Moduli and Friends 17 June 2021
IMAR

Quantum invariants via the topology of configuration spaces

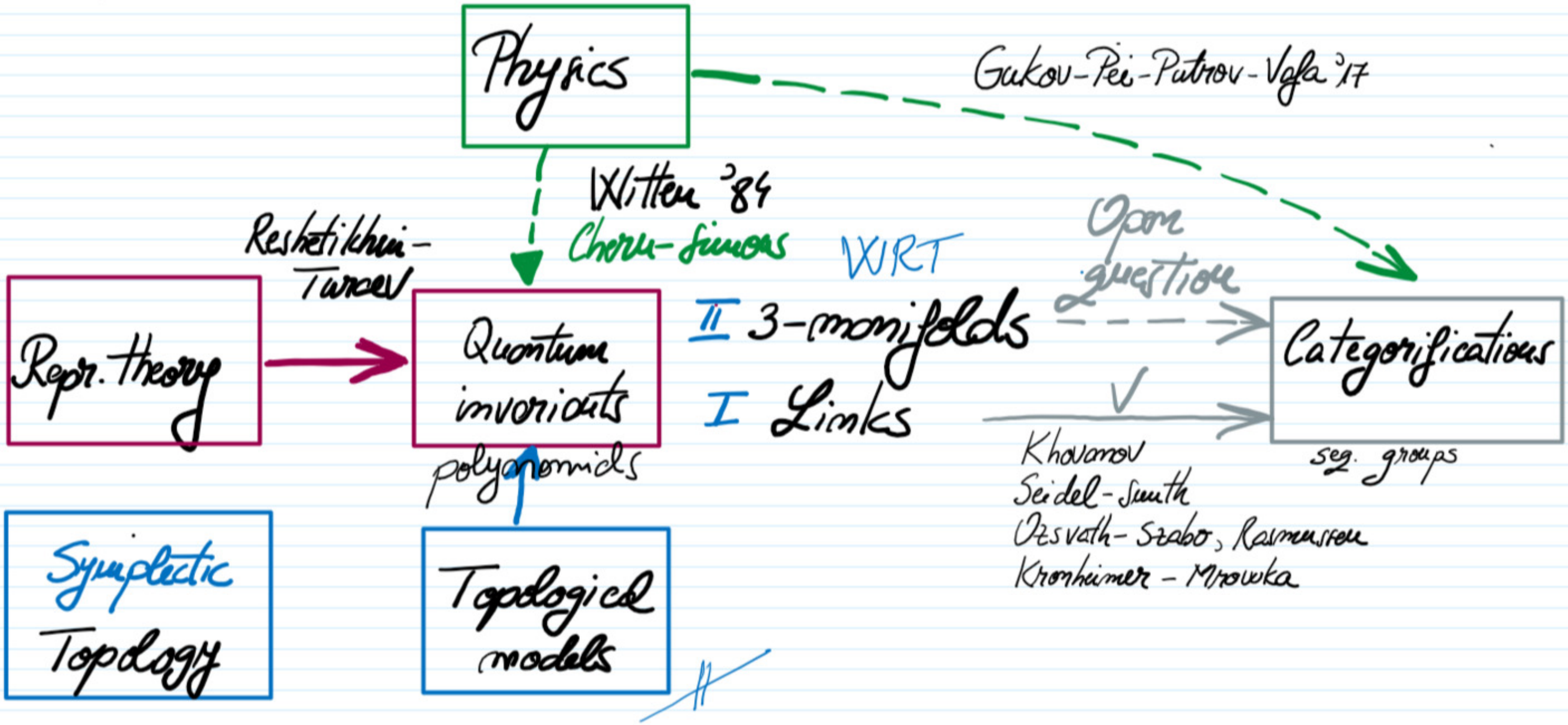
I: Link invariants

$\boxed{\text{Outline}}$ Motivation

I. Homological set-up: local systems
II. Topological model with immersed Lagrangians
III. Topological model with embedded Lagrangians

$\boxed{\text{Motivation}}$ • Aim Define topological models for $U_q(sl(2))$-quantum invariants

$\boxed{\text{Topological model: graded intersection pairing of homology classes in coverings of configuration spaces}}$



• Today: link invariants

		Dowlin ('18) Rasmussen	
Categorifications	Heegaard-Floer Homology	$\xleftarrow{\text{spectral seq}}$	Khovanov Homology
Invariants	$\boxed{\text{Alexander polyn } \Delta(K,t)}$		$\boxed{\text{Jones polyn. } J(K,q)}$
Quantum generalisations (coloured)	$\boxed{\Phi_N(K,\lambda)}$		$\boxed{J_N(K,q)}$

· <u>Aim</u> Bring them together in the same geometric picture //

Th (Bigelow '00, Lawrence)
Noodles and Forks

skein rel

Jones polym.

$N=2$

$N \in \mathbb{N}$

$\left(U_q(sl(2)), \; V_N \right)$
q parameter, N-dim repr

$\xrightarrow{RT}$

Coloured Jones polym.
$J_N(L,q) \in \mathbb{Z}[q^{\pm 1}]$

$\left(U_{\xi_N}(sl(2)), \; V_\lambda \right)$
$\xi_N = e^{2\pi i/2N}$, N-dim repr
$\lambda \in \mathbb{C}$ parameter

Coloured Alexander polym.
$\Phi_N(L,\lambda) \in \mathbb{Z}[\xi_N^{\pm 1}, \xi_N^{\pm \lambda}]$

$1/\Delta_K$

Unified algebraically
Willets '20

Th 1 (A. '20)
Unified topological model
(immersed Lagrangians)

- uses representation theory

Th 2 (A. '20)
Unified model over 3 variables
(embedded Lagrangians)

- uses Th 1
- suitable for computations

- Kohno '12
- Ito '15
- Martel '19

Representation theory

Intersections in configuration spaces

$\boxed{\text{Main result}}$

Fix $N \in \mathbb{N}$ - colour of the quantum invariant

Let L - oriented link ; $L = \hat{\beta}_n$ for $\beta_n \in B_n$

Construction : $\forall\ i_1, \dots, i_{n-1} \in \{0, \dots, N-1\} \rightsquigarrow$ Two Lagrangians

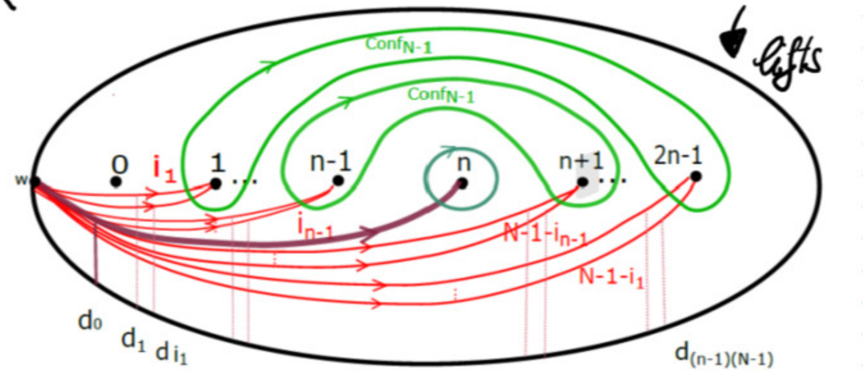
$$\mathcal{H}_*(\underset{x}{\mathbb{Z}} \oplus \underset{d}{\mathbb{Z}} \text{ covering}) \quad \left(\mathcal{F}_{i_1,\dots,i_{n-1}} \in \mathcal{H}^{-n}_{2n,(n-1)(N-1)+1} \circlearrowleft \beta_{2n} \ ; \ \mathcal{L}_{i_1,\dots,i_{n-1}} \in \mathcal{H}^{-n,\partial}_{2n,(n-1)(N-1)+1}\right)$$

|| not $\mathcal{H}_{2n,*}$

$\big\downarrow$

$$Conf_{(n-1)(N-1)+1}(\mathbb{D}_{2n})$$

$\big\downarrow$ lifts



• **Def** (State sum of Lagrangian intersections)

$$\boxed{\Lambda_N(\beta_n) := u^{-w(\beta_n)} \cdot u^{-(n-1)} \sum_{i_1,\dots,i_{n-1}=0}^{N-1} \left\langle (\beta_n \cup \mathbb{I}_n)\, \mathcal{F}_{i_1,\dots,i_{n-1}}, \mathcal{L}_{i_1,\dots,i_{n-1}} \right\rangle}$$

$\in \mathbb{Z}[x^{\pm 1}, d^{\pm 1}, u^{\pm 1}]$

• $\boxed{\text{Th 2}}$ (A'20 Unified model through state sums of Lagrangian intersections)

The polynomial in 3 variables Λ_N recovers the N^{th} Coloured Jones and N^{th} Coloured Alexander polynomials for links:

$$\boxed{J_N(L, q) = \Lambda_N(\beta_n) \Big|_{\psi_{1,q,N}}}$$

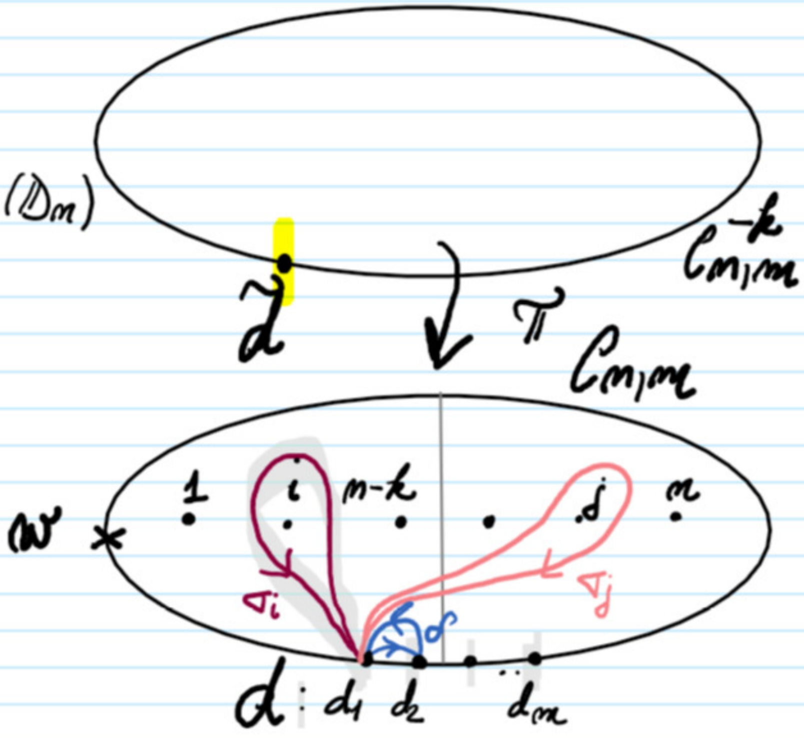
$$\boxed{\Phi_N(L, \lambda) = \Lambda_N(\beta_n) \Big|_{\psi_{1-N, \xi_N, \lambda}}}$$

specialisations of coefficients

I Homological representations

Fix $n, m, k \in \mathbb{N}$

$\mathbb{D}_n := D^2 \setminus \{1, \dots, n\} \rightsquigarrow C_{n,m} = \mathrm{Conf}_m(\mathbb{D}_n)$



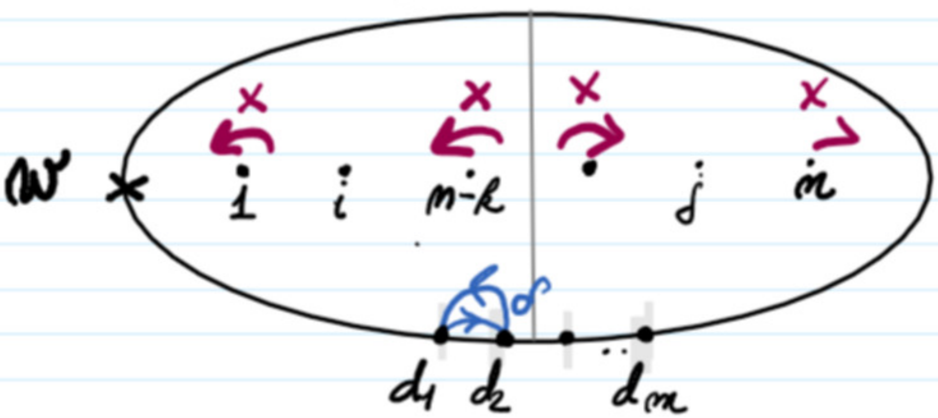
• **Def** (Local system) Fix $0 \le k \le n$

$$\pi_1(C_{n,m})^{ab} \quad \underset{\langle \sigma_i \rangle}{\mathbb{Z}^n} \oplus \underset{\langle \delta \rangle}{\mathbb{Z}} \xrightarrow{\varphi^{-k}} \underset{\langle x \rangle}{\mathbb{Z}} \oplus \underset{\langle d \rangle}{\mathbb{Z}}$$

$$\sigma_i \mapsto \begin{cases} x, & 0 \le i \le n-k \\ -x, & i > n-k \end{cases}$$

$\rightsquigarrow C^{-k}_{n,m}$ covering sp

Monodromy of the local system φ^{-k}



- Let $w \in \partial D_n$; $\tilde{d} \in \pi^{-1}(d)$
- $\boxed{\text{Tools}}$: Homology of this covering sp.

① $\mathcal{H}^{-k}_{n,m} \subseteq H^{lf}_m \left(C^{-k}_{n,m} , \pi^{-1}(w) ; \mathbb{Z} \right)$ ← Borel Moore w.r.t. punctures colissions

$B_n \circlearrowright$ MCG

② $\mathcal{H}^{-k,\partial}_{n,m} \subseteq H^{cell}_m \left(C^{-k}_{n,m} , \partial ; \mathbb{Z} \right)$

③ <u>Prop</u>: (A-Palmer): Intersection pairing:

$$\langle , \rangle : \mathcal{H}^{-k}_{n,m} \otimes \mathcal{H}^{-k,\partial}_{n,m} \to \mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$$

Construction of homology classes

$$E_{n,m} := \{ n\text{-partitions of } m \}$$

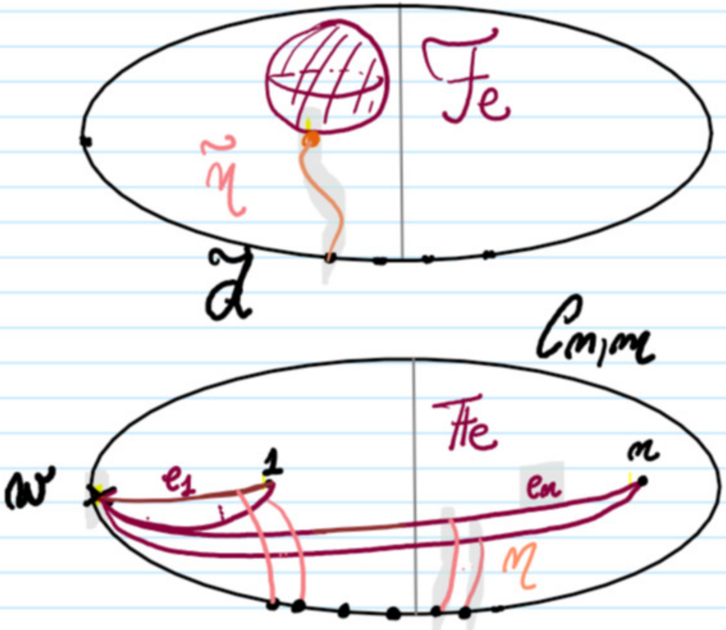
$$\ni\ e = (e_1, \dots, e_n) \rightsquigarrow \mathscr{F}_e \in \mathcal{H}^{-k}_{n,m}$$

$\downarrow$ $e_1 + \dots + e_n = m$ $\qquad\qquad$ $\uparrow$ lift through $\tilde{\eta}(1)$

$$\left(\mathbb{F}_e \subseteq C_{n,m}\ ;\ \eta : d \to \mathbb{F}_e \right) \longrightarrow \tilde{\eta}\text{ - lift through } \tilde{d}$$

given by the product of curves $\qquad$ path in the config.

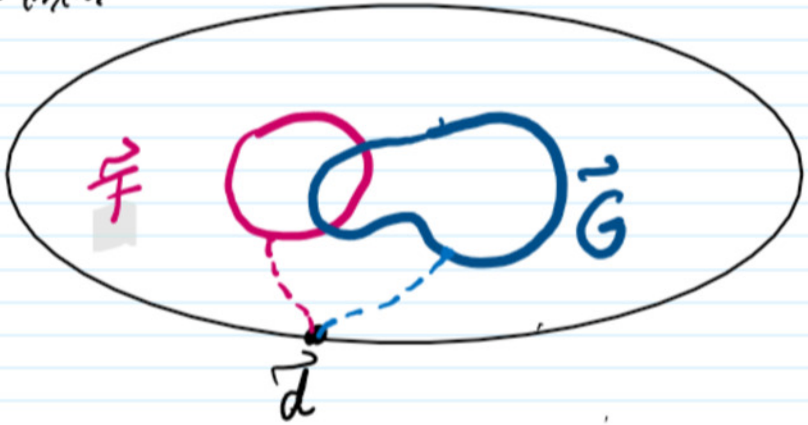
$\therefore$ (geometric support, path to the base point) $\rightsquigarrow$ homology class



$\boxed{\text{Formula for the intersection form}}$

Let $[\vec{\mathcal{F}}] \in H^{-k}_{m,m}$; $[\vec{G}] \in H^{\partial,-k}_{m,m}$

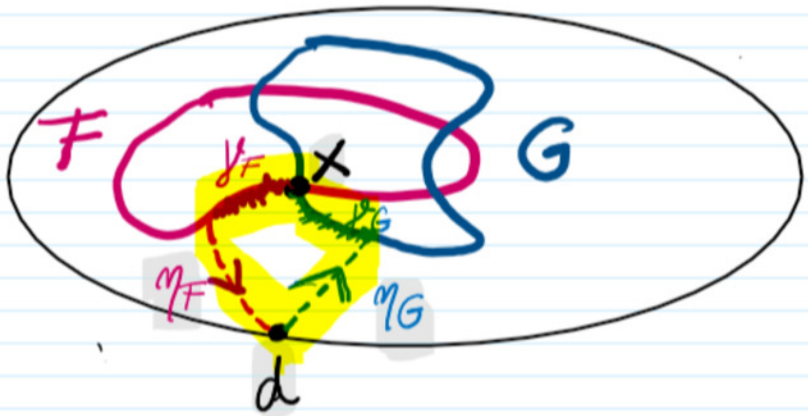
lifts of $\mathcal{F}, G \subseteq C_{m,m}$ through $\tilde{\eta}_{\mathcal{F}}(1), \tilde{\eta}_G(1)$



Suppose $\exists\ \eta_{\mathcal{F}}/\eta_G : d \rightsquigarrow \mathcal{F}/G$ paths

- $\forall\, x \in \mathcal{F} \cap G \rightsquigarrow \ell_x \subseteq C_{m,m}$ loop

- Choose $\gamma_{\mathcal{F}/G} : \eta_{\mathcal{F}/G}(1) \underset{\text{paths}}{\rightsquigarrow} x$



$$\boxed{\ell_x := \eta_G \circ \gamma_G \circ \gamma_{\mathcal{F}}^{-1} \circ \eta_{\mathcal{F}}^{-1}}$$

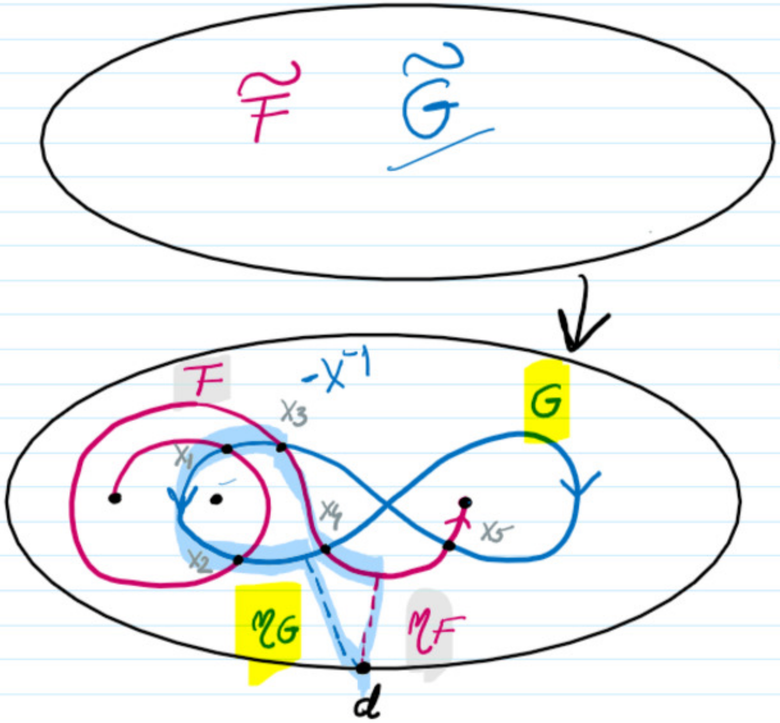
<u>Prop</u> (Formula for $\langle , \rangle$)

$$\boxed{\langle [\vec{\mathcal{F}}], [\vec{G}] \rangle = \sum_{x \in \mathcal{F} \cap G} (\mathcal{F}, G)_x \ \varphi^{k}(\ell_x)}$$

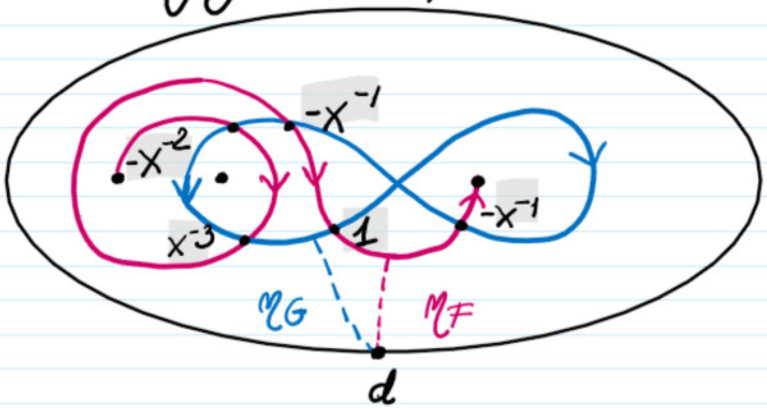
geometric intersection number

grading by the local system

• $\boxed{\text{Examples}}$ $n = 3$; $m = 1$; $k = 0$ $\rightsquigarrow$ Monodromy of the local system



Gradings in the base configuration space



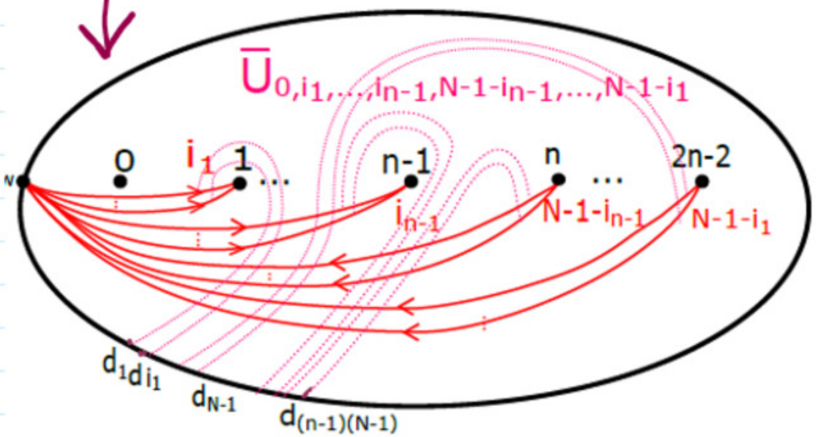
$$\therefore \quad \langle [\tilde{F}], [\tilde{G}] \rangle = 1 - 2x^{-1} - x^{-2} + x^{-3}$$

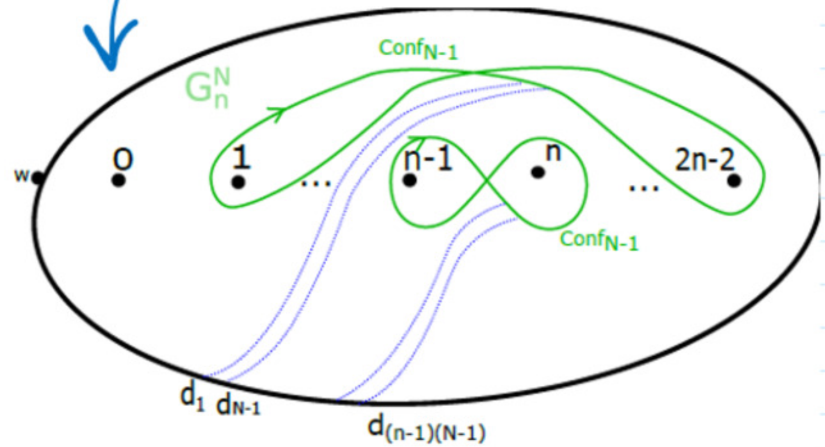
II Topological model with immersed Lagrangians

Context: L - oriented link ; $L = \hat{\beta}_n$ $\beta_n \in B_n$ Fix $N \in \mathbb{N}$

• **Def** (Main classes) $\underline{i} = (i_1, \dots, i_{n-1})$: $i_k \in \{0, \dots, N-1\}$

$\tilde{\mathcal{U}}_{i_1, \dots, i_{n-1}} \in \mathcal{H}^{0}_{2n-1,(n-1)(N-1)}$ $\mathcal{G}^N_n \in \mathcal{H}^{0,\partial}_{2n-1,(n-1)(N-1)}$





• **Def** : (Homology Classes)

$$\left(\mathcal{E}^N := \sum_{i_1, \dots, i_{n-1}=0}^{N-1} d^{-\sum i_k} \cdot \tilde{\mathcal{U}}_{i_1, \dots, i_{n-1}} \; , \quad \mathcal{G}^N_n \right)$$

Not (Specialisation of coefficients) Let $c \in \mathbb{Z}$

$$\psi_{(c),q,\lambda} : \mathbb{Z}[(u^{\pm 1}), x^{\pm 1}, d^{\pm 1}] \to \mathbb{Z}[q^{\pm 1}, q^{\pm \lambda}]$$

$$(u \rightsquigarrow q^{\lambda})$$

$$\begin{cases} x \rightsquigarrow q^{2\lambda} \\ d \rightsquigarrow q^{-2} \end{cases}$$

$\boxed{\text{Th 1}}$ (Topological model via immersed Lagrangians) $L = \hat{\beta_n}$; $N \in \mathbb{N}$

Let $\boxed{I_N(\beta_n) := \langle (\beta_n \cup \mathbb{1}_{n-1}) \mathcal{E}_n^N, \mathcal{G}_n^N \rangle} \in \mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$

Then, I_N recovers the N^{th} col. Jones and col. Alexander polyn.:

$$\boxed{J_N(L,q) = q^{-(N-1)\,wr(\beta_n)} \cdot q^{-(n-1)(N-1)} \cdot I_N(\beta_n)\Big|_{\psi_{q,N-1}}}$$

$\psi_J = \psi_{q,N-1}$

$$\boxed{\Phi_N(L,\lambda) = \xi_N^{(N-1)\cdot\lambda\cdot wr(\beta_n)} \cdot \xi_N^{(N-1)(n-1)\lambda} \cdot I_N(\beta_n)\Big|_{\psi_{\xi_N,\lambda}}}$$

$\psi_\Phi = \psi_{\xi_N,\lambda}$

Construction and idea of proof

$\boxed{\textbf{Algebraic context}}$ $\left(U_q(sl(2)), R\right)$ over $\mathbb{Z}[q^{\pm 1}, s^{\pm 1}]$

$N \in \mathbb{N}$

$$\hat{\mathbb{V}} := \langle v_0, v_1, \ldots, v_{N-1}, v_N, \ldots \rangle \quad \text{Verma module}$$

$$\cup \qquad \mathbb{V}_N$$

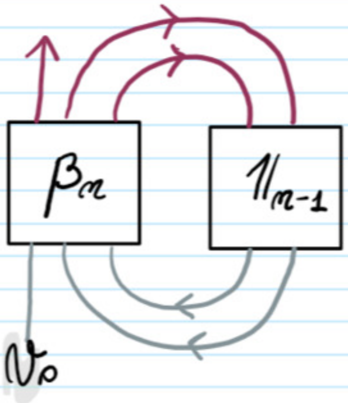
Specialisations of variables

$$\begin{tikzcd}
 & \mathbb{V}_N \arrow[dl, dashed, "s \to q^{N-1}"'] \arrow[dr, dashed, "s \to \xi_N^{\lambda},\ q \to \xi_N"] & \\
V_N \arrow[d, squiggly] & & V_\lambda \arrow[d, squiggly] \\
\underline{\underline{\mathcal{Y}_N}} & & \underline{\underline{\Phi_N}}
\end{tikzcd}$$

$\Psi_{\underline{\underline{\mathcal{Y}}}}$ $\qquad$ $\Psi_{\underline{\underline{\Phi}}}$

Step 1 Definition of $\mathcal{Y}_N$ and Φ_N in this set-up $L = \hat{\beta}_n$

Diagramatically



$ev_{\mathcal{Y}}$: $\curvearrowright$ $v_i \otimes v_{N-1-i} \mapsto s q^{-2i}$ otherwise $0 \rightsquigarrow \mathcal{Y}_N$

ev_{Φ} : $\curvearrowright$ $v_i \otimes v_{N-1-i} \mapsto s^{1-N} q^{-2i}$ otherwise $0 \rightsquigarrow \Phi_N$

coev : $\cup$ $1 \mapsto \sum_i v_i \otimes v_{N-1-i}$

See both invariants from a construction over 2-varibles

Extend $ev_{\mathcal{Y}}$ and ev_{Φ} on all vectors from the Verma module with zero unless they are from W_N

Step 2 Use the weight spaces from the Verma module

$\mathbb{Z}[q^{\pm 1}, s^{\pm 1}]$

ev_y $\qquad$ ev_Φ

ev $\neq 0$ iff $j_k = i_k$

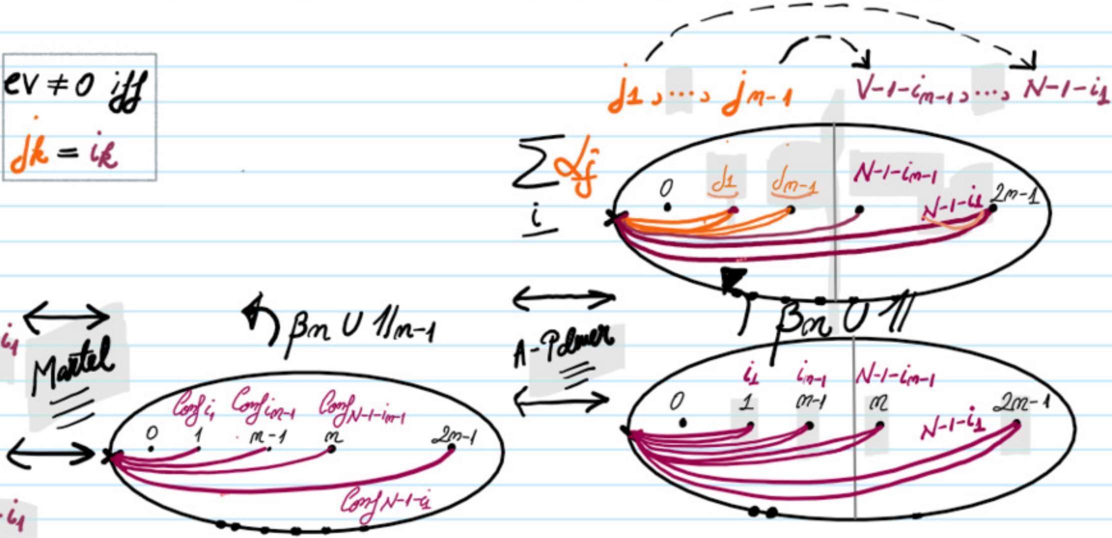
$$\sum_j \alpha_j \, v_0 \otimes v_{j_1} \otimes \dots \otimes v_{j_{m-1}} \otimes v_{N-1-i_{m-1}} \otimes \dots \otimes v_{N-1-i_1}$$

$\beta_m \quad \mathbb{1}_{m-1}$

$$\sum_{i=0}^{N-1} v_0 \otimes v_{i_1} \otimes \dots \otimes v_{i_{N-1}} \otimes v_{N-1-i_{m-1}} \otimes \dots \otimes v_{N-1-i_1}$$

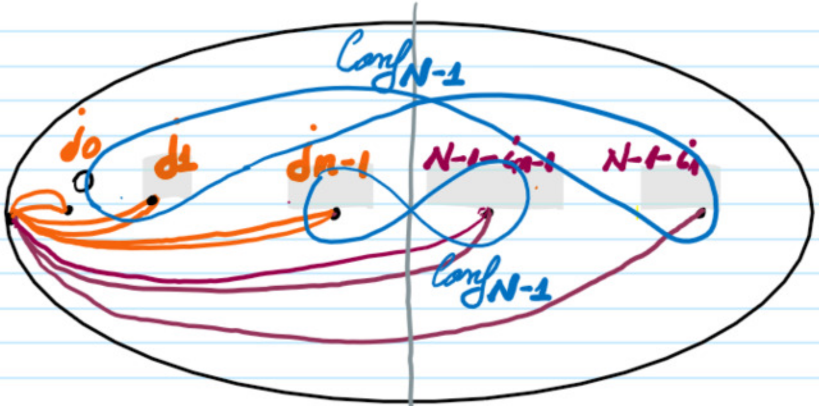
coev

v_0



Step 3 We need a dual class which intersects

$\mathcal{F}_{j_0, j_1, \ldots, j_{m-1}, N-1-i_{m-1}, \ldots, N-1-i_1}$ non-zero iff $\boxed{\begin{array}{l} j_0 = 0 \\ j_k = i_k \end{array}}$



$\mathcal{G}^N_m$

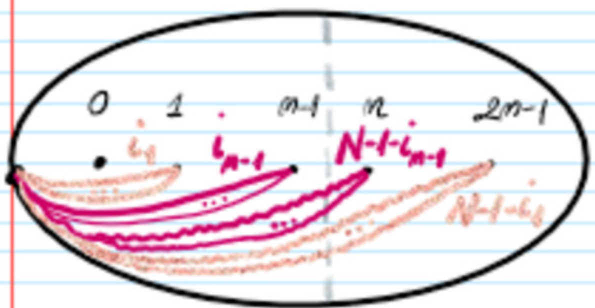
Corollary 1 (Recover Bigelow's model for the Jones polynomial)

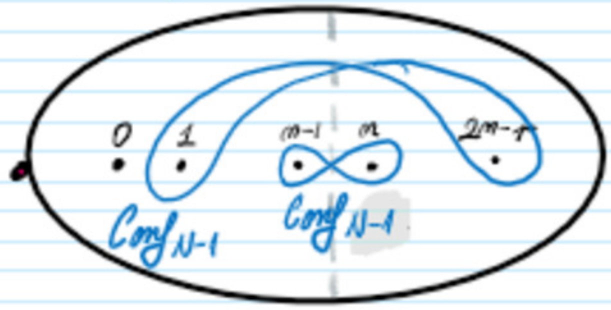
Th 1 for $N=2$ recovers Bigelow's model:

$\mathcal{F}_m^2 \rightarrow$ forks $\quad \mathcal{G}_m^2 \rightarrow$ noodles

Proof For $i_1, \dots, i_{m-1} \in \{0, \dots, N-1\}$

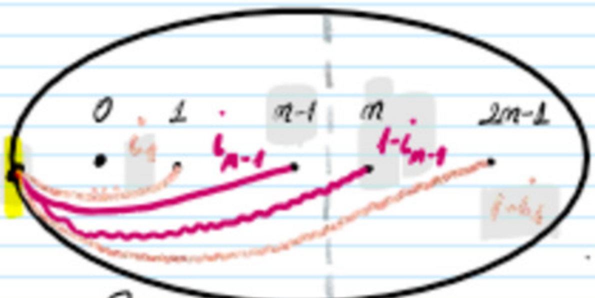
$\tilde{\mathcal{U}}_{i_1,\dots,i_{m-1}} \qquad \mathcal{G}_m^N$



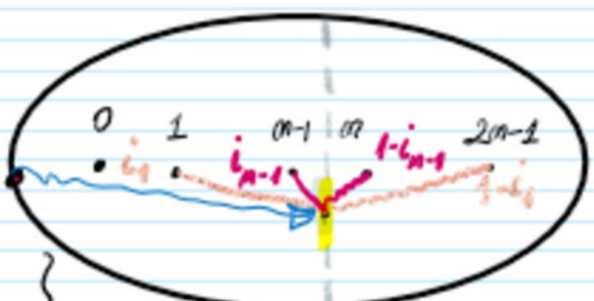


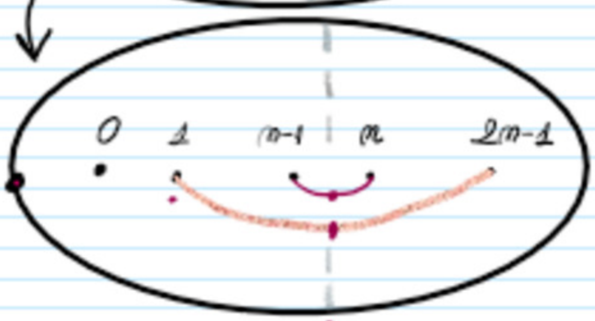
$\boxed{N=2}$ Jones polyn. $i_1, \dots, i_{m-1} \in \{0,1\}$

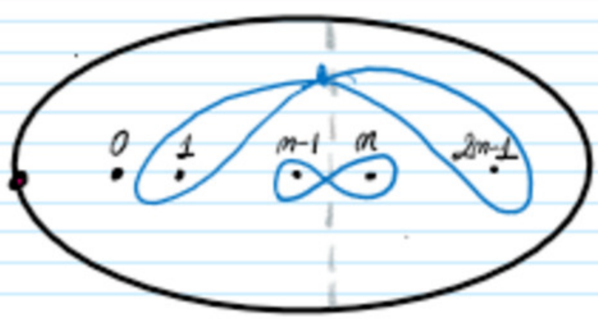
$\mathcal{E}_m^2 = \sum_i d^{-\Sigma i_k} \tilde{\mathcal{U}}_{i_1,\dots,i_{m-1}} \qquad \mathcal{G}_m^2$



up to homotopy and d-coefficients







$\mathcal{E}_m^2$ fork and $\mathcal{G}_m^2$ noodle

• **Corollary 2** (ADO invariants from $\mathbb{Z}\oplus\mathbb{Z}_{2N}$-covering spaces)

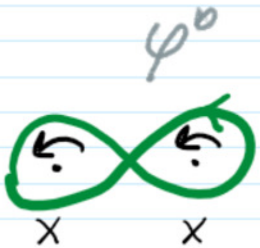
The N^{th} coloured Alexander invariant comes from an intersection pairing in a $\mathbb{Z}\oplus\mathbb{Z}_{2N}$-covering of a conf. sp. in the punctured disk.

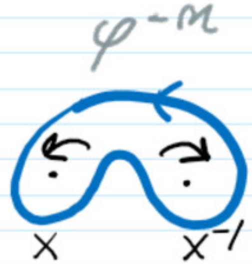
Questions

1. Model with embedded Lagrangians
2. Simple lifts (paths η to the base point)
3. Geometrical meaning of the d-coeff.

III Model with embedded Lagrangians

Construction | Ideas

① ② Replace 

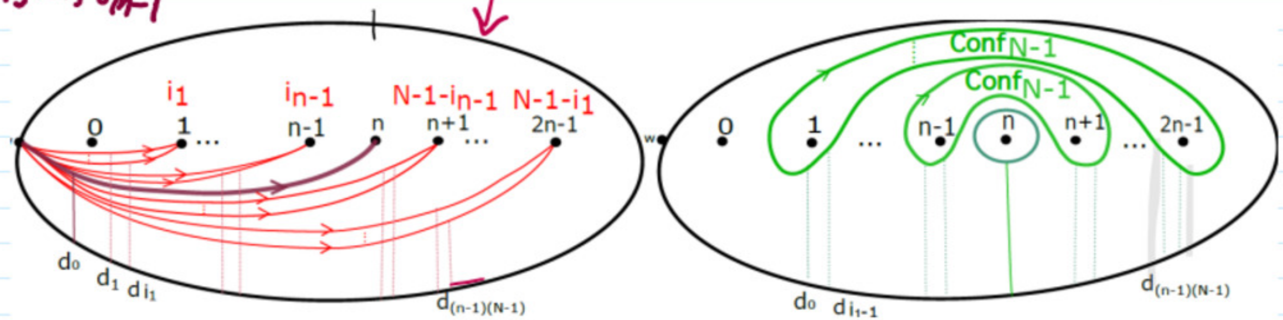
by 

(change the local system)

③ Add an extra puncture

Def (Homology classes) $\underline{i} = (i_1,\dots,i_{n-1})$ $i_k \in \{0,\dots,N-1\}$

$\mathcal{F}_{i_1,\dots,i_{n-1}} \in H^{-m}_{2m,(n-1)(N-1)+1}$ $\mathscr{L}_{i_1,\dots,i_{n-1}} \in H^{-m,\partial}_{2m,(n-1)(N-1)+1}$



• $\boxed{Th2}$ (A'20 Unified model through embedded Lagrangians)

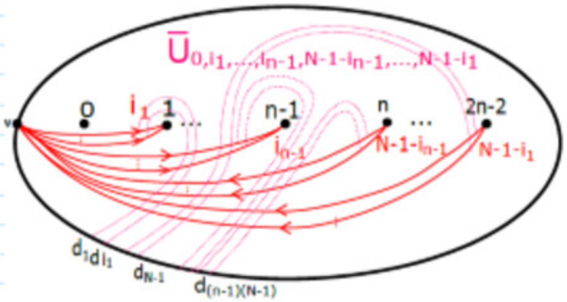
$$\boxed{\Lambda_N(\beta_n) := u^{-w(\beta_n)} \cdot u^{-(n-1)} \sum_{i_1,\ldots,i_{n-1}=0}^{N-1} \left\langle (\beta_n \cup \mathbb{I}_n)\, \mathscr{F}_{i_1,\ldots,i_{n-1}}, \mathscr{L}_{i_1,\ldots,i_{n-1}} \right\rangle} \in \mathbb{Z}[x^{\pm 1}, d^{\pm 1}, u^{\pm 1}]$$

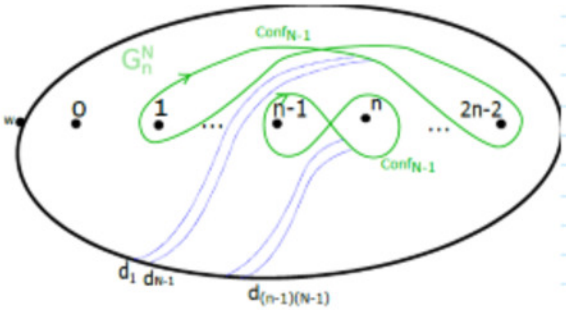
Then : $\begin{cases} \boxed{J_N(L,q) = \Lambda_N(\beta_n)\,|_{\psi_{1,q,N}}} \\ \boxed{\Phi_N(L,\lambda) = \Lambda_N(\beta_n)\,|_{\psi_{1-N,\, q_N,\, \lambda}}} \end{cases}$ specialisations of coefficients

• Proof We have two types of intersection pairings:

(1) Immersed classes $\longrightarrow$ $\boxed{Th1}$

$\tilde{\mathcal{U}}_{i_1,\ldots,i_{n-1}} \in \mathcal{H}^{0}_{2n-1,(n-1)(N-1)}$ $\quad \mathcal{G}^N_n \in \mathcal{H}^{0,\partial}_{2n-1,(n-1)(N-1)}$

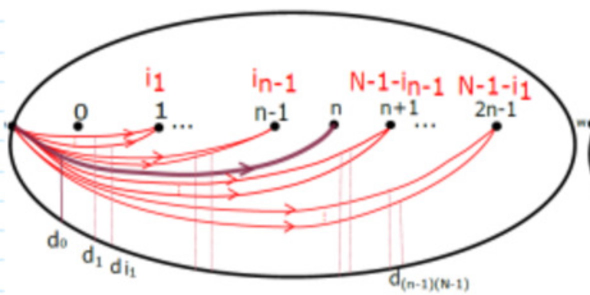


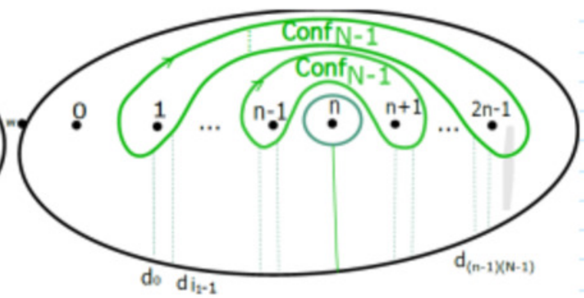


$\langle \, , \, \rangle_{1}$

(2) Embedded classes $\longrightarrow$ $\boxed{Th2}$

$\mathscr{F}_{i_1,\ldots,i_{n-1}} \in \mathcal{H}^{-n}_{2n,(n-1)(N-1)+1}$ $\quad \mathscr{L}_{i_1,\ldots,i_{n-1}} \in \mathcal{H}^{-n,\partial}_{2n,(n-1)(N-1)+1}$





$\langle \, , \, \rangle_{2}$

• **Corollary** $\boxed{N=2}$: Jones and Alexander polyn. from the same geometric/topological picture

• **Example** T = trefoil knot : $\beta_2=\sigma^3$ $\circlearrowleft \sigma^3$

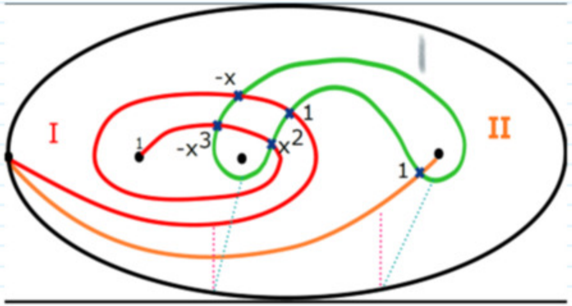
I $\mathcal{F}_1$ × 0 1 2 $(i_1=1)$

II $\mathcal{F}_0$ × 0 1 2 $(i_1=0)$

T 

Th2 Embedded model

Monodromy of the local system

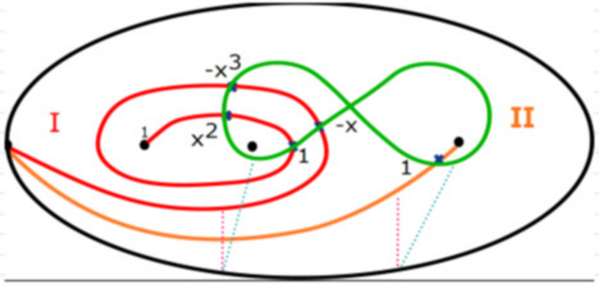


$((\sigma^3\cup\mathbb{1})\mathcal{F}_1,\mathcal{L}_1)$ $((\sigma^3\cup\mathbb{1})\mathcal{F}_0,\mathcal{L}_0)$

$u^3\cdot u\left(\langle(\sigma^3\cup\mathbb{1})\,d\,\mathcal{F}_1,\mathcal{L}_1\rangle+\langle(\sigma^3\cup\mathbb{1})\mathcal{F}_0,\mathcal{L}_0\rangle\right)$

Th1 Immersed model

Monodromy of the local system



$$\boxed{\Lambda_2(\sigma^3) = u^4\left(d\left(-x^3+x^2-x+1\right)+1\right)} \in \mathbb{Z}[u^{\pm1}, x^{\pm1}, d^{\pm1}]$$

$u=q^2$, $x=q^2$, $d=q^{-2}$ $\qquad$ $u=\xi_2^{-\lambda}$, $x=\xi_2^{2\lambda}$; $d=\xi_2^{-2}=-1$

$$\boxed{J(T) = -q^8+q^2+q^6} \qquad \boxed{\Delta(T,x) = x-1+x^{-1}}$$

Jones $\qquad$ Alexander

Heegaard Floer Homology $\xleftarrow{\text{s.s.}}$ Categorification

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