I Homotopy sheaves

Definition: A sheaf on a topological space X is a contravariant functor $\mathcal{F}: Op(X) \longrightarrow Set$ such that, if $U = V \cup W$ the diagram $\mathcal{F}(U) \to \mathcal{F}(V) \times \mathcal{F}(W) \rightrightarrows \mathcal{F}(V \cap W)$ is a limit diagram.

Problem: Some structures feel like they should be sheaves but do not satisfy this definition.

Example: Take X a manifold, consider

$$Vect_n : Op(X)^{op} \longrightarrow Set$$
$$U \longmapsto \text{(isomorphism classes) of vector bundles of dim } n \text{ on } U$$

This is not a sheaf, indeed any vector bundle is locally trivial but not globally.

Fix: $Vect_n : Op(X)^{op} \longrightarrow Groupoids$

$$U \longmapsto \text{groupoid of } n\text{-dim vector bundles on } U.$$

This is still not a sheaf but the diagram

$$Vect_n(U) \longrightarrow Vect_n(V) \times Vect_n(W) \rightrightarrows Vect_n(V \cap W)$$

is a homotopy limit diagram

Equivalently,

$Vect_n(U)$ is equivalent to the groupoid of triples (A, B, ϕ) with $A \in Vect_n(V)$, $B \in Vect_n(W)$ and ϕ is an isomorphism $\phi: A|_{V \cap W} \simeq B|_{V \cap W}$

This induces a short exact sequence

$$\pi_1 \mathrm{Vect}_n(V\cap W) \to \pi_0 \mathrm{Vect}_n(U) \to \pi_0 \mathrm{Vect}_n(V) \times \pi_0 \mathrm{Vect}_n(W) \rightrightarrows \pi_0 \mathrm{Vect}_n(V\cap W)$$

More generally we make the following definition

Definition: A functor $\mathcal{F} : \mathrm{Op}(X)^{op} \to sSet/Top$ is a homotopy sheaf if for all $U \in \mathrm{Op}(X)$ s.t. $U = V \cup W$, the diagram $\mathcal{F}(U) \to \mathcal{F}(V) \times \mathcal{F}(W) \rightrightarrows \mathcal{F}(V\cap W)$ is a homotopy limit diagram.

Meaning that it induces an weak homotopy equivalence from $\mathcal{F}(U)$ to the $sSet/Top$ of triples (a, b, γ) with $a \in \mathcal{F}(V)$, $b \in \mathcal{F}(W)$ and γ is a path between the restrictions of a an b to $V\cap W$.

This induces a long exact sequence

$$\to \pi_{i+1}(\mathcal{F}(V\cap W)) \to \pi_i(\mathcal{F}(U)) \to \pi_i(\mathcal{F}(V)) \times \pi_i(\mathcal{F}(W)) \to \pi_i(\mathcal{F}(V\cap W)) \to \cdots$$

$$\to \pi_1(\mathcal{F}(V\cap W)) \to \pi_0(\mathcal{F}(U)) \to \pi_0(\mathcal{F}(V)) \times \pi_0(\mathcal{F}(W)) \to \pi_0(\mathcal{F}(V\cap W))$$

Proposition

- The inclusion functor Ho(ho. sheaves on X) $\to$ Ho (presheaves on X) has a left adjoint: the (homotopy) sheafification functor. denoted $\mathcal{F} \mapsto \mathcal{F}^{\#}$
- The sheafification functor is homotopically idempotent, i.e. the map $\mathcal{F}^{\#} \to (\mathcal{F}^{\#})^{\#}$ is a weak equivalence of (pre)sheaves
- A map $\mathcal{F} \to \mathcal{G}$ of presheaves induces a weak equivalence $\mathcal{F}^{\#} \to \mathcal{G}^{\#}$ if and only if it is a stalkwise weak equivalence
- There is a (fringed) spectral sequence
$$H^i(X, \pi_j \mathcal{F}) \Rightarrow \pi_{j-i} \mathcal{F}(X)$$
for a homotopy sheaf $\mathcal{F}$

I Immersions

Fix X a manifold and M a manifold (smooth manifolds)
Consider the space $\mathrm{Imm}(X, M)$ of immersions $X \to M$

<u>Observation</u>: The functor $U \mapsto \mathrm{Imm}(U, M)$ is a sheaf on X

<u>Definition</u>: A formal immersion from X to M is a continuous map $f: X \to M$ and the data of linear injective maps $\phi_x : T_x X \to T_{f(x)} M$ depending continuously on x.

We denote by $\mathrm{FImm}(X, M)$ the space of formal immersions from X to M

<u>Proposition</u> The obvious map $\mathrm{Imm}(X, M) \to \mathrm{FImm}(X, M)$ is a homotopy sheafification.

<u>Proof</u> Claim 1: $U \mapsto \mathrm{FImm}(U, M)$ is a homotopy sheaf

Claim 2: $\mathrm{Imm}(D, M) \to \mathrm{FImm}(D, M)$ is a weak equivalence when D is an open disk. In fact both spaces are homotopy equivalent to $\mathrm{Fr}_d(TM)$ (with $d = \dim(D)$) the space of d-frames in TM. $\square$

<u>Theorem</u> [Smale-Hirsch]
The presheaf $U \mapsto \mathrm{Imm}(U, M)$ is a homotopy sheaf if $\dim X < \dim M$

<u>Corollary</u> The space $\mathrm{Imm}(S^2, \mathbb{R}^3)$ is connected

<u>Proof</u> We have $\mathrm{Imm}(S^2, \mathbb{R}^3) \simeq \mathrm{FImm}(S^2, \mathbb{R}^3)$

$$\begin{aligned}
\mathrm{FImm}(S^2, \mathbb{R}^3) &\simeq \{\text{fiberwise injections } TS^2 \to S^2 \times \mathbb{R}^3\} \\
&\simeq \{\text{fiberwise bijections } TS^2 \times \mathbb{R} \to S^2 \times \mathbb{R}^3\}
\end{aligned}$$

$$\begin{aligned}
\text{Indeed } \mathrm{InjLin}(\mathbb{R}^2, \mathbb{R}^3) &\simeq SO(3) \\
&\simeq \{\text{continuous maps } S^2 \to SO(3)\} \quad [TS^2 \times \mathbb{R} \cong S^2 \times \mathbb{R}^3]
\end{aligned}$$

$\pi_0 \operatorname{Imm}(S^2, \mathbb{R}^3) \cong \pi_2 SO(3) \cong \pi_2 SU(2) \cong \{0\}$. $\square$

On the other hand, the same reasoning shows that
$\pi_0(\operatorname{Imm}(S^1, \mathbb{R}^2)) \cong \pi_1 SO(2) \cong \mathbb{Z}$
The sphere eversion problem is solvable in $\mathbb{R}^3$ but not in $\mathbb{R}^2$.

III Embeddings

As before X and M are smooth manifolds
We consider the presheaf on X

$$U \longmapsto \operatorname{Emb}(U, M)$$

This is neither a sheaf nor a homotopy sheaf.

Proposition $\operatorname{Emb}(U,M) \longrightarrow F\operatorname{Imm}(U,M)$ is a homotopy sheafification. In particular, if $\dim(X) < \dim(M)$ $\operatorname{Emb}(U,M) \longrightarrow \operatorname{Imm}(U,M)$ is a homotopy sheafification

Proof $F\operatorname{Imm}(U,M)$ is a homotopy sheaf. Moreover $\operatorname{Emb}(D, M) \longrightarrow Fr_d(TM)$ is a weak equivalence. $\square$

How can we go further?

Definition Let X be any topological space. A J_k-cover of X is a collection of open sets of X $\{U_i\}_{i \in I}$ s.t any collection of k points in X is contained in one of the U_i's.

Remark A J_1-cover is a usual cover

Observation: $U \longmapsto \operatorname{Emb}(U,M)$ is a J_2-sheaf on M

But it is not a J_2-homotopy sheaf.

Definition A J_k-homotopy sheaf on X is a presheaf $\mathcal{F} : \operatorname{Op}(X)^{op} \longrightarrow sSet/Top$ s.t for any open set U and any J_k-cover $\{U_i\}_{i\in I} \longrightarrow U$ the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{\substack{S \subset I \\ |S| = 2}} \mathcal{F}(U_S) \Rrightarrow \prod_{\substack{S \subset I \\ |S| = 3}} \mathcal{F}(U_S)$$

is a homotopy limit diagram

Definition Let X, M be smooth manifolds
The embedding calculus tower is

$$\begin{tikzcd} & \vdots \arrow[d] \\ & T_3 \mathrm{Emb}(X,M) \arrow[d] \\ & T_2 \mathrm{Emb}(X,M) \arrow[d] \\ \mathrm{Emb}(X,M) \arrow[r] \arrow[ur] \arrow[uur] & T_1 \mathrm{Emb}(X,M) \simeq \mathrm{Imm}(X,M) \end{tikzcd}$$

where $T_k \mathrm{Emb}(X,M)$ is the value on X of the $\mathcal{J}_k$ homotopy sheafification of $U \longmapsto \mathrm{Emb}(U,M)$

Theorem [Goodwillie - Klein - Weiss]
The embedding calculus tower converges in codimension ≥ 3
More precisely, the map
$\mathrm{Emb}(X,M) \to T_k \mathrm{Emb}(X,M)$ is $(3 - \dim M + k-1)(\dim M - \dim X - 2))$ connected.

Under the codimension ≥ 3 assumption we obtain a spectral sequence

$$\pi_{t-s} L_s \Rightarrow \pi_{t-s} \mathrm{Emb}(X,M)$$

where L_s is the homotopy fiber of $T_s \to T_{s-1}$ can be explicitly computed in some cases.

IV Applications of embedding calculus

Theorem [Arone - Lambrechts - Volić]
let $2m \leq n-1$ The rational homology of $\overline{\mathrm{Emb}}(M, \mathbb{R}^n)$ with M of dim m depends only on the rational homology of M

$$\overline{\mathrm{Emb}}(M, \mathbb{R}^n) \simeq \mathrm{hofib}\left[\mathrm{Emb}(M, \mathbb{R}^n) \to \mathrm{Imm}(M, \mathbb{R}^n)\right]$$

Theorem [Arone Turchin / Fresse Turchin Willwacher]
"Computation" of $\pi_* \operatorname{Emb}_\partial(D^m, D^n) \otimes \mathbb{Q}$ if $n - m \geq 3$

Theorem [Weiss]
The map $H^*(BTop, \mathbb{Q}) \to H^*(BTop(n); \mathbb{Q})$ has "exotic" classes in its ~~kernel~~ image
An exotic class is a class that is not zero but is sent to zero by the map $H^*(BTop(n), \mathbb{Q}) \to H^*(BO(n), \mathbb{Q})$

Theorem [Kupers]
For $n \neq 4, 5, 7$, the ~~group~~ space $BTop(n)$ has finitely generated homotopy groups.

Ingredients for these theorem
- For $m \neq 4$ $\quad \operatorname{Diff}_\partial(D^m) \simeq \Omega^{m+1}(Top(m)/O(m))$ [Morlet]

 $Top(m)/O(m)$ is the homotopy fiber of $BO(m) \to BTop(m)$
- For any manifold M with boundary, one has a fiber sequence

$$B\operatorname{Diff}_\partial(D^n) \to B\operatorname{Diff}_\partial(M) \to B\operatorname{Emb}^{\cong}_{1/2\partial}(M, M)$$

If we apply this to $M = W_{g,1}$ ~~[illegible]~~ we can get informations about $B\operatorname{Diff}_\partial(D^{2n})$

$$W_{g,1} = \left(\#_g S^n \times S^n\right) - D$$

Theorem [Knudsen - Kupers]
Knot theory is not a good invariant of manifolds of dim 4:
If $M \cong M'$ are homeomorphic 4-manifolds then
$\operatorname{Emb}(S^1, M) \simeq \operatorname{Emb}(S^1, M')$