Moduli and Friends
IMAR

30 June 2021

Quantum invariants via the topology of configuration spaces

II: 3-manifold invariants

Outline

1. Motivation
2. Topological model for the WRT invariants for 3-mflds
3. Proof of the intersection model

Motivation

• Aim: Describe Witten–Reshetikhin–Turaev invariants as graded intersection pairings between Lagrangians in configuration spaces $\dashrightarrow$ **Categorification**

Representation theory

Jones polym.

Intersections in configuration spaces

Links $N \in \mathbb{N}$ colour

$\left(U_q(sl(2)), V_N \right) \xrightarrow{RT}$

q parameter

V_N N-dim repr

$N_1, \dots, N_l \in \mathbb{N}$

Coloured Jones polym.

$J_N(L, q) \in \mathbb{Z}[q^{\pm 1}]$

$J_{N_1, \dots, N_l}(L, q)$

$\longrightarrow$ Th (A. '20)

linear combination

Reshetikhin-Turaev

$\left(U_\xi(sl(2)), V_1, \dots, V_{\mathcal{N}-1} \right) \rightsquigarrow$

$\xi = e^{2\pi i / 2\mathcal{N}}$

WRT invariant

$\tau_{\mathcal{N}}(M^3) \in \mathbb{C}$

$\longrightarrow$ Th (A. '21)

Topological model from states of Lagrangian intersections

3-manifolds $\mathcal{N} \in \mathbb{N}$ level

Witten

Chern-Simons

Physics

Invariants	Categorifications	
	Representation theory	Symplectic geometry
Jones polyn.	Khovanov Homology	Seidel - Smith '06 Manolescu
Coloured Jones polyn.	Khovanov	?
	Categorified quantum groups (Khovanov - Lauda - Rouquier '10) (Webster '13)	
WRT invariant	Open problem	
	(Gukov - Pei - Putrov - Vafa '17) BPS states Categorification of (3d $\mathcal{N}=2$) partition functions	
	Physics	

Topological intersection

Classification of smooth structures on 4-mfds

$\boxed{\text{Main result}}$

Fix $\mathcal{N} \in \mathbb{N}$ - level of the quantum invariant

Let M - closed, oriented 3-mfd

$$M = S^3(L) \quad ; \quad L\text{ - framed oriented link}$$

$$L = \hat{\beta}_n \quad \beta_n \in B_n$$

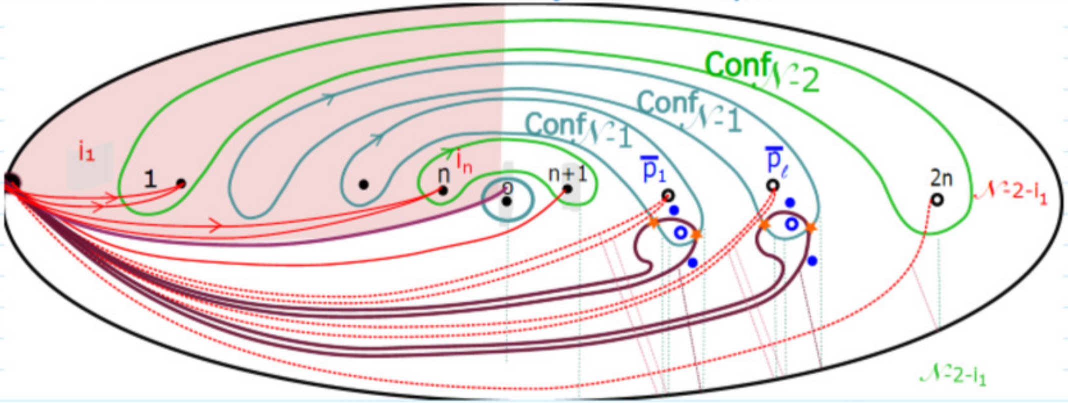
Construction: $\forall\, i_1, \dots, i_n \in \{0, \dots, \mathcal{N}-2\} \rightsquigarrow$ Two Lagrangians

$\bar{i} = (i_1, \dots, i_n)$

$\mathcal{H}_*(\mathbb{Z}^{2\ell} \oplus \mathbb{Z}$ covering$)$ $\qquad \mathcal{F}_{\bar{i}}^{\mathcal{N}} \in \mathcal{H}_{\ell,n}^{\mathcal{N}} \qquad \mathscr{L}_{\bar{i}}^{\mathcal{N}} \in \mathcal{H}_{\ell,n}^{\mathcal{N},\partial}$

$\downarrow$ || not $\mathcal{H}_{\ell,n}^{\mathcal{N}}$

$\mathrm{Conf}_{n(\mathcal{N}-2)+\ell+1}(\mathbb{D}_{2n+3\ell+1})$



• **Def** (Lagrangian intersection) $\bar{i} \in \{0, \mathcal{N}-2\}$

$$\boxed{\Lambda_{\bar{i}}(\beta_m) := \widetilde{\prod_{k=1}^{l}} x_{C(p_k)}^{\left(f_{p_k} - \sum_j lk_{p_k,j}\right)} \cdot \widetilde{\prod_{k=1}^{n}} x_{C(k)}^{-1} \left\langle (\beta_m \cup \mathbb{1}_{n+3l+1}) \mathcal{F}_{\bar{i}}^{\mathcal{N}}, \mathcal{L}_{\bar{i}}^{\mathcal{N}} \right\rangle} \in \mathbb{Z}[x_1^{\pm1}, \dots, x_l^{\pm1}, y_1^{\pm1}, \dots, y_l^{\pm1}, d^{\pm1}]$$

$$\mathbb{Z}[x_1^{\pm1}, \dots, x_l^{\pm1}, y_1^{\pm1}, \dots, y_l^{\pm1}, d^{\pm1}] \xrightarrow{\psi_{\xi, N_1, \dots, N_l}} \mathbb{C}$$

• **Th** (A. '21 Topological state sum model for WRT invariant)

Fix $\mathcal{N} \in \mathbb{N}$ level and $M = S^3(L)$ framed link. Let $L = \hat{\beta}_m$

Then:
$$\boxed{\tau_{\mathcal{N}}(M) = \frac{\{1\}_\xi^{-1}}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot \sum_{\underbrace{i_1, \dots, i_m = 0}_{\bar{i}}}^{\mathcal{N}-2} \left(\sum_{\substack{1 \le N_1, \dots, N_l \le \mathcal{N}-1 \\ \bar{i} < C(N)}} \Lambda_{\bar{i}}(\beta_m) \Big|_{\psi_{\xi, N_1, \dots, N_l}} \right)}$$

intersection in a fixed configuration sp.

- **Witten-Reshetikhin-Turaev invariants**

$$\left.\begin{array}{l} M \text{- closed, oriented 3-mfd} \\ \mathcal{N} \in \mathbb{N} \text{ level } ; \ \xi = e^{2\pi i/2\mathcal{N}} \end{array}\right\} \xrightarrow{\text{WRT}} \tau_{\mathcal{N}}(M) \in \mathbb{C}$$

- **Definition from $Rep(U_q(sl(2))$** (Reshetikhin-Turaev)

Look at $M = S^3(L)$; L - framed oriented link $L = L_1 \cup \dots \cup L_e$

Notations $(lk_{ij})_{i,j}$ = linking matrix of $L \to b, b_+, b_- = \#$ zero, pos, neg eigenval

- $lk_{ij} = \begin{cases} lk(L_i, L_j) & , i \neq j \\ f_i = \text{framing}(L_i) & , i = j \end{cases}$

- $\boxed{\Delta_{+/-} = J_\Omega(\mathcal{U}_{+/-}, \xi)}$

  unknot w/ framing $+1/-1$

- $\mathcal{D} = |\Delta_+|$

Def: (Kirby colour) $\begin{array}{c} N \in \mathbb{N} \\ N < \mathcal{N} \end{array} \rightsquigarrow V_N \in Rep(\mathcal{U}_\xi(sl(2)))$
N-dim. representation

$$\boxed{\Omega := \sum_{N=1}^{\mathcal{N}-1} q\dim(V_N) \cdot V_N = \sum_{N=1}^{\mathcal{N}-1} [N]_\xi \cdot V_N}$$

$$\begin{cases} [x]_q = \dfrac{q^x - q^{-x}}{q - q^{-1}} \\ \{x\}_q = q^x - q^{-x} \end{cases}$$

$$\rightsquigarrow \boxed{\tau_{\mathcal{N}}(M) = \frac{1}{\mathcal{D}^b \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot J_{\Omega, \dots, \Omega}(L, \xi)}$$

II Topological model for the WRT invariants

II.1 Local system and homology groups

Fix $n \in \mathbb{N}$

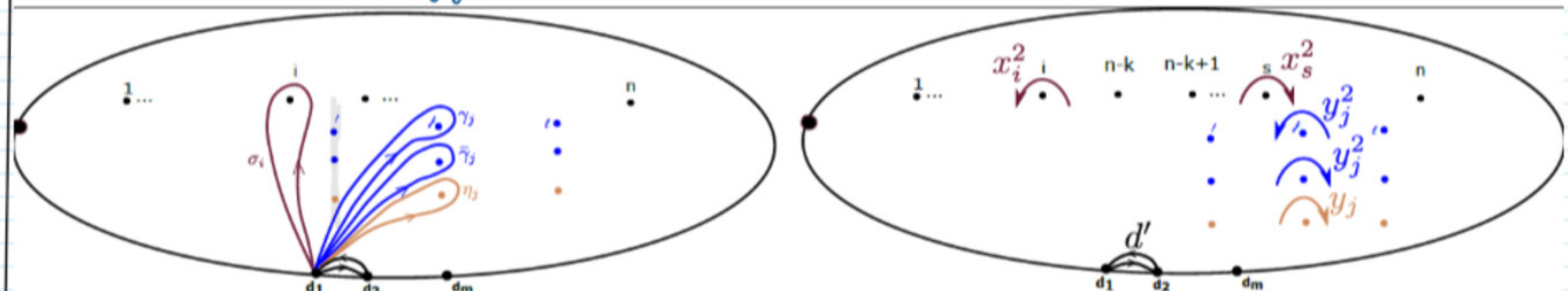
$m, \bar{e}, k \in \mathbb{N}$ $\rightsquigarrow$ use $\mathrm{Conf}_m(D_{n+3\bar{e}})$

# punctures	blue punctures	weight	opposite monodromy
n ;	e ;	m ,	k

Def : (Local system)

$$\pi_1(C_{n+3\bar{e},m}) \xrightarrow{ab} \underset{\langle \sigma_i \rangle}{\mathbb{Z}^n} \oplus \underset{\left\langle \gamma_j ; \bar{\gamma}_j \right\rangle}{\mathbb{Z}^{2\bar{e}}} \oplus \underset{\langle \eta_j \rangle}{\mathbb{Z}^{e}} \oplus \underset{\langle \delta \rangle}{\mathbb{Z}} \longrightarrow \underset{\langle x_i \rangle}{\mathbb{Z}^n} \oplus \underset{\langle y_j \rangle}{\mathbb{Z}^{\bar{e}}} \oplus \underset{\langle d \rangle}{\mathbb{Z}}$$

Φ



$\rightsquigarrow$ covering sp. $\tilde{C}_{n+3\bar{e},m}$

$\rightsquigarrow$ $H_m(\tilde{C}_{n+3\bar{e},m}) = \mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1}, y_1^{\pm1},\dots,y_{\bar{e}}^{\pm1}, d^{\pm1}]$-mod

- $\boxed{\text{Colourings inherited from braid closures}}$

$B_n = MCG(\mathbb{D}_n)$ $\qquad$ B_n^C : braids which preserve the colouring C

$L = \hat{\beta}_n$ link w/ ℓ comp $\rightsquigarrow$ colouring of the braid $C:\{1,\dots,n\}\to\{1,\dots,\ell\}$

$$\rightsquigarrow \mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1},y_1^{\pm1},\dots,y_{\bar{\ell}}^{\pm1},d^{\pm1}] \xrightarrow{f_C} \mathbb{Z}[x_1^{\pm1},\dots,x_\ell^{\pm1},y_1^{\pm1},\dots,y_{\bar{\ell}}^{\pm1},d^{\pm1}]$$

- $\boxed{\text{Tools}}$: Homology of this covering sp.

# punctures	blue punctures	weight	opposite monodromy
n ;	$\bar{\ell}$;	m ,	k

① $\mathcal{H}^{-k}_{n,\bar{\ell},m} \subseteq H^{\mathcal{G}}_m(\widetilde{C}_{n+3\bar{\ell},k}\ \pi^{-1}(w);\mathbb{Z})/_{f_C}$

$B_n^C \circlearrowright MCG$

② $\mathcal{H}^{-k,\partial}_{n,\bar{\ell},m} \subseteq H^{\mathrm{cell}}_m(\widetilde{C}_{n+3\bar{\ell},k},\partial;\mathbb{Z})/_{f_C}$

③ <u>Prop</u>: (A-Palmer) Intersection pairing:

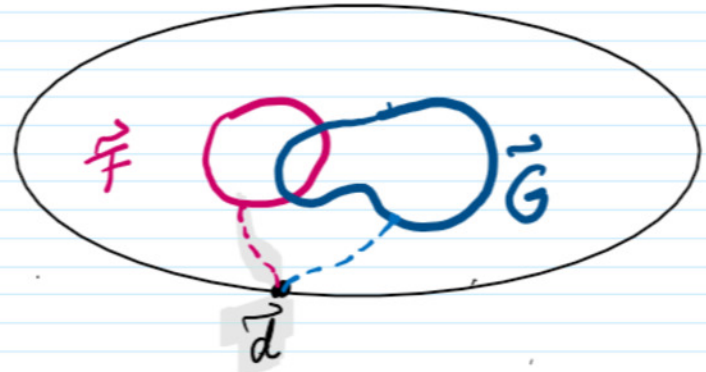
$$\langle\ ,\ \rangle : \mathcal{H}^{-k}_{n,\bar{\ell},m} \otimes \mathcal{H}^{-k,\partial}_{n,\bar{\ell},m} \to \mathbb{Z}[x_1^{\pm1},\dots,x_\ell^{\pm1},y_1^{\pm1},\dots,y_{\bar{\ell}}^{\pm1},d^{\pm1}]$$

Formula for the intersection form

Let $[\tilde{\mathcal{F}}] \in \mathcal{H}^{-k}_{n,\bar{e},m}$; $[\tilde{G}] \in \mathcal{H}^{-k,2}_{n,\bar{e},m}$

lift through $\tilde{\eta}_{\mathcal{F}}(1)$ — through $\tilde{\eta}_G(1)$

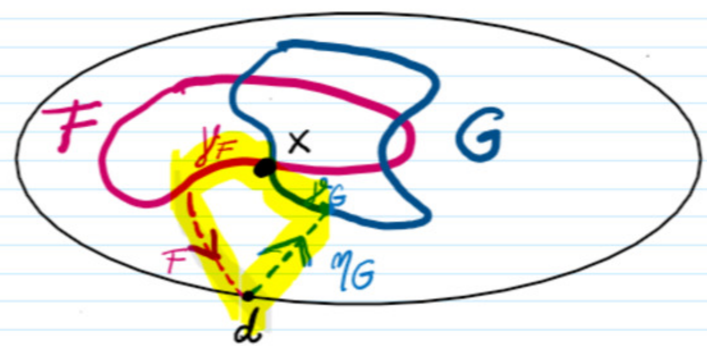
$\begin{pmatrix} \mathcal{F} \\ \eta_{\mathcal{F}} \end{pmatrix}$ $\mathcal{F} \subseteq C_{n+3\bar{e},m}$, $\eta_{\mathcal{F}}$ path $d \to \mathcal{F}$; $\begin{pmatrix} G \\ \eta_G \end{pmatrix}$ $G \subseteq C_{n+3\bar{e},m}$, η_G path $d \to G$



→ Denote by $\tilde{\eta}_{\mathcal{F}/G}$ = lifts of $\eta_{\mathcal{F}/G}$ through $\tilde{d}$

- $\forall x \in \mathcal{F} \cap G \rightsquigarrow$ loop $\ell_x \subseteq C_{n+3\bar{e},m}$ ~~H~~

→ Choose $\gamma_{\mathcal{F}/G}$ paths $\eta_{\mathcal{F}/G}(1) \to x$



$$\boxed{\ell_x := \eta_{\mathcal{F}}^{-1} \circ \gamma_{\mathcal{F}}^{-1} \circ \gamma_G \circ \eta_G}$$

Prop (Formula for $\langle , \rangle$)

$$\boxed{\langle [\tilde{\mathcal{F}}], [\tilde{G}] \rangle = \sum_{x \in \mathcal{F} \cap G} (\mathcal{F}, G) \cdot \Phi(\ell_x)}$$

geometric intersection number — grading by the local system

① (geometric support ; path to the base point d) $\rightsquigarrow$ homology class

∴ $\mathcal{F}$ $\eta_{\mathcal{F}}$. $[\tilde{\mathcal{F}}]$

② • intersection points $x \in \mathcal{F} \cap G$ $\leftrightsquigarrow$ intersection pairing

• graded by the local system encoded $\langle [\tilde{\mathcal{F}}], [\tilde{G}] \rangle$

via the paths to the base points

$\boxed{\text{Base configuration space}}$ $\boxed{\text{Covering space}}$

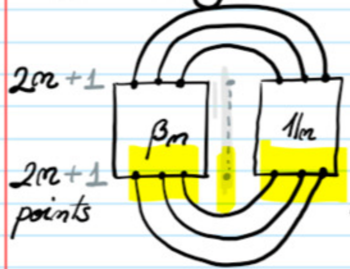
II.2 Construction of the homology classes

Let $\mathcal{N} \in \mathbb{N}$ level. $M = S^3(L)$; $L = \widehat{\beta_n}$

link with ℓ comp. $\quad \beta_n \in B_n$

- We will work in $\mathrm{Conf}_{n(\mathcal{N}-2)+\ell+1}\left(\mathbb{D}_{2n+3\ell+1}\right)$

- $\boxed{\text{Colouring}}$



Let the induced colouring

$$\begin{cases} C : \{1, \dots, 2n\} \to \{1, \dots, \ell\} \\ C(0) := 1 \end{cases}$$

$\rightsquigarrow \; C : \{0, 1, \dots, 2n\} \to \{1, \dots, \ell\}$ colouring of $2n+1$ pts.

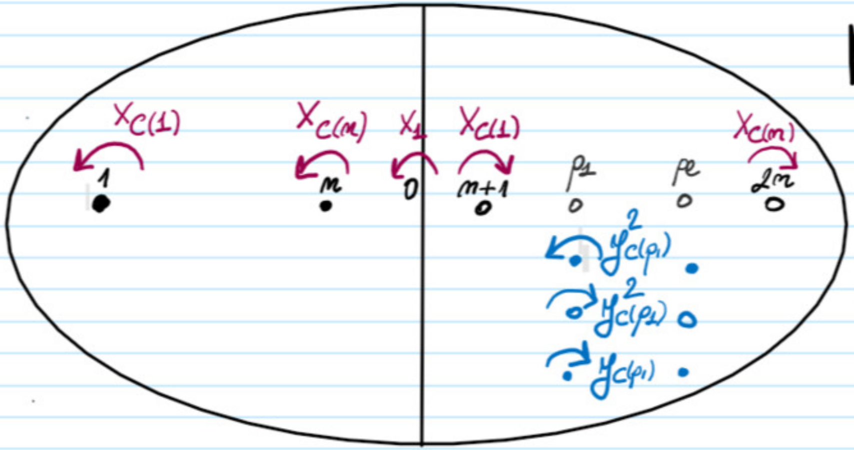
# punctures	blue punctures	weight	opposite monodromy
$n \to 2n+1$;	ℓ ;	$m \to m(\mathcal{N}-2)+1$;	$k \to -m$

$\rightsquigarrow \boxed{\mathcal{H}^{-k}_{n,\bar{\ell},m}}$

Homology groups

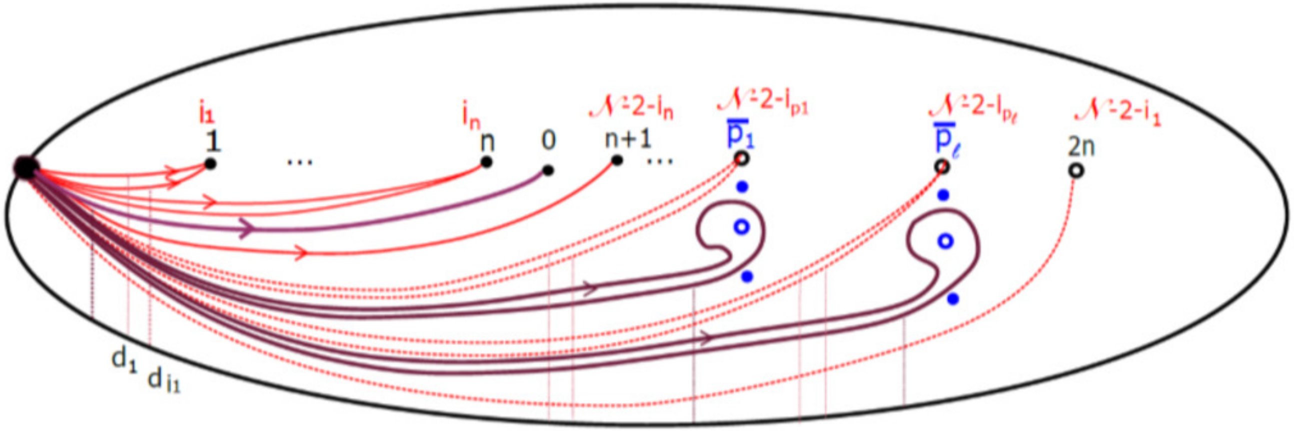
$$\mathcal{H}^{-m}_{2m+1,\ell,m(N-2)+1} \text{ and } \mathcal{H}^{-m,\partial}_{2m+1,\ell,m(N-2)+1} = \mathbb{Z}[x_1^{\pm 1},\dots,x_\ell^{\pm 1},y_1^{\pm 1},\dots,y_\ell^{\pm 1},d^{\pm 1}]\text{-modules}$$

$\mathcal{H}^{N}_{\ell,m}$ $\qquad$ $\mathcal{H}^{N,\partial}_{\ell,m}$

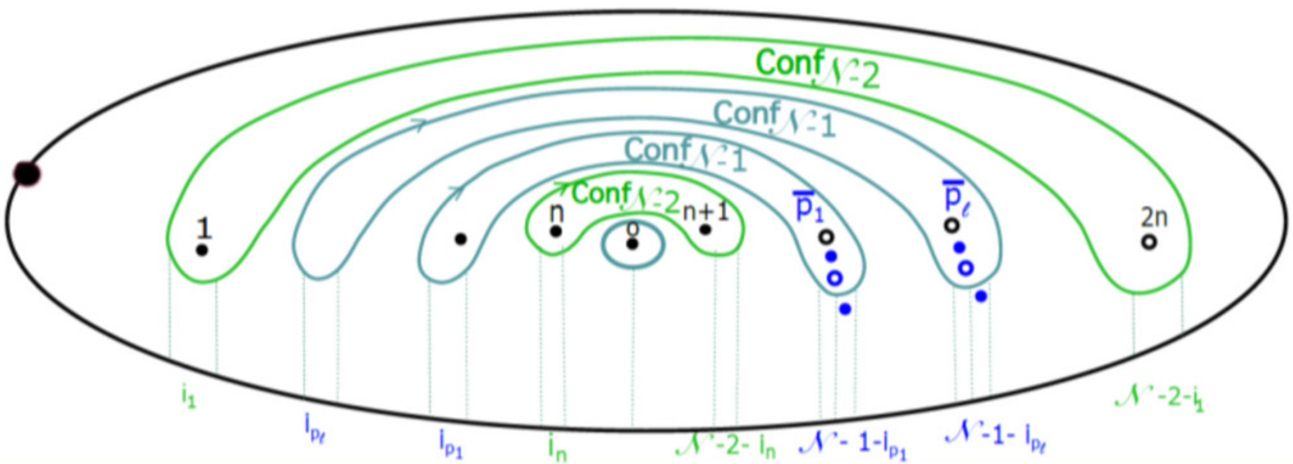


$\mathrm{Conf}_{m(N-2)+\ell+1}(\mathbb{D}_{2m+3\ell+1})$

• Def (First homology class) $\mathscr{F}_{\bar{i}}^{\mathcal{N}} \in \mathcal{H}_{e,m}^{\mathcal{N}}$ $\bar{i} = i_1, \dots, i_n \in \{0, \mathcal{N}-2\}$



• Def (Second homology class) $\mathscr{L}_{\bar{i}}^{\mathcal{N}} \in \mathcal{H}_{e,m}^{\mathcal{N},\partial}$



• **Def** (Lagrangian intersection) $\bar{i} \in \{0, N-2\}$

$$\boxed{\begin{gathered}\Lambda_{\bar{i}}(\beta_n) := \widetilde{\prod_{k=1}^{l}} x_{C(p_k)}^{\left(i_{p_k} - \sum_j lk_{p_k,j}\right)} \cdot \widetilde{\prod_{k=1}^{n}} x_{C(k)}^{-1} \\ \left\langle (\beta_n \cup \mathbb{I}_{n+3l+1}) \mathcal{F}_{\bar{i}}^{N}, \mathcal{L}_{\bar{i}}^{N} \right\rangle \end{gathered}} \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1^{\pm 1}, \dots, y_l^{\pm 1}, d^{\pm 1}] \xrightarrow{\psi_{\xi, N_1, \dots, N_l}} \mathbb{C}$$

• **Def** (Specialisation of coefficients)

$N_1, \dots, N_l \in \mathbb{N} \rightsquigarrow \psi_{\xi, N_1, \dots, N_l}$

$$\begin{cases} x_i \to \xi^{N_i - 1} \\ y_i \to \xi^{N_i} \\ d \to \xi^{-2} \end{cases}$$

• $\boxed{\text{Th}}$ (Topological state sum model for WRT invariant)

Fix $N \in \mathbb{N}$ level and $M = \underset{\text{framed link}}{S^3(L)}$. Let $L = \widehat{\beta_n}$.

Then:
$$\boxed{\tau_N(M) = \frac{\{1\}_\xi^{-1}}{\mathcal{D}^{b} \cdot \Delta_+^{b_+} \cdot \Delta_-^{b_-}} \cdot \sum_{\substack{i_1, \dots, i_n = 0 \\ \bar{i}}}^{N-2} \left(\sum_{\substack{1 \le N_1, \dots, N_l \le N-1 \\ \bar{i} < C(N)}} \Lambda_{\bar{i}}(\beta_n) \Big|_{\psi_{\xi, N_1, \dots, N_l}} \right)}$$

$(i_k < N_{C(k)} - 1)$

III Proof of the intersection model

· Reminder $\left(\Omega := \sum_{N=1}^{\mathcal{N}-1} [N]_\xi \cdot V_N, \quad \tau_{\mathcal{N}}(M) = \frac{1}{\mathcal{D}^b \cdot \Delta_+^{b+} \cdot \Delta_-^{b-}} \cdot \mathcal{J}_{\Omega,\dots,\Omega}(L,\xi) \right)$

Kirby colour

$$\rightsquigarrow \boxed{\tau_{\mathcal{N}}(M) = \frac{1}{\mathcal{D}^b \cdot \Delta_+^{b+} \cdot \Delta_-^{b-}} \cdot \sum_{\bar{N}=0}^{\mathcal{N}-1} [N_1]_\xi \cdots [N_l]_\xi \, \mathcal{J}_{N_1,\dots,N_l}(L,\xi)}$$

$\bar{N} = (N_1,\dots,N_l)$

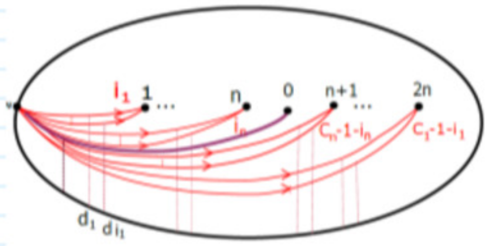
Step 1 Topological model for coloured Jones polynomials for links coloured with different colours: $\bar{N} = (N_1,\dots,N_l)$ $N_i \in \mathbb{N}$

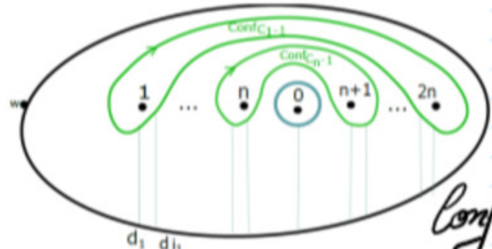
$\boxed{\bar{e}=0}$

$\rightsquigarrow$ colouring on β_n : $C^{\bar{N}} = (C_1,\dots,C_n)$

· Def: (Coloured homology classes) $\bar{i} \in \{0, C^{\bar{N}}\}$ $0 \leq i_k \leq C_k - 1$ $k \in \overline{0,n}$

$\mathcal{F}_{\bar{i}}^{C^{\bar{N}}} \in \mathcal{H}^{-m}_{2m+1, 0, \sum (C_i - 1)+1}$ $\mathscr{L}_{\bar{i}}^{C^{\bar{N}}} \in \mathcal{H}^{-m,\partial}_{2m+1, 0, \sum (C_i - 1)+1}$





$\mathrm{Conf}_{\sum_{i=1}^{m} (C_i - 1)+1}(D_{2m+1})$

$\cdot \boxed{Th}$ (A.) $$\boxed{\begin{aligned} \mathcal{J}_{N_1,\dots,N_e}(L) = &\, q^{\sum\left(f_i - \sum_{j\neq i} ek_{i,j}\right)(N_i-1)} \cdot \\ &\left(\sum_{\bar{i}\in C(\bar{N})} \prod_{i=1}^{m} x_{c(i)}^{-1} \cdot \left\langle (\beta_m \cup 1\!\!1)\, \mathcal{F}_{\bar{i}}^{C^{\bar{N}}}, \mathcal{L}_{\bar{i}}^{C^{\bar{N}}} \right\rangle\right)\Big|_{\psi_{q},\, N_1,\dots,N_e} \end{aligned}}$$

Translate the def. of WRT from repr. theory using this formula

$$\boxed{\begin{aligned} \tau_{\mathcal{N}}(M) = &\, \frac{1}{D^{b}\cdot\Delta_+^{b_+}\cdot\Delta_-^{b_-}} \cdot \sum_{\bar{N}=1}^{\mathcal{N}-1} q^{\sum\left(f_i - \sum_{j\neq i} ek_{i,j}\right)(N_i-1)} \cdot [N_1]_\xi \cdot \ldots \cdot [N_e]_\xi \\ &\left(\sum_{\bar{i}\in C(\bar{N})} \prod_{i=1}^{m} x_{c(i)}^{-1} \cdot \left\langle (\beta_m \cup 1\!\!1)\, \mathcal{F}_{\bar{i}}^{C^{\bar{N}}}, \mathcal{L}_{\bar{i}}^{C^{\bar{N}}} \right\rangle\right)\Big|_{\psi_{q},\, N_1,\dots,N_e} \end{aligned}}$$

intersection in $\mathrm{Conf}_{\sum_{i=1}^{m}(c_i-1)+1}(D_{2m+1})$

$\rightsquigarrow$ # particles varies depending on $\bar{N} \in \{0, \mathcal{N}-1\}$

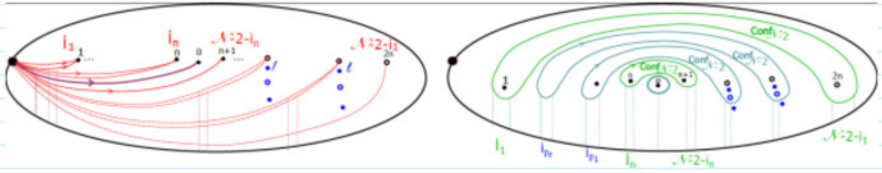
• $\boxed{\text{Step 2}}$ **Construct classes in a fixed configuration space**

• complete the geometric supports to the right s.t the # symmetric segments becomes $\mathcal{N}-2$

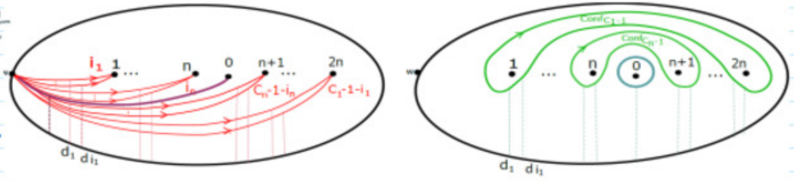
$\boxed{\mathcal{Y}_{N_1,\dots,N_\ell}(L)}$ $\quad C^{\bar{N}}$ induced colouring

• **Def** (Global homology classes) $\bar{i} \in \{0, \mathcal{N}-2\}$

$\mathcal{F}_{\bar{i}}^{\mathcal{N},\ell} \in \mathcal{H}^{-m}_{2m+1,\ell,m(\mathcal{N}-2)+1}$ $\quad \mathcal{L}_{\bar{i}}^{\mathcal{N},\ell} \in \mathcal{H}^{-m,\partial}_{2m+1,\ell,}$ $\qquad$ add arcs in the right $\qquad \mathcal{F}_{\bar{i}}^{C^{\bar{N}}} \in \mathcal{H}^{-m}_{2m+1,0,\Sigma(C_i-1)+1}$ $\quad \mathcal{L}_{\bar{i}}^{C^{\bar{N}}} \in \mathcal{H}^{-m,\partial}_{2m+1,0,\Sigma(C_i-1)+1}$



add 3ℓ punctures



$\mathrm{Conf}_{m(\mathcal{N}-2)+1}(D_{2m+3\ell+1})$

• **Prop**: This change does not affect the intersection form.

$$\boxed{\langle (\beta_m \cup \mathbb{1})\, \mathcal{F}_{\bar{i}}^{C^{\bar{N}}}, \mathcal{L}_{\bar{i}}^{C^{\bar{N}}} \rangle = \langle (\beta_m \cup \mathbb{1})\, \mathcal{F}_{\bar{i}}^{\mathcal{N},\ell}, \mathcal{L}_{\bar{i}}^{\mathcal{N},\ell} \rangle}\,.$$

$$\Rightarrow \boxed{\begin{aligned}\tau_{\mathcal{N}}(M) = \frac{1}{\mathcal{D}^{b}\cdot\Delta_+^{b_+}\cdot\Delta_-^{b_-}} \cdot \sum_{\overline{N}=1}^{\mathcal{N}-1} [N_1]_\xi \cdot \ldots \cdot [N_\ell]_\xi \\ \sum_{\overline{i}\in C(\overline{N})} \left(\prod_{k=1}^{\ell} x_{C(p_k)}^{\left(f_{p_k} - \sum_j lk_{p_k,j}\right)} \prod_{i=1}^{m} x_{c(i)}^{-1} \cdot \langle (\beta_m \cup \mathbb{1})\, \mathcal{F}_{\overline{i}}^{\mathcal{N},\ell}, \mathcal{L}_{\overline{i}}^{\mathcal{N},\ell} \rangle \right) \Big|_{\psi_{\xi, N_1,\ldots,N_\ell}}\end{aligned}}$$

intersection in $\mathrm{Conf}_{\underbrace{m(\mathcal{N}-2)+1}_{\text{fixed \# particles}}}(D_{2m+3\ell+1})$

- $\boxed{\text{Step 3}}$ $\boxed{Q}$ How to encode the Kirby colour? $[N_1]_\xi \cdot \ldots \cdot [N_\ell]_\xi$

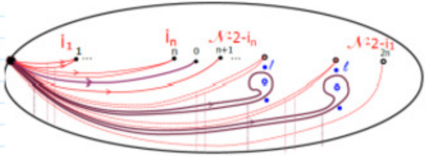
- Rk : $\boxed{[N_1]_\xi \cdot \ldots \cdot [N_\ell]_\xi = \{1\}_\xi^{-\ell}\ \psi_{\xi, N_1,\ldots,N_\ell}\left(\prod_{i=1}^{\ell} (y_i - y_i^{-1}) \right)}$ $\quad y_i \to q^{N_i}$

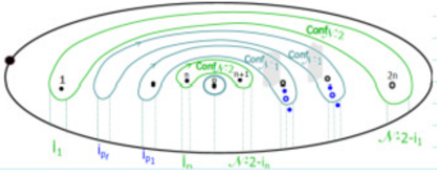
$\rightsquigarrow$ we'll obtain this from the twisted coefficients of intersection points

Idea: Add ℓ circles to the supports of the classes

(WRT homology classes)

$\mathcal{F}_{\bar{i}}^{N} \in \mathcal{H}^{-m}_{2m+1,\, \ell,\, m(N-2)+3\ell+1}$ $\quad$ $\mathcal{L}_{\bar{i}}^{N} \in \mathcal{H}^{-m,\partial}_{2m+1,\ell,\, m(N-1)+3\ell+1}$

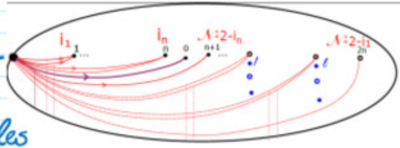


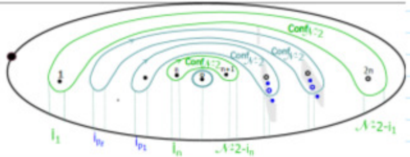


add 3ℓ circles

(Global classes)

$\mathcal{F}_{\bar{i}}^{N,\ell} \in \mathcal{H}^{-m}_{2m+1,\, \ell,\, m(N-2)+1}$ $\quad$ $\mathcal{L}_{\bar{i}}^{N,\ell} \in \mathcal{H}^{-m,\partial}_{2m+1,\ell,}$





Prop: This change captures precisely the coefficients coming from the Kirby colour:

$$\boxed{\langle (\beta_m \cup 1\!\!1)\, \mathcal{F}_{\bar{i}}^{N}, \mathcal{L}_{\bar{i}}^{N} \rangle = \prod_{i=1}^{\ell} (y_i - y_i^{-1})\, \langle (\beta_m \cup 1\!\!1)\, \mathcal{F}_{\bar{i}}^{N,\ell}, \mathcal{L}_{\bar{i}}^{N,\ell} \rangle .} \quad (*)$$

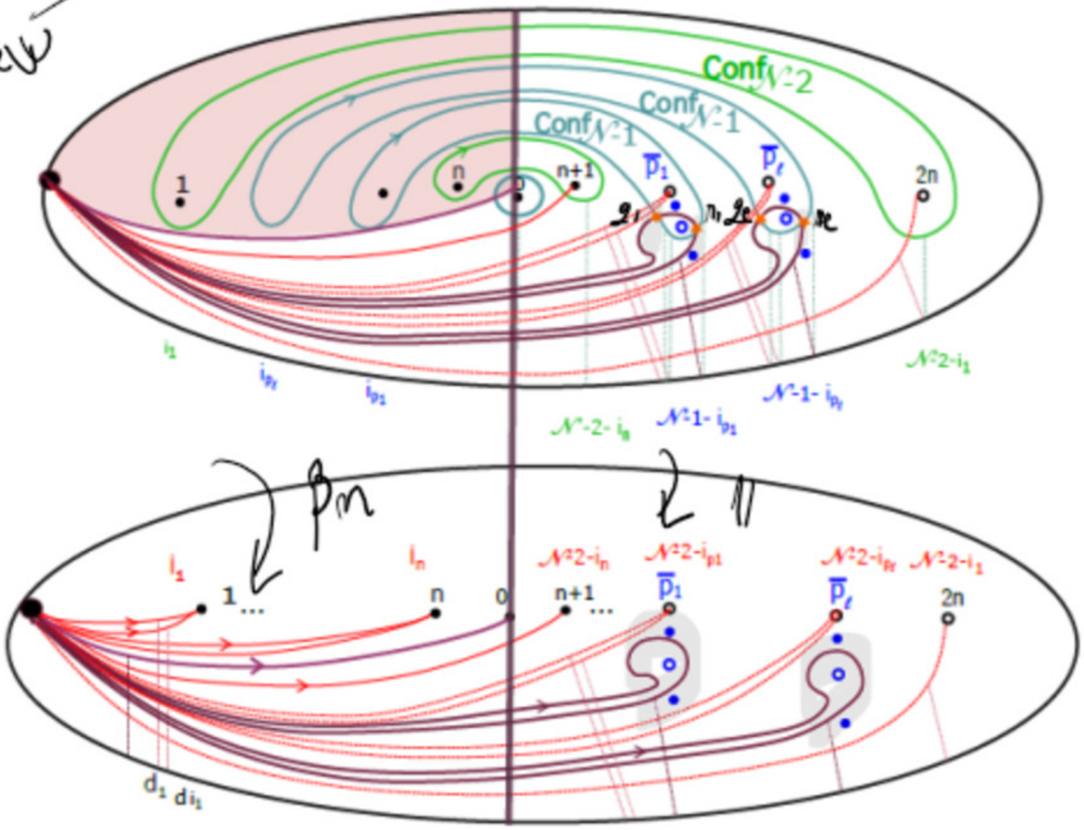
↳ Let $(q_k, r_k)_{k=\overline{1,e}}$ at the intersections between the new circles from $\mathcal{F}_{\bar{i}}^{\mathcal{N}} \cap \mathcal{L}_{\bar{i}}^{\mathcal{N}}$

We have a correspondence:

$$\boxed{\left((\beta_n \cup \mathbb{1})\, \mathcal{F}_{\bar{i}}^{\mathcal{N}} \cap \mathcal{L}_{\bar{i}}^{\mathcal{N}}\right) \xleftrightarrow{1:1} \left((\beta_n \cup \mathbb{1})\, \mathcal{F}_{\bar{i}}^{\mathcal{N},e} \cap \mathcal{L}_{\bar{i}}^{\mathcal{N},e}\right) \times \{q_1, r_1\} \times \dots \times \{q_e, r_e\}}$$

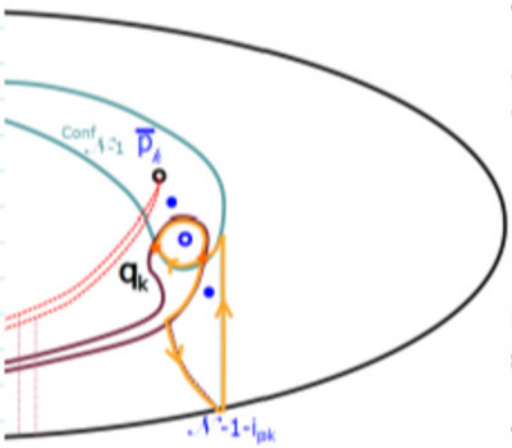
new

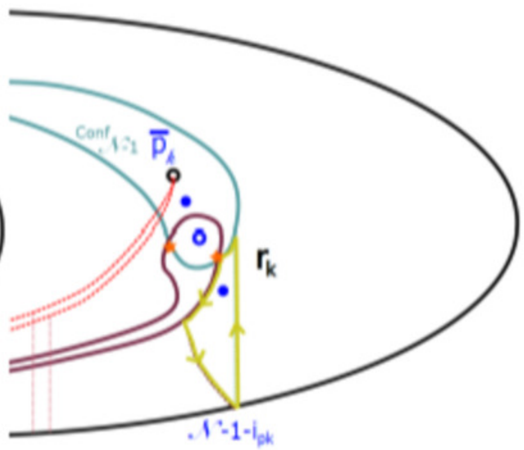
old



· $\boxed{Q}$ What about the gradings?

$$\boxed{\underbrace{\left((\beta_m \cup \mathbb{1})\, \mathscr{F}_{\bar{i}}^{\mathcal{N}} \cap \mathscr{L}_{\bar{i}}^{\mathcal{N}}\right)}_{\text{new}} \xleftrightarrow[\text{Intersection points}]{1:1} \left((\beta_m \cup \mathbb{1})\, \mathscr{F}_{\bar{i}}^{\mathcal{N},e} \underset{\text{old}}{\cap} \mathscr{L}_{\bar{i}}^{\mathcal{N},e}\right) \times \{q_1, r_1\} \times \dots \times \{q_e, r_e\}}$$





Monodromy of Φ

y_k^2

y_k^2

y_k

$\boxed{q_k}$ $y_k^2 \cdot y_k^{-1} = y_k$ $\quad \boxed{r_k}$ $-y_k^{-1}$

coefficients of q_k, r_k coming from the local system

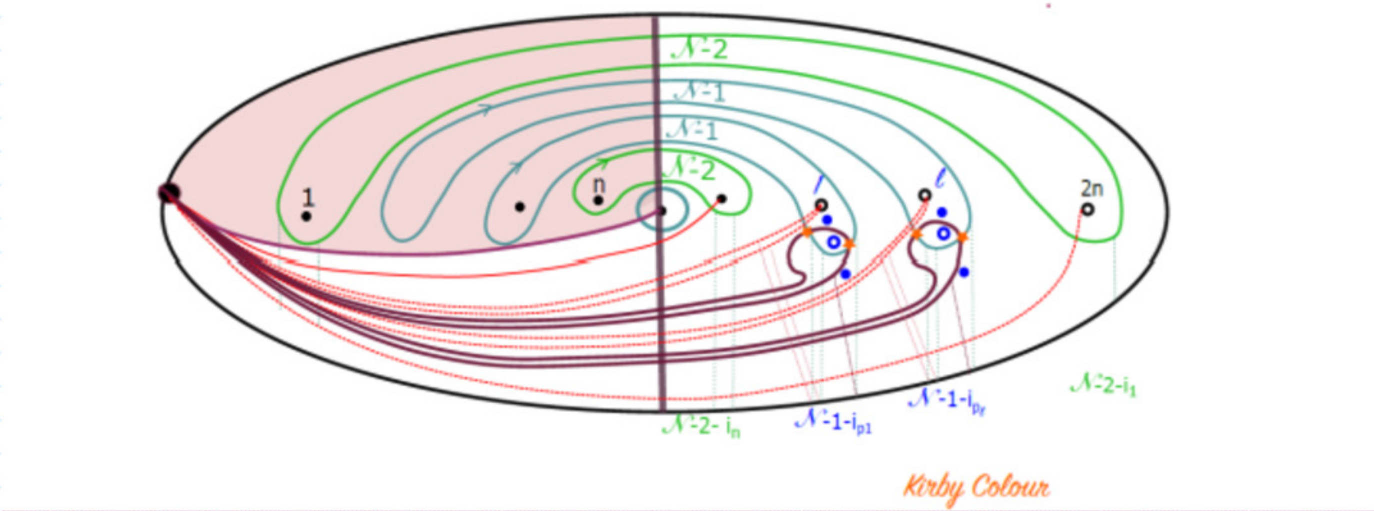
$\Longrightarrow$ (q_k, r_k) contributes by $(y_k - y_k^{-1})$ to the grading

$$\Longrightarrow \langle (\beta_m \cup \mathbb{1})\, \mathscr{F}_{\bar{i}}^{\mathcal{N}}, \mathscr{L}_{\bar{i}}^{\mathcal{N}} \rangle = \overbrace{\prod_{i=1}}^{e} (y_i - y_i^{-1}) \cdot \langle (\beta_m \cup \mathbb{1})\, \mathscr{F}_{\bar{i}}^{\mathcal{N},e}, \mathscr{L}_{\bar{i}}^{\mathcal{N},e} \rangle . \quad (*)$$

Then $(*)$ leads to the intersection model for $\Gamma_{\mathcal{N}}^{(\mathcal{K})}$ $\Longrightarrow$ q.e.d.

· $\boxed{\text{Corollary}}$

$$\tau_{\mathcal{N}}(M) \longleftrightarrow \sum_{\bar{i}}^{\mathcal{N}-2} \underset{\bar{i}<c(\mathcal{N})}{\overset{\text{specialisations of}}{\text{the intersection}}} \; \Lambda_{\bar{i}}(\beta_n)\Big|_{\psi_{\xi, N_1,\dots,N_e}}$$



$$(\beta_n \cup 1\!\!1)\, \mathcal{F}_{\bar{i}}^{\mathcal{N}} \cap \mathcal{L}_{\bar{i}}^{\mathcal{N}}$$

$\tau_{\mathcal{N}}(M) \longleftrightarrow$ encoded by the intersection points between these Lagrangians

$\bar{i} \in \{0, \mathcal{N}-2\}$