, bend right, "\text{glue}"'] & \Gamma_{g,2} \arrow[r, "\alpha"] & \Gamma_{g+1,1} \arrow[r, "\beta"] \arrow[d, "\delta"] & \cdots \\
& \Gamma_{g,0} & & \Gamma_{g+1,0} &
\end{tikzcd}$$

Thm [Harer '85] + Ivanov, Boldsen, Randal-Williams

$$H_i(B\Gamma_{g,b}) \xrightarrow{\alpha_*} H_i(B\Gamma_{g+1,b-1}) \quad \text{isom. for } i \leq \tfrac{2}{3}(g-1)$$

$$H_i(B\Gamma_{g,b}) \xrightarrow{\beta_*} H_i(B\Gamma_{g,b+1}) \quad \text{isom. for } i \leq \tfrac{2}{3}g$$

$$H_i(B\Gamma_{g,b}) \xrightarrow{\delta_*} H_i(B\Gamma_{g,b-1}) \quad \text{isom. for } i \leq \tfrac{2}{3}g$$

Idea of proof

Strategy due to Quillen:

G_i —— family of groups & inclusions (indexed by some totally-ordered set)

construct $G_i \curvearrowright X_i$ — simplicial complex

such that
- stabilisers of simplices of X_i are (conjugate to) G_j for $j < i$.
- $\pi_*(X_i) = 0$ for $* \leq f(i)$ [$f(i) \to \infty$ as $i \to \infty$] "highly-connected"

⤳ spectral sequence

$$\bigoplus_{\substack{\text{orbits of}\\ p\text{-simplices}}} H_q(\mathrm{stab}(p\text{-simplex})) \Rightarrow H_*(X_i) = 0 \text{ in a range of degrees.}$$

⟶ inductive argument

In this case:

$$X_{g,b} = \left[\begin{array}{ll} \text{vertices} & : \text{ certain (isotopy classes of) arcs on } \Sigma_{g,b} \\ \text{simplices} & : \text{ collections of arcs that may be realised disjointly} \end{array} \right.$$

- (roughly) $\mathrm{stab}\underbrace{(a_0, \dots, a_p)}_{\text{arcs on } \Sigma_{g,b}} \cong$ MCG of $\underbrace{\Sigma_{g,b} \text{ cut along } a_0, \dots, a_p}_{\text{surface of lower complexity}}$

- $X_{g,b} \xhookrightarrow{(*)} Y_{g,b}$ ← bigger complex, where more general arcs are allowed.

[Hatcher] $Y_{g,b}$ deformation retracts onto $\underbrace{\mathrm{star}(\text{vertex})}_{\text{contractible}} \subseteq Y_{g,b}$

(complicated inductive argument) $\Rightarrow$ $(*)$ induces $\cong$ on π_* up to degree $g-2$.

(3) Stable (co)homology

MMM-classes (Miller-Morita-Mumford)

$B\Gamma_{g,b} \simeq B\mathrm{Diff}^+_\partial(\Sigma_{g,b})$ classifies smooth, oriented $\Sigma_{g,b}$-bundles (whose fibrewise boundary is the <u>trivial</u> $\partial\Sigma_{g,b}$-bundle)

$$\begin{tikzcd} T^v E \arrow[d] & \\ E \arrow[r, "f"] \arrow[d, "\pi"] & BSO(2) \simeq \mathbb{CP}^\infty \\ B\mathrm{Diff}^+_\partial(\Sigma_{g,b}) & \end{tikzcd}$$

rank-2 oriented vector bundle, built from $T(\pi^{-1}(x))$ for all x.

universal $\Sigma_{g,b}$-bundle

classifies rank-2 oriented (real) vector bundles.

$$H^*(BSO(2)) \cong \mathbb{Z}[e] \qquad \deg(e) = 2 \qquad \underline{\text{Euler class}}$$

$$\kappa_i := \int_{\text{fibre}} f^*(e^{i+1}) \in H^{2i}(B\Gamma_{g,b}) \qquad \underline{i^{th} \text{ Miller-Morita-Mumford class}}$$

<u>Now switch to $\mathbb{Q}$ coefficients</u>

$$\kappa_i \in H^{2i}(B\Gamma_{g,b};\mathbb{Q})$$

graded ring homomorphism

$$\mathbb{Q}[\kappa_1,\kappa_2,\dots] \longrightarrow H^*(B\Gamma_{g,b};\mathbb{Q})$$

<u>Fact</u> (easy check) This is compatible with stabilisation:

$$\begin{tikzcd}
 & & \mathbb{Q}[\kappa_1,\kappa_2,\dots] \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
\cdots & H^*(B\Gamma_{g,1};\mathbb{Q}) \arrow[l] & & H^*(B\Gamma_{g+1,1};\mathbb{Q}) \arrow[ll] & \cdots \arrow[l]
\end{tikzcd}$$

So if we define $B\Gamma_{\infty,1} := \operatorname{hocolim}_{g\to\infty} B\Gamma_{g,1}$

We get $\mathbb{Q}[\kappa_1,\kappa_2,\dots] \longrightarrow H^*(B\Gamma_{\infty,1};\mathbb{Q})$

understood

stable cohomology of MCGs

<u>Conjecture</u> (Mumford '83) This is an <u>isomorphism</u>.

<u>Thm</u> (Miller '86) It is <u>injective</u>.

$$\begin{tikzcd}
 & \mathbb{Q}[\kappa_1,\kappa_2,\dots] \arrow[dl, "(*)"'] \arrow[dr, "\text{injective}"] & \\
H^*(B\Gamma_{g,1};\mathbb{Q}) & & H^*(B\Gamma_{\infty,1};\mathbb{Q}) \arrow[ll]
\end{tikzcd}$$

$\cong$ for $* \leq \frac{2}{3}(g-1)$

but in higher degrees $(*)$ has a non-trivial kernel $\longrightarrow$ [Grigoriev]

Recall from last time:

Def (Quillen): $(X) \longmapsto \left(\begin{array}{c} X \\ \downarrow \\ X^+ \end{array}\right)$ is the unique operation characterised by

space

map of spaces

- $H_*(X) \to H_*(X^+)$ isom $\forall$ local coeffs.
- $\pi_1(X) \longrightarrow \pi_1(X^+) \cong \pi_1(X) / \text{max. perfect subgroup}$, with $\pi_1(X) \twoheadrightarrow \pi_1(X)/\text{max. perfect subgroup}$

Thm (Quillen) $BGL_\infty(R)^+$ is an infinite loop space

$$\left.\begin{array}{l} X_0 \simeq \Omega X_1 \\ X_1 \simeq \Omega X_2 \\ X_2 \simeq \Omega X_3, \ldots \end{array}\right] \quad X_0 \text{ infinite loop space.}$$

$\xleftrightarrow{1:1}$ generalised homology theories

& hence "easy" to compute with.

Def (Quillen) $K_i(R) := \pi_i(BGL_\infty(R)^+)$

Analogue for MCGs:

Thm [Tillmann '97] $(B\Gamma_{\infty,1})^+$ is an infinite loop space.

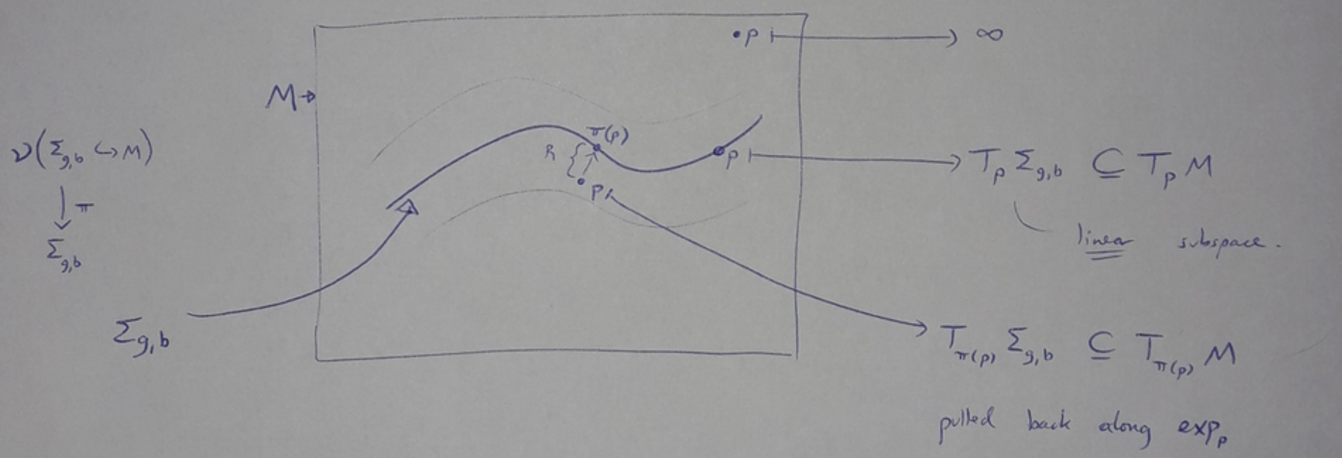
$$BDiff^+_\partial(\Sigma_{g,b}) \simeq \underbrace{\operatorname*{colim}_{N\to\infty} Emb_\partial(\Sigma_{g,b}, \mathbb{R}^N) \Big/ Diff^+_\partial(\Sigma_{g,b})}_{\mathcal{E}_\partial(\Sigma_{g,b},\, \mathbb{R}^N).}$$

<u>Scanning map</u>. (analogous to scanning map for config spaces — last time)

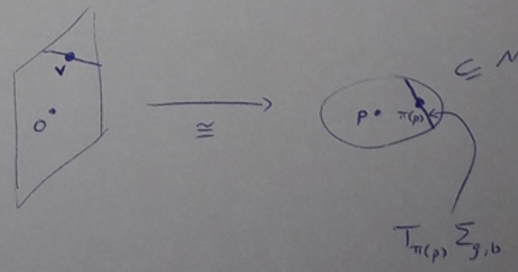
$$\mathcal{E}_\partial(\Sigma_{g,b}, \overset{M}{\cancel{}}) \longrightarrow \Gamma^{cpt}\left(\begin{array}{c}\dot{A}_2(TM)\\ \downarrow \\ M\end{array}\right)$$

fibre over $p \in M$ is $\dot{A}_2(T_pM)$

$= (\text{affine } 2\text{-planes in } T_pM)^*$ ← "1-pt compact'n"



$$exp_p : T_pM \hookrightarrow M$$



$M = \mathbb{R}^N$

$$\Gamma^{cpt}\left(\begin{matrix}\dot{A}_2(T\mathbb{R}^N)\\ \downarrow \\ \mathbb{R}^N\end{matrix}\right) \cong \mathrm{Map}^{cpt}\left(\mathbb{R}^N, \dot{A}_2(\mathbb{R}^N)\right)$$

$$\simeq \mathrm{Map}_*\left(S^N, \dot{A}_2(\mathbb{R}^N)\right)$$

$$\simeq \Omega^N \dot{A}_2(\mathbb{R}^N).$$

scan : $\mathcal{E}_\partial(\Sigma_{g,b}, \mathbb{R}^N) \longrightarrow \Omega^N \dot{A}_2(\mathbb{R}^N)$

$\boxed{\Omega^N \dot{A}_2(\mathbb{R}^N) \to \Omega^{N+1}\dot{A}_2(\mathbb{R}^{N+1})}$

Let $N \to \infty$.

scan : $B\mathrm{Diff}^+_\partial(\Sigma_{g,b}) \longrightarrow \underbrace{\operatorname{colim}_{N\to\infty} \Omega^N \dot{A}_2(\mathbb{R}^N)}_{\Omega^\infty MTSO(2)}$

infinite loop space.

These are compatible with stabilisation, so let $g \to \infty$

$$\begin{matrix} B\mathrm{Diff}^+_\partial(\Sigma_{\infty,b}) & \xrightarrow{\text{scan}} & \Omega^\infty MTSO(2) \\ \text{IS} & & \\ B\Gamma_{\infty,b} & & \end{matrix}$$

Conj [Madsen] $\boxed{\text{This is the Quillen +-construction of } B\Gamma_{\infty,b}.}$

"Easy" calculation:

$$\begin{tikzcd} & B\mathrm{Diff}^+_\partial(\Sigma_{\infty,b})^+ \arrow[d, "\wr"] \\ B\mathrm{Diff}^+_\partial(\Sigma_{\infty,b}) \arrow[ur, "H_*\cong"] \arrow[r, "\text{scan}"] & \Omega^\infty MTSO(2) \\ B\Gamma_{\infty,b} & \\ & \mathbb{Q}[\kappa_1, \kappa_2, \dots] \arrow[u, dotted] \arrow[ul, dashed] \end{tikzcd}$$

Madsen Conj

$\cong$ on $H^*(-;\mathbb{Q})$

MMM-classes

$\dashrightarrow$ = map def^d after taking $H^*(-;\mathbb{Q})$

Madsen Conj

$\Downarrow$

Mumford Conj

Thm (Madsen–Weiss '02 / '07) The Madsen conj. is true.

Cor
- Mumford conj. $H^*(B\Gamma_{\infty,1};\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\dots]$
- [Galatius '04] Computation of $H^*(B\Gamma_{\infty,1};\mathbb{Z}/p)$.

Extensions — embedded surfaces. / higher dimensions.

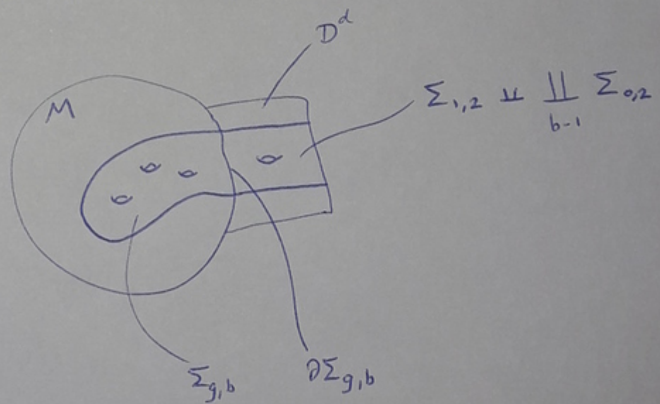
④ Embedded surfaces

Recall that $B\mathrm{Diff}^+_\partial(\Sigma_{g,b}) \simeq \operatorname{colim}_{N\to\infty} \mathcal{E}_\partial(\Sigma_{g,b}, \mathbb{R}^N)$

We can consider $\mathcal{E}_\partial(\Sigma_{g,b}, M)$ for any fixed manifold M^d.

$\partial M \neq \emptyset$

fix $\partial\Sigma_{g,b} \hookrightarrow \partial M$



$$\begin{tikzcd} \mathcal{E}_\partial(\Sigma_{g,b}, M) \arrow[rr, "\text{stab}"] \arrow[dr, "\text{scan}"'] & & \mathcal{E}_\partial(\Sigma_{g+1,b}, M) \arrow[dl, "\text{scan}"] \\ & \Gamma^{cpt}\left(\begin{matrix} \mathcal{L}_2(TM) \\ \downarrow \\ M \end{matrix}\right) & \end{tikzcd}$$

Thm [Cantero-Randal-Williams '17]

For $\pi_1(M) = 1$ and $d = \dim(M) \geq 5$,

$(\mathrm{stab})_*$ and $(\mathrm{scan})_*$ are isomorphisms on homology in deg $\leq \frac{2}{3}(g-1)$

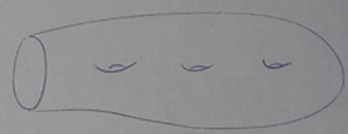
Note: If M splits as $M = \mathbb{R}^i \times N^{d-i}$

then $\Gamma^{cpt}\left(\begin{array}{c} \dot{A}_2(TM) \\ \downarrow \\ M \end{array} \right)$ is an <u>i-fold loop space</u>,

but in general it is not a loop space.

⑤ **Higher dimensions**

Let $W_{g,1} := \#_g (S^n \times S^n) \setminus \mathrm{int}(D^{2n})$



$$\begin{tikzcd}
\cdots \arrow[r] & B\mathrm{Diff}_\partial(W_{g,1}) \arrow[rr, "\mathrm{stab}"] \arrow[dr, "\mathrm{scan}"'] & & B\mathrm{Diff}_\partial(W_{g+1,1}) \arrow[r] \arrow[dl, "\mathrm{scan}"] & \cdots \\
& & \Omega^\infty MT\theta^n & &
\end{tikzcd}$$

defined like $\Omega^\infty MTSO(2)$, but with

$$\dot{A}_2 = \{\text{oriented affine } 2\text{-planes}\}^*$$
$$\downarrow$$
$$\{\text{affine } 2n\text{-planes} + \theta^n\text{-structure}\}^*$$

$$\begin{array}{ccc}
BSO(2) = Gr_2^+(\mathbb{R}^\infty) & & X \\
\text{univ. cover} \downarrow & & \downarrow n\text{-connected covering} \\
BO(2) = Gr_2(\mathbb{R}^\infty) & & BO(2n) \\
& & = Gr_{2n}(\mathbb{R}^\infty)
\end{array}$$

Eg θ^1-structure $\longrightarrow$ orientation
θ^2 / θ^3-structure $\longrightarrow$ spin
θ^4-structure $\longrightarrow$ "string" ...

Thm [Galatius-Randal-Williams '14]

- $(\mathrm{stab})_*$ and $(\mathrm{scan})_*$ are isomorphisms on homology in deg $\leq \frac{1}{2}(g-3)$
- $H^*(\Omega^\infty MT\theta^n ; \mathbb{Q}) \cong \mathbb{Q}$ [certain analogues of the MMM-classes]

$$\begin{tikzcd}
T^v E \arrow[d] & & X \arrow[d] \\
E \arrow[d] \arrow[urr] \arrow[rr, "f"] \arrow[drr] & & BSO(2n) \arrow[d] \\
BDiff_\partial(W_{g,1}) & & BO(2n)
\end{tikzcd}$$

$$\int_{\text{fibre}} f^*(c)$$

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⑥ A different analogue:

$(\partial M \neq \emptyset)$

Fix closed manifold P^d

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{unl}}\left(\coprod_n P, M\right) \arrow[rr, "\mathrm{stab}"] \arrow[dr, "\mathrm{scan}"'] & & \mathcal{E}^{\mathrm{unl}}\left(\coprod_{n+1} P, M\right) \arrow[dl, "\mathrm{scan}"] \\
& \Gamma^{cpt}\left(\begin{matrix} \dot{A}_d(TM) \\ \downarrow \\ M \end{matrix}\right) &
\end{tikzcd}$$

"unlinked" := isotopic to a fixed embedding.

Thm [P, '18] If $d = \dim(P) \leq \frac{1}{2}(\dim(M) - 3)$

then $(\mathrm{stab})_*$ is an isomorphism on homology in deg $\leq \frac{1}{2}(n-1)$

Note: [Kupers '13/'20] proves this in the (disjoint) special case of $P = S^1$, $\dim(M) = 3$.

BUT

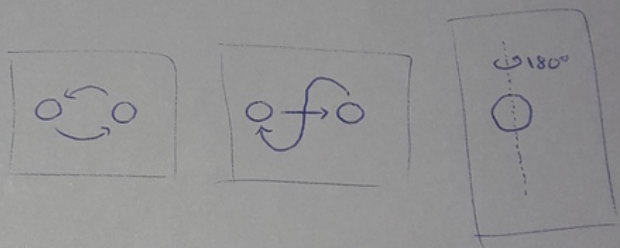
<u>Obs</u> $(\mathrm{scan})_*$ is <u>not</u> an isomorphism on homology when $n \to \infty$!!

<u>Eg</u> $\left.\begin{array}{l} M = \mathbb{R}^3 \\ P = S^1 \end{array}\right\} \longrightarrow$ space of unlinks in $\mathbb{R}^3$

<u>Calc^n</u> $\boxed{H_1\left(\Gamma^{cpt}\left(\begin{array}{c}\mathcal{L}_1(TM)\\ \downarrow \\ M\end{array}\right)\right) \text{ is } \underline{\text{infinite}}}$

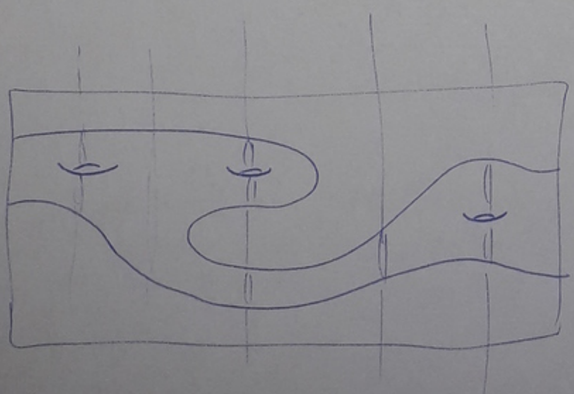
[Brendle-Hatcher '10] compute $\pi_1\left(\mathcal{E}^{unl}\left(\coprod_n S^1, \mathbb{R}^3\right)\right)$

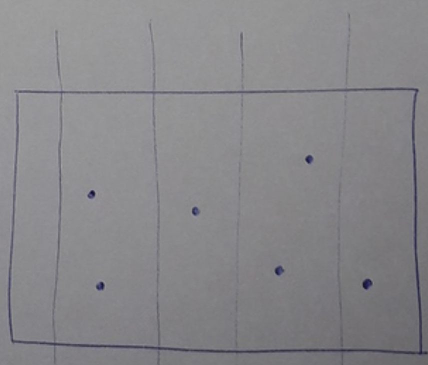
$\hookrightarrow \boxed{H_1\left(\mathcal{E}^{unl}\left(\coprod_n S^1, \mathbb{R}^3\right)\right) \cong (\mathbb{Z}/2)^3}$ $(n \geq 2)$



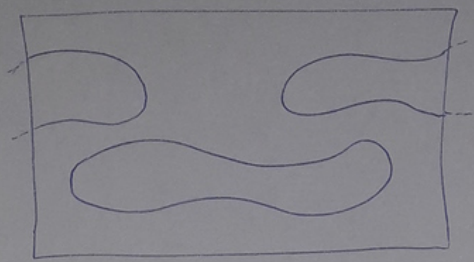
<u>What goes wrong?</u>

The proofs for $\mathcal{E}_\partial(\Sigma_{g,b}, M)$ and $\mathcal{E}\left(\coprod_n *, M\right) = C_n(M)$ use the fact that <u>non-compact configurations of surfaces / points</u> in $\mathbb{R}^N$ can be cut into standard pieces by hyperplanes:





However, for $\mathcal{E}^{\mathrm{unl}}(\coprod_n P, \mathbb{R}^N)$, this is not true:



$P = S^1$

<u>Work in progress [P. '21]</u>

Construct a model for the stable H_* of $\mathcal{E}^{\mathrm{unl}}(\coprod_n P, \mathbb{R}^N)$

by <u>building in</u> this property.

$$\mathcal{E}^{\mathrm{unl}}\Big(\coprod_\infty P, \mathbb{R}^N\Big) \xrightarrow{H_* \cong} \Omega^N \underbrace{X_P(\mathbb{R}^N)}$$

$\approx$ non-cpt submflds of $(-1,1)^N$ of the form

$Q \cap (-1,1)^N$, $Q \subset \mathbb{R}^N$

$Q \cong \coprod_n P$

s.t. $\exists$ <u>separating hyperplanes</u>.