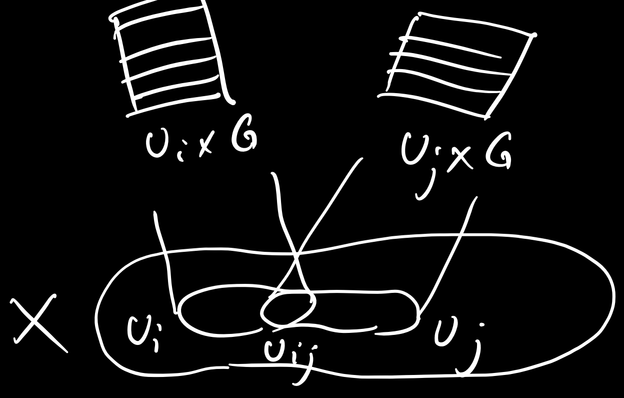
Bounded & Unbounded cohomology of diffeomorphism groups

jt N. Monod

Milnor's work :

Let G be a Lie group

$$\left\{ \begin{array}{c} \text{flat principle} \\ G\text{-bdles over} \\ X \end{array} \right\} \Big/ \text{iso} \longrightarrow \left\{ \begin{array}{c} \text{principle } G\text{-bdles} \\ \text{over } X \end{array} \right\} \Big/ \text{iso}$$

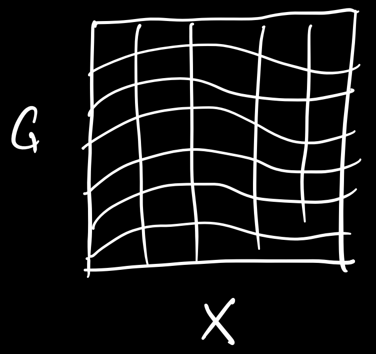


$$U_{ij} := U_i \cap U_j \xrightarrow{\varphi_{ij}} G$$

satisfies cocycle condition

Def If all φ_{ij}'s are constant the bundle is called flat.

Rk:



on flat bundles we have global foliation on total space $\pitchfork$ fibers.

On the level classifying spaces:

$$\eta : BG^\delta \longrightarrow BG$$

G^δ is G made discrete

Conj (Milnor) η^* induces an iso on $H(-;\mathbb{F}_p)$.

Rk. He proved for soluble Lie gps

Rk. Suslin proved it for $GL_n(\overset{\mathbb{R}}{\mathbb{C}})$

when $* \leq n$

Thm (Milnor) If G is cpct or complex semi-simple

$$\underline{\underline{H^*(BG;\mathbb{R})}} \longrightarrow \underline{\underline{H^*(BG^\delta;\mathbb{R})}}$$

is trivial.

Rk. because Chern-Weil theory.

Thm (Gromov) If G is a real algebraic group

$$\begin{tikzcd}
\eta^* : H^*(BG;\mathbb{R}) \arrow[r] & H^*(BG^\delta;\mathbb{R}) \cong H^*(G^\delta;\mathbb{R}) \\
& H_b^*(G;\mathbb{R}) \arrow[u]
\end{tikzcd}$$

$\operatorname{Im} \eta^*$ lands in $H_b^*(G;\mathbb{R})$

Ex: $G = GL_2^+(\mathbb{R}),\ PSL_2(\mathbb{R})$

$$BG \simeq BS^1 \simeq \mathbb{CP}^\infty$$

$$H^*(BG;\mathbb{R}) = \mathbb{R}[\underset{\substack{\uparrow \\ \text{Euler class}}}{e}]$$

$$\mathbb{R}[e] \longrightarrow H^*(BGL_2^+(\mathbb{R})^\delta ; \mathbb{R})$$

<u>Thm (Milnor)</u> Suppose

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] & E \arrow[d, "\pi"] \\ & \Sigma_g \end{tikzcd}$$

is flat $GL_2^+(\mathbb{R})$-bdle

$$\left| \langle e(\pi), [\Sigma_g] \rangle \right| \leq g-1$$

<u>Ex</u>. The only flat surface is Σ_1.

<u>Thm</u> (Wood) $G = PSL_2(\mathbb{R})$ or $\mathrm{Homeo}_0(S^1)$

$\circlearrowright_{S^1}$

Suppose

$$\begin{tikzcd} S^1 \arrow[r] & E \arrow[d, "\pi"] \\ & {} \end{tikzcd}$$

is flat

"

$$\Sigma_g$$

$$\left| \langle e(\pi), [\Sigma_g] \rangle \right| \leq |\chi(\Sigma_g)| = 2g-2$$

Rk. $\|e\| = \frac{1}{2}$

Rk. (Bucher - Monod) $\|e\| = \frac{1}{2^n}$

$G = GL^{+}_{2n}(\mathbb{R})$

Rk. $e^2 = 0$ in $H^4(PSL_2(\mathbb{R}); \mathbb{R})$

Thm (Thurston)

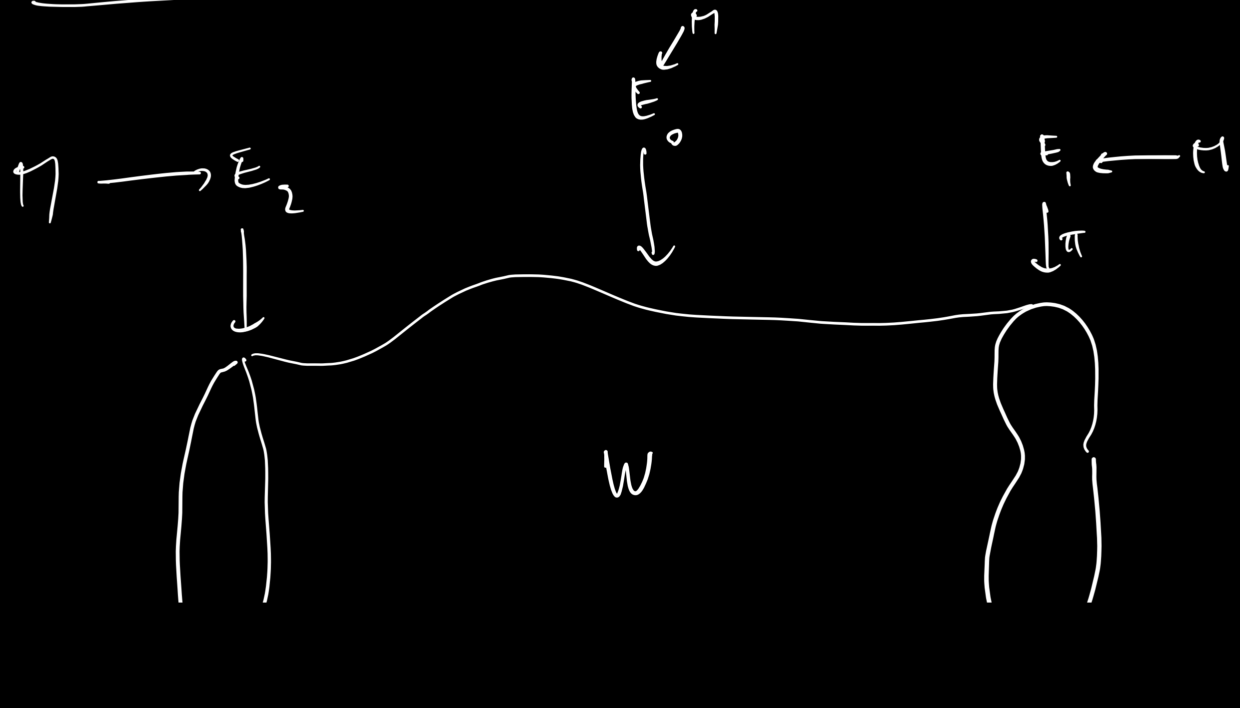
$$B\mathrm{Homeo}(M)^{\delta} \longrightarrow B\mathrm{Homeo}(M)$$

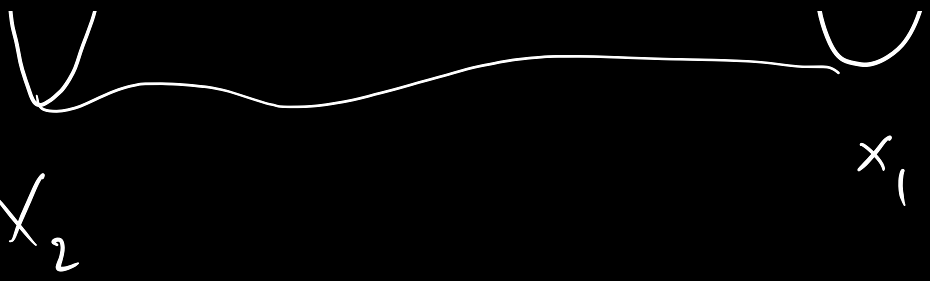
induces H_*-iso w/ any coefficient

Ex. $\mathrm{Homeo}_0(S^1) \simeq S^1$

Thurston $\Longrightarrow$ $H^*(\mathrm{Homeo}_0(S^1)^{\delta}; \mathbb{R}) \cong \mathbb{R}[e]$

Freedman's point of view :





$$E_0\big|_{X_i} = E_i$$

$$\exists \begin{tikzcd} E_0 \arrow[d] \\ W \end{tikzcd} \text{ s.t. } \begin{tikzcd} E_2 \arrow[d] \\ X_2 \end{tikzcd} \text{ is flat.}$$

Thm (Freedman) if X_1 is 3-fold one can choose W to be semi-s-cobordism.

$$X_2 \xrightarrow[1]{\deg} X_1$$

is H_*-iso but X_2 has different π_1.

Rk. $\rho : \pi_1(X) \longrightarrow \mathrm{Homeo}(M) / _{\mathrm{conj}}$

$$\Updownarrow$$

$$M \longrightarrow \tilde{X} \times_{\pi_1} M \longrightarrow X$$

Bounded cohomology

:

$X_\bullet$ is semi-simplicial set

$$\begin{tikzcd} X_p \arrow[r, shift left=3, "d_0"] \arrow[r, shift left=1] \arrow[r, phantom, shift right=1, "\vdots"] \arrow[r, shift right=3, "d_p"'] & X_{p-1} \end{tikzcd}$$

satisfying standard identities

$$H_b^*(X_\bullet) := H^*\Big(0 \longrightarrow \underset{\substack{\uparrow \\ \text{bounded} \\ \text{function}}}{\ell^\infty(X_0)} \xrightarrow[d_0^* - d_1^*]{d} \ell^\infty(X_1) \longrightarrow \cdots\Big)$$

<u>Def</u>. For a group G

$$H_b^*(G) := H^*\big(0 \longrightarrow \ell^\infty(G)^G \longrightarrow \ell^\infty(G \times G)^G \longrightarrow \cdots \big)$$

$$H_b^*(G) \longrightarrow H^*(G)$$

<u>Rk</u>. There is no f.g gp Γ known for which $\tilde{H}_b^*(\Gamma; \mathbb{R}) \neq 0$ and we can

Compute $H_b^*(\Gamma; \mathbb{R})$

Rk. If Γ is amenable, $\widehat{H}_b^*(\Gamma;\mathbb{R}) = 0$

Rk. Thm (Brooks) $\dim H_b^2(F_2 ; \mathbb{R}) = \infty$

$$\dim H_b^3(F_2 ; \mathbb{R}) = \infty$$

$$H_b^*(F_2 ; \mathbb{R}) = ? \qquad * > 3$$

2- applications for H_b^*

$$\begin{tikzcd} \Sigma_g \arrow[r] & E \arrow[d, "\pi"] \\ & \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d] \\ X \end{tikzcd}$$

$$\begin{tikzcd} k_i := \pi_!\left(e^{i+1}(T_\pi E)\right) \in H^{2i}(\mathrm{Mod}(\Sigma_g);\mathbb{R}) \\ H_b^{2i}(\mathrm{Mod}(\Sigma_g);\mathbb{R}) \arrow[u] \end{tikzcd}$$

Thm (Hamenstädt, Morita) k_i's are bounded.

Rk.

$$\begin{tikzcd} \Sigma_g \arrow[r] & E \arrow[d] & \Longrightarrow k_1 = 0 \\ & T^2 & \end{tikzcd}$$

Ghys' thm:

$$\Gamma \longrightarrow \mathrm{Homeo}_\circ(S^1)$$

$$\begin{tikzcd} e \in H^2(\Gamma, \mathbb{Z}) \\ e_b \in H^2_b(\Gamma, \mathbb{Z}) \arrow[u] \end{tikzcd}$$

If $e_b = 0$ then $\Gamma \circlearrowright S^1$ has a fixed pt.

Thm (Ghys, / Burger-Monod)

$\Gamma < SL_n(\mathbb{R})$ $n > 2$

if $\Gamma \circlearrowright S^1$ then the action has a finite orbit.

$G = \mathrm{Homeo}_0(S^1), \mathrm{Diff}_0(S^1)$

Thm (Monod_N)

$$H_b^*\big(\underset{\mathrm{Homeo}_0(S^1)}{\mathrm{Diff}_0(S^1)};\mathbb{R}\big) \cong \mathbb{R}[e]$$

Rk. Thurston's thm

$$\begin{tikzcd} H^*(\mathrm{Homeo}_0(S^1);\mathbb{R}) \cong \mathbb{R}[e] \\ H_b^*(\mathrm{Homeo}_0(S^1);\mathbb{R}) \arrow[u, "\cong"'] \end{tikzcd}$$

Thurston's thesis

$$H_2(\mathrm{Diff}(S^1);\mathbb{Z}) \twoheadrightarrow \underset{\uparrow}{\mathbb{Z}} \oplus \underset{\uparrow}{\mathbb{R}}$$

Euler Class GV

$$H_b^2(\mathrm{Diff}(S^1)) \xrightarrow{\neq} H^2(\mathrm{Diff}(S^1);\mathbb{R})$$

$$\mathrm{Diff}_0(D^2) \longrightarrow \mathrm{Diff}_0(S^1)$$

Thm (Morita) $e^k \neq 0$ in $H^{2k}(\mathrm{Diff}(S^1);\mathbb{R})$

Morita, $e^4 = 0$ in $H^8(\mathrm{Diff}(D^2);\mathbb{R})$

Thm (Monod-N) $H_b^*(\mathrm{Diff}_0(D^2);\mathbb{R}) \cong \mathbb{R}[e]$

Generalizing Milnor–Wood inequality

$$\begin{tikzcd} G = PSL_2(\mathbb{R}) \arrow[d] \\ G = GL^+_{2n}(\mathbb{R}) \end{tikzcd}$$

$G = PSL_2(\mathbb{R})$, $\underline{\underline{Homeo(S^1)}}$

$G = GL^+_{2n}(\mathbb{R})$, $e \in H^{2n}_b(GL^+_{2n}(\mathbb{R}); \mathbb{R})$

(Sullivan–Smillie
Bucher–Monod)

Ghys' question

Suppose $\rho : \pi_1(M^4) \longrightarrow Homeo_0(S^3)$

$$\begin{tikzcd} S^3 \arrow[r] & E \arrow[d, "\pi"] \\ & {} \end{tikzcd}$$

gives $\downarrow$ which is flat

M^4

$$\left|\langle e(\pi), [M^4] \rangle\right| \le \underset{\substack{\uparrow \\ \text{depending only} \\ \text{on } M.}}{c(M)}$$

Thm (Monod - N) $e \in H^4(\mathrm{Homeo}_0(S^3); \mathbb{R})$

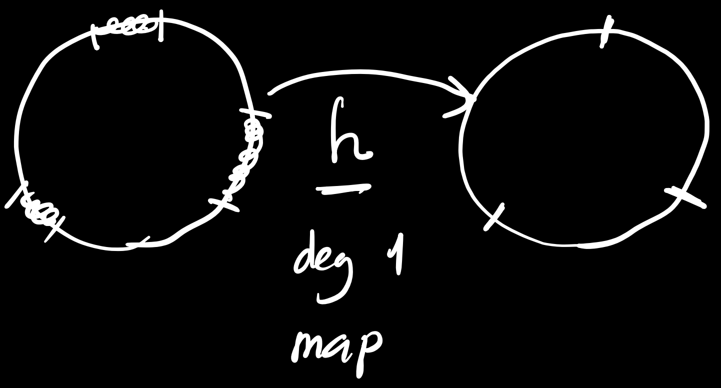
is not a bounded class.

In fact, $H^4_b(\mathrm{Homeo}_0(S^3); \mathbb{R}) = 0$.

$\mathrm{Homeo}(S^1)$

$$\begin{tikzcd}
 & \mathrm{Homeo}(S^1) \\
\Gamma \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\
 & \mathrm{Homeo}(S^1)
\end{tikzcd}$$

__Def.__ φ_1 & φ_2 are semi-conj



$$\varphi_1 h = h \varphi_2$$

__Ghys' thm.__ φ_1 & φ_2 are semi-conj

$$e_b(\varphi_1) = e_b(\varphi_2)$$

Thm (Bowden - Hensel - Webb)

$$\dim H_b^2(\mathrm{Homeo}_0(\Sigma_g); \mathbb{R}) = \infty$$

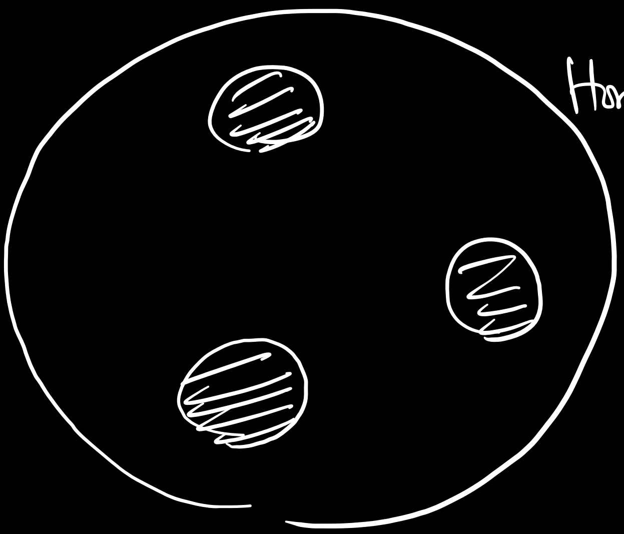
$g > 0$

Thurston $H^2(\mathrm{Homeo}_0(\Sigma_g)) = 0.$

$g > 1$

Q. $H_b^4(\mathrm{Homeo}_0(S^2); \mathbb{R}) = ?$

$H^4(\mathrm{Homeo}_0(S^2); \mathbb{R}) = \mathbb{R} \quad (P_1)$



$\mathrm{Homeo}_0(S^3 - 3 \text{ balls})$

is a uniformly perfect gp.

(Polterovich – Burago – Ivanov)

$\mathrm{Homeo}_0(S^2 - 3 \text{ balls})$

is probably <u>not</u> uniformly perfec