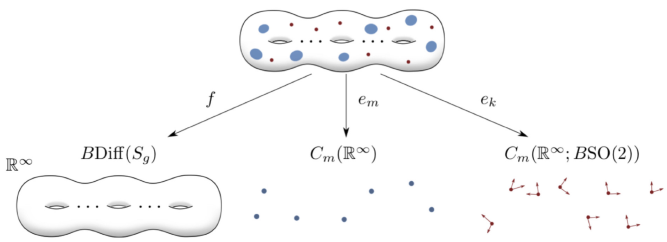
Decoupling Moduli of Configurations on Surfaces

Luciana Basualdo Bonatto, Max Planck Institute for Mathematics, Bonn

"Decoupling": properties associated to oriented surfaces $F_{g,b}$ with marked points that can be understood by looking separately at surfaces and at points in $\mathbb{R}^\infty$.



Thm (Bödigheimer – Tillmann, '01): the map

$$C_k(F_{g,b}) /\!\!/ \mathrm{Diff}(F_{g,b}) \longrightarrow B\mathrm{Diff}(F_{g,b}) \times C_k(\mathbb{R}^\infty; BGL_2^+)$$

is a homology isomorphism in degrees $\leq \frac{2}{3} g$.

GOAL FOR TODAY

- This also holds for generalised configuration spaces
  - → Labelled configurations
  - → Configurations with partially summable labels
  - → Submanifold configurations
- This gives new homological stability results → for instance about factorization homology
- This gives a splitting on monoids of moduli of configuration spaces.

I. Background and Notation

I.1. Monoids of Configurations

Defn: • The ordered configuration space of k points in a space M is

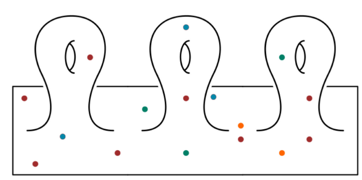
$$\tilde{C}_k(M) := \{(x_1,\dots,x_k) \in M^k \mid x_i \neq x_j \text{ for } i \neq j\}$$

• The unordered configuration space of k points in X is

$$C_k(M) := \tilde{C}_k(M)/\Sigma_k .$$

• For a well-pointed path-connected space $(X,*)$, the configuration space of M with labels in X

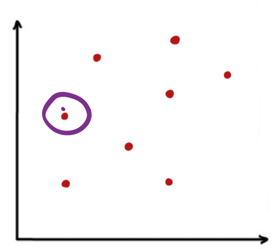
$$C(M;X) := \left. \coprod_k \tilde{C}_k(M) \times_{\Sigma_k} X^k \middle/ \sim \right.$$



$C(F_{3,1};$ colours$)$

$$(x_1,\dots,x_k; z_1,\dots,*) \sim (x_1,\dots,x_{k-1}; z_1,\dots,z_{k-1}), \quad z_k = *$$

Scanning results: For X path-connected



• Segal ('73): $C(\mathbb{R}^n; X) \simeq \Omega^n \Sigma^n X$

• McDuff ('75), Bödigheimer ('87): more general manifolds

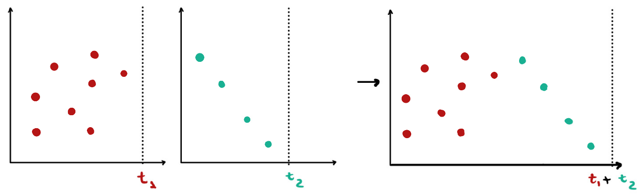
• McDuff ('75): configurations of positive and negative particles

• Salvatore ('09): partially summable labels

SEGAL'S ORIGINAL IDEA

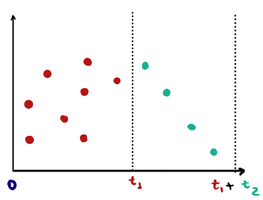
Consider the monoid of configuration spaces

$$C'(\mathbb{R}^n; X) \simeq C(\mathbb{R}^n; X)$$



X path-connected

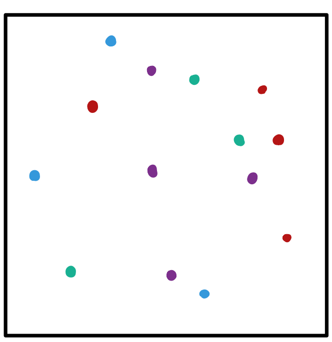
$$C'(\mathbb{R}^n; X) \simeq \Omega B\, C'(\mathbb{R}^n; X) \simeq \Omega\, C'(I \times \mathbb{R}^{n-1}, \partial I \times \mathbb{R}^{n-1}; X) \simeq \dots$$



$$\simeq \Omega^n C'(I^n, \partial I^n; X)$$

$$\simeq \Omega^n \Sigma^n X$$

Tablecloth argument



$$C(I^n, \partial I^n; X) \simeq \Sigma^n X$$

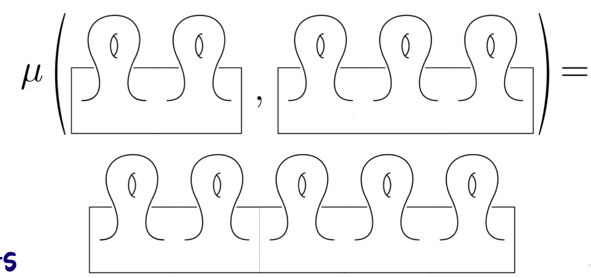
I.2. Monoids of manifolds

• Oriented Surfaces

Miller ('86), Tillmann ('00)

Let $F_{g,b}$ denote a surface of genus g and b boundary components

Representatives



$$B\,\mathrm{Diff}(F_{g,1}) \times B\,\mathrm{Diff}(F_{n,1}) \longrightarrow B\,\mathrm{Diff}(F_{g+n,1})$$

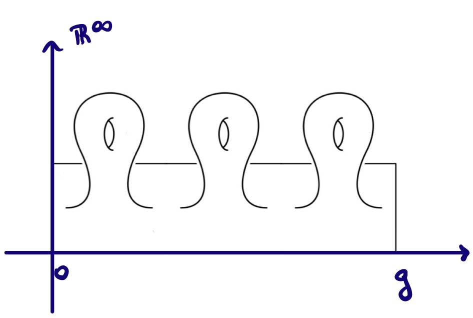
$$\mathcal{S} = \coprod_{g \geq 1} B\mathrm{Diff}(F_{g,1}) \quad \longrightarrow \text{monoid of oriented surfaces}$$

A Model for $B\mathrm{Diff}(F_{g,1})$: $\quad B\mathrm{Diff}(F_{g,1}) \simeq E\mathrm{Diff}(F_{g,1}) / \mathrm{Diff}(F_{g,1})$

Take $E\mathrm{Diff}(F_{g,1}) = \mathrm{Emb}_\partial(F_{g,1}\,;\,[0,g] \times \mathbb{R}^\infty)$

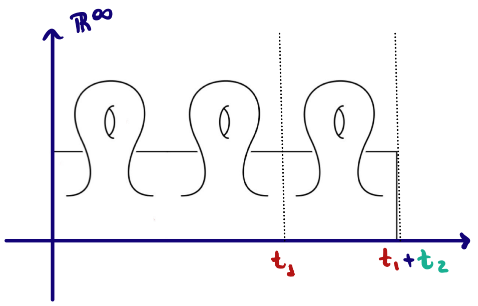
↑ Good boundary condition.

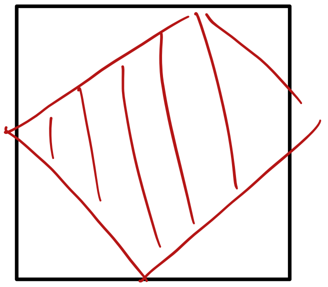
So we have $E\mathrm{Diff}(F_{g,1}) \times E\mathrm{Diff}(F_{n,1}) \longrightarrow E\mathrm{Diff}(F_{g+n,1})$



Using Segal's ideas it was possible to study

$$\Omega B \mathcal{C} \simeq \quad . \; . \; . \quad \simeq \mathbb{Z} \times \Omega^\infty MTSO(2)$$





[Galatius-Madsen-Tillmann-Weiss, Madsen-Weiss, Galatius-Randal-Williams]

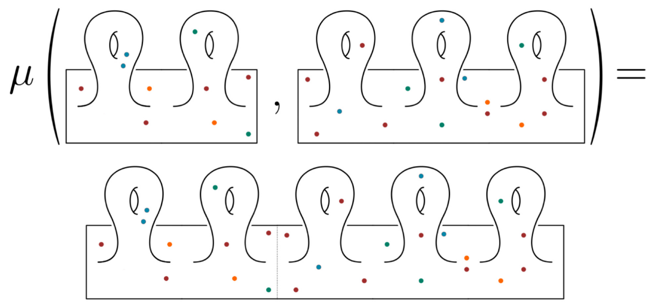
- Higher Dimensions

Galatius — Randal-Williams (09)

↳ Crucial to the study of the classifying space of the cobordism category.

II. Monoid of Configurations on Surfaces

$$\frac{EDiff(F_{g,1}) \times C(F_{g,1}; X)}{Diff(F_{g,1})}$$



↳ That is a model for the homotopy quotient (Borel construction)

$$C(F_{g,1}; X) /\!\!/ Diff(F_{g,1})$$

Monoid of configurations: $MC(X) := \coprod_{g \geq 0} C(F_{g,1}; X) /\!\!/ Diff(F_{g,1})$

Thm (BB, '22):

$$\Omega B\, MC(X) \simeq \underbrace{\mathbb{Z} \times \Omega^\infty MTSO(2)}_{\Omega B\, \mathfrak{S}} \times \underbrace{\Omega^\infty \Sigma^\infty (BSO(2)_+ \wedge X)}_{\Omega B\, C'(\mathbb{R}^\infty;\, BSO(2)_+ \wedge X)}$$

Idea of proof:

- Maps of monoids

$$\begin{tikzcd} & \mathfrak{S} \\ MC(X) \arrow[ur] \arrow[dr] & \\ & C'(\mathbb{R}^\infty; BSO(2)_+ \wedge X) \end{tikzcd}$$

$\mathfrak{S}$: Forgets the config.

$C'(\mathbb{R}^\infty; BSO(2)_+ \wedge X)$: Forgets the surface, but remembers an infinitesimal configuration of the particles

induce maps on group completions.

Since loop spaces are simple, to check the weak equivalence we only need to verify the map induces a homology equivalence.

$$\Omega B\, MC(X) \xrightarrow[H_*\cong]{} \Omega B\, S \times \Omega B\, C'(\mathbb{R}^\infty; BSO(2)_+ \wedge X)$$

- By the Group Completion Theorem it is enough to show

$$C(F_\infty; X)/\!\!/_{Diff(F_\infty)} \longrightarrow BDiff(F_\infty) \times C'(\mathbb{R}^\infty; (BSO(2)_+ \wedge X))$$

is a homology equivalence.

Recall:

Thm (Bödigheimer - Tillmann, '01): the map

$$C_k(F_{g,b})/\!\!/_{Diff(F_{g,b})} \longrightarrow BDiff(F_{g,b}) \times C_k(\mathbb{R}^\infty, BSO(2))$$

is a homology isomorphism in degrees $\leq \frac{2}{3} g$.

In a bit more generality: For Y a Σ_k-space

$$\tilde{C}_k(F_{g,b}) \times_{\Sigma_k} Y /\!\!/_{Diff(F_{g,b})} \longrightarrow BDiff(F_{g,b}) \times \left(\tilde{C}_k(\mathbb{R}^\infty) \times BSO(2)^k \times Y\right)/_{\Sigma_k}$$

is a homology isomorphism in degrees $\leq \frac{2}{3} g$.

Thm (BB, '22): For any X

$$C(F_{g,b};X)/\!\!/_{\mathrm{Diff}(F_{g,b})} \longrightarrow B\mathrm{Diff}(F_{g,b}) \times C(\mathbb{R}^\infty; BSO(2)_+ \wedge X)$$

is a homology isomorphism in degrees $\leq \frac{2}{3} g$.

Proof:

- Filtration by the number of points in the configuration

$$Y_n := C_{\leq n}(F_{g,b}; X)/\!\!/_{\mathrm{Diff}(F_{g,b})}$$

$$Z_n := B\mathrm{Diff}(F_{g,b}) \times C'_{\leq n}(\mathbb{R}^\infty; BSO(2)_+ \wedge X)$$

Cofibration sequences

$$\begin{tikzcd}
E\mathrm{Diff}(F_{g,b}) \times_{\mathrm{Diff}(F_{g,b})} C_n(F_{g,b}) \arrow[r] \arrow[d, "(\mathrm{I})"] & E\mathrm{Diff}(F_{g,b}) \times_{\mathrm{Diff}(F_{g,b})} \left(\tilde{C}_n(F_{g,b}) \times_{\Sigma_n} X^{\wedge n}\right) \arrow[r] \arrow[d, "(\mathrm{II})"] & Y_n/Y_{n-1} \arrow[d, "(\mathrm{III})"] \\
B\mathrm{Diff}(F_{g,b}) \times C_n(\mathbb{R}^\infty) \arrow[r] & B\mathrm{Diff}(F_{g,b}) \times \tilde{C}_n(\mathbb{R}^\infty) \times_{\Sigma_n} (BSO(2)_+ \wedge X)^{\wedge n} \arrow[r] & Z_n/Z_{n-1}
\end{tikzcd}$$

By (BT00, BB21), (I) and (II) are homology isos

$\Rightarrow$ (III) is a homology iso. $\blacksquare$

Corollary (BB, '22): For $b \geq 1$, the stabilization map

$$C(F_{g,b};X)/\!\!/_{\mathrm{Diff}(F_{g,b})} \longrightarrow C(F_{g+1,b};X)/\!\!/_{\mathrm{Diff}(F_{g+1,b})}$$

is a homology isomorphism in degrees $\leq \frac{2}{3}g$.

Proof:

$$\begin{tikzcd}
C(F_{g,b};X)/\!\!/_{\mathrm{Diff}(F_{g,b})} \arrow[r, "\mathcal{D}", "H_*\cong_{\frac{2}{3}g}"'] \arrow[d] & B\mathrm{Diff}(F_{g,b}) \arrow[d, "H_*\cong_{\leq\frac{2}{3}g}"', "\text{Harer}"] & \times & C(\mathbb{R}^\infty; X\wedge(BGL_2)_+) \arrow[d] \arrow[d, equal, shift left=6] \\
C(F_{g+1,b};X)/\!\!/_{\mathrm{Diff}(F_{g+1,b})} \arrow[r, "\mathcal{D}"', "H_*\cong_{\frac{2}{3}g}"'] & B\mathrm{Diff}(F_{g+1,b}) & \times & C(\mathbb{R}^\infty; X\wedge(BGL_2)_+)
\end{tikzcd}$$

$\square$

III. CONFIGURATIONS WITH PARTIALLY SUMMABLE LABELS

III.1. Partially Summable Labels

d-monoid $\longleftrightarrow$ algebra over E_d-operad

$(M, *)$ $\qquad\qquad E_d(k) \times_{\Sigma_k} M^k \longrightarrow M$

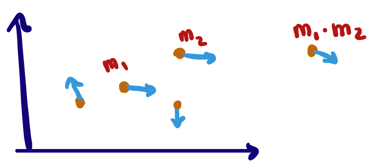
Partial d-monoid $\quad \forall\, k \geq 0$

$$\begin{tikzcd} \mathrm{Comp}_k \arrow[r, hook] \arrow[dr, "\text{mult.}"'] & E_d(k) \times_{\Sigma_k} M^k \arrow[d, "\not"] \\ & M \end{tikzcd}$$

+ conditions

Examples: ① $\{-, 0, +\}$, $\quad \mathrm{Comp}_2 = \{(0,+), (0,-), (+,-), (0,0),$
$(+,0), (-,0), (-,+)\}$

② M a d-monoid.



$P = M \times \mathbb{R}^2_*$

For P a partial d-monoid, we define

$$C_\Sigma(M^d; P)$$

to be the configuration space of points in M labelled in P, in which points are allowed to collide if their labels are summable.

Making this precise:

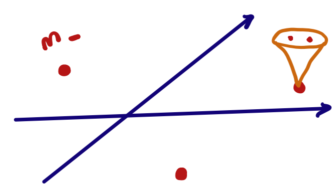
- Fulton-MacPherson configuration spaces: $M \subset \mathbb{R}^n$

$$F : \tilde{C}_k(M) \longrightarrow M^k \times \underbrace{(S^{n-1})^{\binom{k}{2}}}_{\text{pair wise direction}} \times \underbrace{(0,\infty]^{\binom{k}{3}}}_{\text{relative distances}}$$

Then $\tilde{C}_k[M] := \text{Closure}(\text{Im}\, F)$.

↳ Smooth manifold with corners

↳ Interior is $C_k(M)$



- For a partial d-monoid

$$C_\Sigma(M; P) := \left(\coprod_k \tilde{C}_k[M] \times_{\Sigma_k} P^k \right) \Big/ \sim$$

↑ Fulton-MacPherson Operad

As before, we can look at the Borel constructions

$$C_\mathcal{E}(M;P) /\!\!/ \mathrm{Diff}(M)$$

Proposition: For a d-monoid P,

$$C_\mathcal{E}(M;P) \simeq \int_M \Omega^d B^d P$$

Factorization homology

As before, we can create a topological monoid

$$MC_\mathcal{E}(P) := \coprod_{g \geq 0} C_\mathcal{E}(F_{g,1};P) /\!\!/ \mathrm{Diff}(F_{g,1})$$

Thm (BB, '22): For $b \geq 1$, the stabilization map

$$C_\mathcal{E}(F_{g,b};X) /\!\!/ \mathrm{Diff}(F_{g,b}) \longrightarrow C_\mathcal{E}(F_{g+1,b};X) /\!\!/ \mathrm{Diff}(F_{g+1,b})$$

is a homology isomorphism in degrees $\leq \frac{2}{3} g$.

Thm (BB '22):

Does not depend on the surfaces

$$\Omega B\, MC_\mathcal{E}(P) \simeq \underbrace{\mathbb{Z} \times \Omega^\infty MTSO(2)}_{\Omega B S} \times \Omega B \boxed{}$$

"Points in $\mathbb{R}^\infty_+$" ...

! THE PROOF BEFORE DOES NOT WORK !

Idea: Add MUCH more data → semi-simplicial resolution

II.2. The semi-simplicial resolution

Recall: a semi-simplicial space is a functor

$$\Delta^{op}_{inj} \longrightarrow \text{Spaces}$$

(no degeneracies just face maps)

Nice properties:

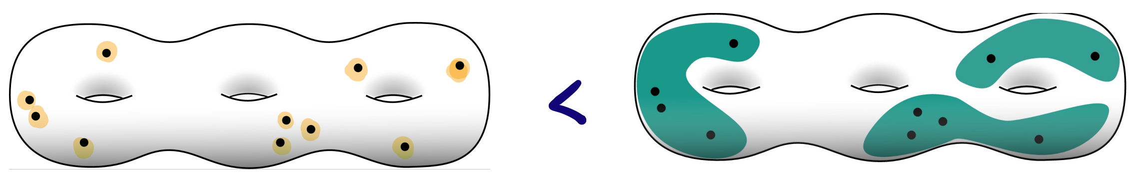
(1) Homology equivalences can be checked levelwise

$$X_\bullet \longrightarrow Y_\bullet \quad \text{that} \quad X_n \xrightarrow[H_*\cong]{} Y_n,\ \forall n \Rightarrow |X_\bullet| \xrightarrow[H_*\cong]{} |Y_\bullet|$$

(2) Galatius — Randal-William's "Simplicial Technique" ('14): good for constructing realizations.

$$X_\bullet \longrightarrow Y \quad \text{s.t.} \quad |X_\bullet| \xrightarrow{\simeq} Y$$

A Semi-simplicial realization for $C_\Sigma(M;P)$:



$D_\Sigma(M;P)$: space of configurations surrounded by embedded discs

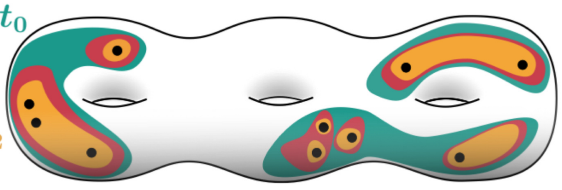
$$D_\Sigma(M;P) := \left(\coprod_{n\geq 0} \mathrm{Emb}\left(\coprod_n \mathbb{R}^d; M\right) \underset{\Sigma_n \wr GL_d}{\times} C_\Sigma\left(\coprod_n \mathbb{R}^d; P\right)\right) \Big/ \sim$$

$D_\Sigma(M;P)$ is a poset with $(c,e) < (c',e')$ iff

$c = c'$ and $\mathrm{Im}\, e \supset \mathrm{Im}\, e'$

$D_\Sigma(M;P)_\bullet$:= semi-simp. nerve of $D_\Sigma(M;P)$

2-simplex
$(c,e_0) < (c,e_1) < (c,e_2)$



Proposition (BB '22): $|D_\Sigma(M;P)_\bullet| \simeq C_\Sigma(M;P)$

Relation to Factorization homology

Thm (BB '21):

$|D_\Sigma(F_{g,b};P)_\bullet| /\!\!/ \mathrm{Diff}(F_{g,b}) \longrightarrow B\mathrm{Diff}(F_{g,b}) \times |D^2_\Sigma(\mathbb{R}^\infty;P)_\bullet|$

Embedded 2-discs in $\mathbb{R}^\infty$

induces $H_* \cong$ in $* \leq \frac{2}{3}g$.

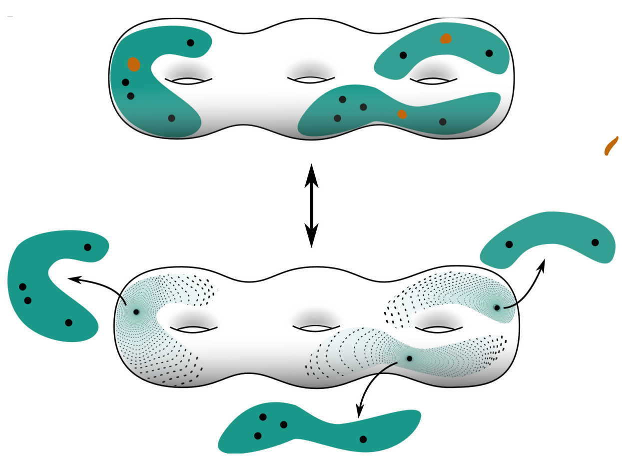
Idea of proof: Work levelwise.

On 0-simplices:

$$D_\Sigma(F_{g,b};P)_0 \longrightarrow B\mathrm{Diff}(F_{g,b}) \times D^2_\Sigma(\mathbb{R}^\infty;P)_0 \qquad (\mathrm{I})$$

Key observation:

$D_\Sigma(F_{g,b};P)_0 \xrightarrow{\simeq} C(F_{g,b};C_\Sigma(D^2;P))$

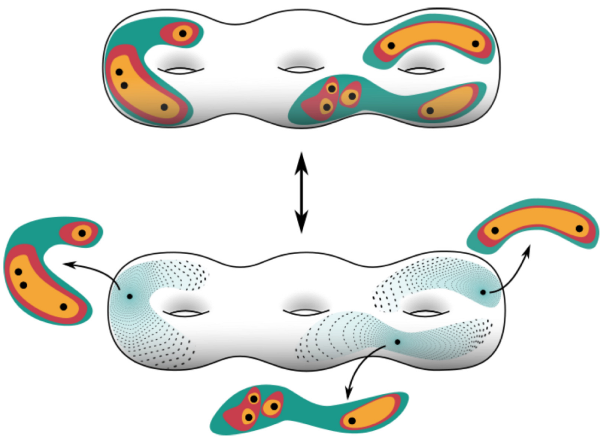


Then (I) induces H_* iso for $* \leq \frac{2}{3}g$ because of the Decoupling theorem for labelled configurations.

Same for higher simplices!

On 2-simplices:

$$D_\Sigma(F_{g,b};P)_2 \longrightarrow BDiff(F_{g,b}) \times D^2_\Sigma(\mathbb{R}^\infty;P)_2$$



Then we can again apply the Decoupling theorem for labelled configurations.

...

$\square$