Configuration spaces and manifold calculus

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Esquisse d'un programme

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Grothendieck's idea: Try to understand $\Gamma_{\mathbb{Q}}$ via its action on a collection of (profinite completions) of homotopy types of varieties and a collection of maps between them (Teichmüller tower).

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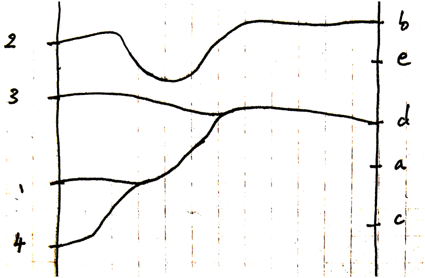

Remark

We have $\mathrm{con}(\mathbb{R}^2)^{\cong} \simeq \sqcup_n \pi_{\leq 1}(\mathrm{Emb}(n, \mathbb{R}^2))$

A homotopical interpretation of $\widehat{GT}$

Theorem (H.)

The Grothendieck-Teichmüller is the group of homotopy automorphisms of the profinite completion of $\mathrm{con}(\mathbb{R}^2)$*.*

Three ingredients.

Theorem (Drinfeld)

The Grothendieck-Teichmüller group is a subgroup of the group of homotopy automorphisms of the operad of profinite parenthesized braids $\widehat{PaB}$*.*

Theorem (H.)

The Grothendieck-Teichmüller group is the group of homotopy automorphisms of the profinite completion of the little 2-disks operad.

Theorem (Boavida-Weiss)

The little 2-disks operad contains the same homotopical data as the configuration category of $\mathbb{R}^2$*.*

This result suggests an algebro-geometric construction of $\mathrm{con}(\mathbb{R}^2)$. This has been done recently by Vaintrob.

More generally, if X is a smooth algebraic variety over a number field, then $\mathrm{con}(X^{an})$ should be the Betti realization of an algebro-geometric object (work in progress with Boavida de Brito and Kosanović).

In particular, there is an action of the absolute Galois group of the field on the profinite completion of $\mathrm{con}(X^{an})$.

Given two differentiable manifolds M and N, there is a map

$$Emb(M,N) \to Map_{/Fin}(\mathrm{con}(M), \mathrm{con}(N))$$

Theorem (Boavida de Brito-Weiss)

Let M and N be two smooth manifolds. Then, there is a homotopy cartesian square

$$\begin{array}{ccc} T_\infty Emb(M,N) & \longrightarrow & Map_{/Fin}(\mathrm{con}(M), \mathrm{con}(N)) \\ \downarrow & & \downarrow \\ Imm(M,N) & \longrightarrow & \Gamma \end{array}$$

Configuration categories and differential topology (Manifold calculus)

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with Γ the space of sections of a fiber bundle over M whose fiber over m is the space of pairs (n, α) with $n \in N$ and $\alpha : \mathrm{con}(T_m M) \to \mathrm{con}(T_n N)$ a map of configuration categories.

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(There is a map $Imm(M,N) \to \Gamma'$ with Γ' the space of section of a fiber bundle over M whose fiber over m is the space of pairs $(n.\beta)$ with β an injective linear map $T_m M \to T_n N$.)

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Recall that the obvious map $Emb(M,N) \to T_\infty Emb(M,N)$ is a weak equivalence if $\dim(N) - \dim(M) \geq 3$.

Theorem (Boavida de Brito-Weiss)

Let M and N be two smooth manifolds. Then, for all $k \geq 0$ there is a homotopy cartesian square

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Long knots

Definition

Fix a linear embedding $j : \mathbb{R} \to \mathbb{R}^3$*. The space of* *long knots**, denoted* $Emb_c(\mathbb{R}, \mathbb{R}^3)$ *is the space of embeddings from* $\mathbb{R}$ *to* $\mathbb{R}^3$ *that coïncide with* j *outside of a compact subset of* $\mathbb{R}$*.*

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Knot invariants

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A knot invariant with values in an abelian group A is a map

$$f : \mathcal{K} := \pi_0(Emb_c(\mathbb{R}, \mathbb{R}^3)) \to A.$$

The knot invariant I is said to be additive if it satisfies the formula

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A singular knot is a compactly supported immersion $\mathbb{R} \to \mathbb{R}^3$ whose only singularities are a finite number of double points at which the two tangent lines are distinct. We denote by $\mathcal{S}$ the set of singular knots up to isotopy.

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Any knot invariant $v : \mathcal{K} \to A$ may be extended to $\mathcal{S}$ using the following formula

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We denote by I_n the subgroup of $\mathbb{Z}[\mathcal{K}]$ generated by resolutions of singular knots with $n+1$ double points. An equivalent definition is to say that a knot invariant is of type $\leq n$ if it factors through $\mathbb{Z}[\mathcal{K}]/I_{n+1}$.

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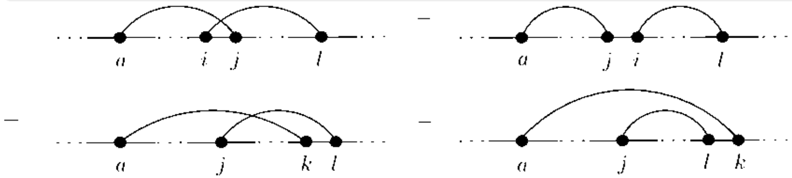
Proposition (Vassiliev)

There is a surjective map $\mathcal{A}_n \to I_n/I_{n+1}$ where $\mathcal{A}_n$ is the free abelian group generated by chord diagrams with n-chords modulo the 4T-relation.



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We denote by $\sim_n$ this equivalence relation. An additive invariant of type $\leq n$ is exactly the data of a monoid homomorphism $\mathcal{K}/\sim_n \to A$. There is a similarly defined group of indecomposable chord diagrams $\mathcal{A}_n^I$ with a surjective map

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The map $\mathcal{A}_n \to I_n/I_{n+1}$ and the map $\mathcal{A}_n^I \to J_n/J_{n+1}$ induce an isomorphism after tensoring with $\mathbb{Q}$.

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Conjecture

There are isomorphisms

$$\oplus_{k\leq n}\mathcal{A}_k \cong \mathbb{Z}[\mathcal{K}]/I_{n+1}, \quad \oplus_{k\leq n}\mathcal{A}_k^I \cong \mathcal{K}/\sim_{n+1}$$

Definition (Gusarov, Stanford)

A map $\pi_0(Emb_c(\mathbb{R},\mathbb{R}^3))) \to A$ with A an abelian group is an additive invariant of degree $\leq k$ if it is a monoid homomorphism and it is invariant under infection by pure braids lying in $\gamma_{k+1}(P_n)$.

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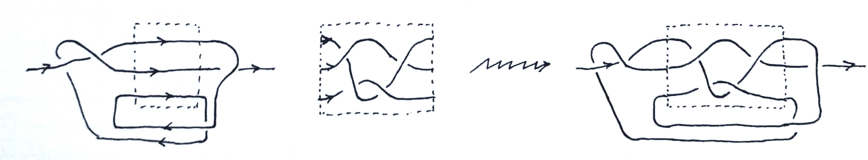

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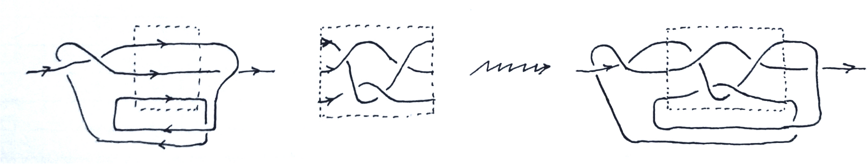

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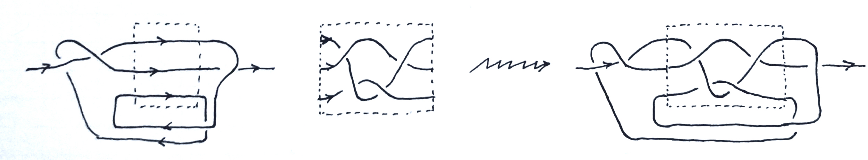

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True after tensoring with $\mathbb{Q}$ (Kontsevich integral). The map ev_{k+1} is a degree $\leq k$ invariant (Budney-Conant-Koytcheff-Sinha, Kosanović-Shi-Teichner)

Theorem (Kosanović)

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- *If $d \geq 4$, we can remove T_∞.*
- *This is a corollary of the previous theorem, using the fact that the space at the top right corner in the cartesian square is contractible in this case (Alexander trick).*

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- *This is a corollary of the previous theorem, using the fact that the space at the top right corner in the cartesian square is contractible in this case (Alexander trick).*

We define $T_k Emb_c(\mathbb{R}, \mathbb{R}^d)$ by the fiber sequence

$$T_k Emb_c(\mathbb{R}, \mathbb{R}^d) \to Imm_c(\mathbb{R}, \mathbb{R}^d) \to \Omega Map_{/Fin_{\leq k}}(\mathrm{con}(\mathbb{R}, k), \mathrm{con}(\mathbb{R}^d, k))$$

We specialize the general theory of manifold calculus to $Emb_c(\mathbb{R}, \mathbb{R}^d)$.

Theorem (Boavida de Brito-Weiss)

There is a fiber sequence

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We have a weak equivalence

$$T_\infty Emb_c(\mathbb{R}, \mathbb{R}^d) \simeq \mathrm{holim}_k T_k Emb_c(\mathbb{R}, \mathbb{R}^d)$$

We write $T_k = T_k Emb_c(\mathbb{R}, \mathbb{R}^d)$. The tower of fibrations $\ldots \to T_k \to T_{k-1} \to \ldots$ induces a spectral sequence

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Theorem (Goodwillie-Weiss,Göppl)

We have $\pi_{t-s}(L_s) = \bigcap_{i=0}^{s-1} \ker(\pi_t(s^i))$ *with*

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This can be computed completely in terms of homotopy groups of spheres using the fiber sequence

$$\bigvee_{s-1} S^{d-1} \to Emb(\underline{s}, \mathbb{R}^d) \to Emb(\underline{s-1}, \mathbb{R}^d)$$

Theorem (Boavida-H.)

Let p be a prime. Let $E^r_{-s,t}$ be the Goodwillie-Weiss spectral sequence for $T_\infty Emb(\mathbb{R},\mathbb{R}^d)$. In the spectral sequence $E^r_{-s,t}\otimes\mathbb{Z}_{(p)}$, in the range $t<2p-2+(s-1)(d-2)$, the only possibly non-zero differential are the d^r with $r-1$ a multiple of $(p-1)(d-2)$.

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Corollary

- *For $n\leq(p-1)(d-2)+3$ and $i\leq 2p-6+2(d-2)$:*

$$\pi_i(T_nEmb_c(\mathbb{R},\mathbb{R}^d))\otimes\mathbb{Z}_{(p)}\cong\oplus_{t-s=i}E^2_{-s,t}(T_n)\otimes\mathbb{Z}_{(p)}$$

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- *For $d > 4$ (resp. $d = 4$) and $i < 2p + 2d - 4$ (resp. $i < 2p$) :*

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Definition

Let X be a simply connected finite type CW-complex. There exists a unique space up to homotopy L_pX called the p-completion of X with a map $X \to L_pX$ such that

- *The map $X \to L_pX$ induces an isomorphism in $H_*(-, \mathbb{F}_p)$*
- *The map $X \to L_pX$ induces p-completion at the level of homotopy groups.*

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We denote by $T \otimes \mathbb{Z}_p$ the tower that we get by replacing $\mathrm{con}(\mathbb{R}^d)$ by its p-completion. The associated spectral sequence is simply the Goodwillie-Weiss spectral sequence tensored with $\mathbb{Z}_p$.

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Theorem (Boavida, H.)

There is a non-trivial action of $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the tower $\{T_n \otimes \mathbb{Z}_p\}_{n \in \mathbb{N}}$. This action is what forces some of the differentials to be zero.

Main theorem, sketch of proof

Let $\chi : \Gamma_{\mathbb{Q}} \to \hat{\mathbb{Z}}^{\times} \cong Aut(\mu_{\infty})$ be the cyclotomic character.

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- *The $\Gamma_{\mathbb{Q}}$-action on $E^1_{-s,n(d-2)+1} \otimes \mathbb{Z}_p$ is cyclotomic of weight n.*

We construct this action in several steps.

- Start from the $\Gamma_{\mathbb{Q}}$-action on the profinite completion of $\mathrm{con}(\mathbb{R}^2)$.
- This induces a $\Gamma_{\mathbb{Q}}$-action on the profinite completion of $\mathrm{con}(\mathbb{R}^d)$. (Follows from a general construction of Boavida-Weiss $\mathrm{con}(M \times N) \simeq \mathrm{con}(M) \otimes \mathrm{con}(N)$).
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Thank you !