On the decomposable Orlik - Solomon algebra (OS)

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- quotient of ext. algebra assoc. to finite simple matroids; in particular to arr. of hyperplanes

- $\mathcal{A} \rightsquigarrow \mathcal{L}(\mathcal{A})$ (intersection lattice)

$\cap$

$V/_{\mathbb{C}} = n \rightsquigarrow X(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$ (complement of $\mathcal{A}$)

($\mathbb{C}$ v. space)

- algebraic properties of this comb. object (decomp OS alg) are connected to the topology of the complement.

<u>Thm</u> (M. Mata Papadima). Let $\mathcal{A}$ be a hypersolvable and not supersolvable complex hyperplane arrangement with complement $X(\mathcal{A})$, and $p > 1$ minimal with the property that $\pi_p(X(\mathcal{A}))$ is non-trivial. Then the following are equivalent:

(1) The second component of the graded module

$\bigoplus_{q \geq 0} \left(\pi_p I^q / \pi_p I^{q+1} \right) =: gr_I^* \pi_p(X(\mathcal{A}))$ has no torsion;

(2) The decomposable OS algebra $A_+^*(\mathcal{A})$ is free in degree $(p+2)$.

$\Lambda^* =$ free exterior algebra on n generators $/ \mathbb{Z}$
$\longrightarrow \Lambda(e_1, \dots, e_n)$

Λ^* = graded algebra

$$\Lambda^m = \langle e_X := e_{i_1} \wedge \dots \wedge e_{i_m} \;/\; X \subset [n] \rangle$$
$$X = \{i_1 < \dots < i_m\}$$

$\partial : \Lambda^* \to \Lambda^{*-1} \qquad \partial(e_i) = 1, \quad i \in [n] \qquad$ (degree (-1))

(boundary map)

M <u>simple</u> matroid on $[n]$
$\Updownarrow$
no circuits of length 1 or 2.

f. <u>matroid</u> : $M = (E, I)$, E finite set, $I \subset$ Subsets (E)
$\downarrow$ the set of independents.

- I has the properties: • $\emptyset \in I$; •• $J_0 \subset I_0$, $I_0 \in I \Rightarrow J_0 \in I$.

••• $A, B \in I$, $|A| > |B| \Rightarrow \exists\, x \in A \setminus B$ s.th. $B \cup \{x\} \in I$

$S \subset E$ <u>dependent</u> $\Leftrightarrow$ $S \notin I$.

$C \subset E$ <u>circuit</u> $\Leftrightarrow$ C minimally dependent; $\mathcal{C}(M)$ set of circuits of M.

. OS ideal of M, $I(M) : \langle \partial(e_C) \;/\; \in \mathcal{C}(M) \rangle$

. decomposable OS ideal of M $\quad I^+(M) = \Lambda^+ I(M)$,

where Λ^+ subalg of elements in Λ^* of degree ≥ 1.

- OS algebra of M, $A^*(M) := \Lambda^* / I^*(M)$

- decomposable OS alg of M $A_+^*(M) := \Lambda^* / \Lambda^+ I^+(M)$

(one can define all of the above over an arbitrary field of coefficients $\Bbbk$. - not: $-_{\Bbbk}$)

- we will be interested in matroids coming from central arrangements of hyperplanes.

$$\mathcal{L}(\mathcal{A}) := \{ \bigcap_{H \in S} H \;/\; S \subseteq \mathcal{A} \}$$

$\downarrow$

matroid on $\mathcal{A}$, with indep defined as linear indep.

$\mathcal{L}(\mathcal{A})$ is a geometric lattice $\leftrightarrow$ of simple matroid.

$\Updownarrow$

(finite, atomistic, semimodular)

- $\mathcal{C}(\mathcal{A}) = \{ C \subset \mathcal{A} \;/\; C$ circuit (min. dep. subset) in $\mathcal{A} \}$.

$I(\mathcal{A})$, $A^*(\mathcal{A})$, $I^+(\mathcal{A})$; $A_+^*(\mathcal{A})$

- $A^*(\mathcal{A})$ is torsion free, it admits a basis (more later) expressed in terms of circuits of $\mathcal{A}$, that does not dep on $\Bbbk$

- (Orlik - Solomon) $A^*(\mathcal{A}) \cong H^*(X(\mathcal{A}); \mathbb{Z})$

iso of alg. $\searrow$ in particular, combinatorial

NBC basis

- a broken circuit is a circuit with it's smallest element deleted.
- a set $I \subset [n]$ is called NBC if it does not contain any broken circuit.
- NBC $\Rightarrow$ indep
- $\Lambda^* \twoheadrightarrow A^*(M) = \Lambda^* / I(M)$

  the monomials e_I, I = NBC

  form a basis (through the above map)
- in each degree p, NBC monomials $|I|=p$

  $$\Lambda^p \twoheadrightarrow A^p(M)$$ give a basis here.

· combinatoriality does not necessarily hold for homotopy groups. ($\pi_1(X(\mathcal{A}))$ is not comb., $\pi_{*>1}$ are difficult to compute)

$\pi_1 := \pi_1(X(\mathcal{A}))$; $\pi_p := \pi_p(X(\mathcal{A}))$

module over the group ring

$R := \mathbb{Z}\pi_1$; $I \subset R$ augm. ideal.

$$gr^*_I(\pi_p) := \bigoplus_{\ell \geq 0} \left(\pi_p I^\ell \big/ \pi_p I^{\ell+1} \right)$$

graded module over the graded ring $gr^*_I R$

(Kohno) it's rational assoc. graded Lie algebra, $gr^*(\pi_1(X(\mathcal{A})) \otimes \mathbb{Q}$. is comb

given by quotients of the lower central series

* $\pi_1(X(\mathcal{A}))$ abelian $\Leftrightarrow$ nilpotent $\Leftrightarrow$ hyperplanes are in general position in codim 2. (elements of rank 2 in the lattice appear as intersection of at most 2 hyp.)

· next.. are $\pi_p(X(\mathcal{A}))$ combinatorial? or some graded objects assoc. to them?

Supersolvability & hypersolvability

- supersolvable arr. (lattice is supersolvable)

$X \in \mathcal{L}(\mathcal{A})$ is called modular $\Leftrightarrow \forall Y \in L(\mathcal{A})$ $rk(X) + rk(Y) = rk(X \vee Y) + rk(X \wedge Y)$

$\mathcal{A}$ is supersolvable $\Leftrightarrow \exists$ a maximal chain of modular elements of length equal to $rk(\mathcal{A})$:

$$X_0 = V < X_1 \cdots < X_{rk\mathcal{A}}$$

equiv to fiber-type (Falk - Randell)

- $X(\mathcal{A})$ - $K(\pi,1)$ space, $\pi_1(X(\mathcal{A}))$ = semidirect product of free groups.

- $A^*(\mathcal{A})$ - quadratic (det. by dep. rel in rank 2)

$$\Downarrow$$

$$\Lambda^+ I =: I^+ \; ; \; I^q = I^q_+ \; , \; \forall q \geq 3 \quad ; \; I_+^{q \leq 2} = 0$$

$$0 \to \left(I/I^+\right)^q \to \Lambda^q_+(\mathcal{A}) \to A^q(\mathcal{A}) \to 0$$

(Jambu - Papadima) generalize s.s to hypersolvability

(hypersolvable arr. admit deformations to s.s. arr)

- the actual def is inductive ($\exists$ composition series) ext of arr defined by conditions on $\mathcal{L}_{\leq 2}(\mathcal{A})$.

- the s.s. deformation has the same π_1 as the initial arrangement ; a h.s and not s.s arrangement is never $K(\pi,1)$:

$$\mathcal{A} \text{ h.s and } K(\pi,1) \Leftrightarrow \mathcal{A} \text{ s.s}$$

$\Rightarrow \exists\, p$ minimal such that $\pi_p(X(\mathcal{A})) \neq 0$, for ($\mathcal{A}$ h.s & not s.s.) (Papadima-Suciu)

comb. determined

- $gr^*(\pi_1) := \bigoplus_i \left(\Gamma_i \pi_1 / \Gamma_{i+1} \pi_1 \right)$ is a f.g. free ab group, i, of ranks comb det.

$\Gamma_1 \pi_1 = \pi_1$; $\Gamma_{i+1} \pi_1 := [\pi_1, \Gamma_i \pi_1]$

(Dimca-Papadima) it is also the order of π_1-connectivity of the space $X(\mathcal{A})$ ($\stackrel{\text{def}}{=}$ max value p for which the $\mathbb{Q}$-betti numbers of $X(\mathcal{A})$ coincide (up to p) to those of $\pi_1(X(\mathcal{A}))$

$$gr_I^*(\pi_p) := \bigoplus_{q \geq 0} \left(\pi_p I^q / \pi_p I^{q+1} \right)$$

→ module over the graded ring $gr_I^* R$ → comb, tors. free

$\pi_p := \pi_p(X(\mathcal{A}))$, $\pi_1 := \pi_1(X(\mathcal{A}))$; $gr_I^0(\pi_p) = \pi_p / \pi_p I$ comb, torsion free

$R = \mathbb{Z}\pi_1$, $I \subset R$ aug. ideal

- in the hypothesis of the theorem, the minimal length of a circuit $C \in \mathcal{C}(\mathcal{A})$, $|C| > 3$ is $(k+2)$.

<u>Gröbner bases (in ext. algebra context).</u>

- $\Lambda^r_{\mathbb{k}}$ = the ext. algebra over n gen. $/_{\mathbb{k}}$ gen by $\{e_1, \dots, e_n\}$
- a <u>monomial order</u> on $\Lambda^r_{\mathbb{k}}$ is a total order on the set of monomials $\mathrm{Mon}(\Lambda^r_{\mathbb{k}})$ of $\Lambda^r_{\mathbb{k}}$ such that:

  (1). $1 < e_X$, for any monomial $e_X \neq 1$

  (2) If $e_X < e_Y$ then $e_X \wedge e_Z < e_Y \wedge e_Z$, $\forall$ e_X, e_Y, e_Z monomials such that $e_X \wedge e_Z \neq 0 \neq e_Y \wedge e_Z$.
- a monomial e_X <u>divides</u> a mon. e_Y $\Leftrightarrow$ $X \subset Y$.
- $f = \sum_{i=1}^{m} a_i e_{X_i} \in \Lambda^r_{\mathbb{k}}$, $a_i \in \mathbb{k} \setminus \{0\}$, $e_{X_i} \in \mathrm{Mon}(\Lambda^r_{\mathbb{k}})$

  <u>supp f</u> $:= \{e_{X_1}, \dots, e_{X_m}\}$ $\to$ support of f.
- $\mathrm{in}_<(f) = \max_<{}_i (e_{X_i})$ $\to$ initial monomial of f
- $I \subset \Lambda^r_{\mathbb{k}}$ nonzero ideal. A <u>Gröbner basis</u> for I

is a finite set of elements $\mathcal{G}_< \subset I$ such that $\forall f \in I$ $\exists g \in \mathcal{G}_<$ such that $in_<(g)$ divides $in_<(f)$. $\mathcal{G}_<$ is called <u>reduced</u> if for all $g \in \mathcal{G}_<$, the coeff of $in_< g = 1$ and, $\forall g \neq g' \in \mathcal{G}_<$, no monomial in $supp(g')$ is divisible by $in_<(g)$.

<u>Rem</u>: a G.B. is also a system of generators.

- $\{\partial e_C \mid C \in \mathcal{C}(M)\}$ is a Gröbner basis (G.B) for the OS ideal $I(M)$.

  does not depend on the order $<$

  the set of circuits of M, simple matroid on $[n]$.

- $\mathcal{C}_q(M) := \{C \in \mathcal{C}(M) \mid |C| = q\}$

$$
\begin{tikzcd}
\langle e_C \mid C \in \mathcal{C}_{q+1}(M)\rangle_{\mathbb{K}} \arrow[r, "\partial_q"] \arrow[rr, bend right, "\bar{\partial}_q"'] & I(M)^q_{\mathbb{K}} \arrow[r, two heads] & \left(I(M)/\Lambda^+ I(M)\right)^q_{\mathbb{K}}
\end{tikzcd}
$$

<u>idea</u>: to restrict the set of generators such that the map $\bar{\partial}_q$ is bijective.

- a circuit C is called <u>chordless</u> if $\nexists$ $i \in [n]$ such that $C = C' \sqcup C''$ and $C' \sqcup \{i\}$; $C'' \sqcup \{i\}$ dependent.

$\mathcal{C}^{NC}_{\ell+1}$ – chordless length $(\ell+1)$ circuits.

- $\bar{\partial}_\ell \big|_{\mathcal{C}^{NC}_{\ell+1}}$ = bij. , $\forall \ell$, for graphic arrangements.

$\downarrow$

subarr. in the braid arr.

$\mathcal{A}_\ell : \prod_{1 \le i < j \le \ell} (x_i - x_j)$

- defined via graphs. Γ $\overset{i \quad j}{\bullet\!\!-\!\!\!-\!\!\bullet} \rightsquigarrow x_i - x_j$

- signed graphic arr $\leftrightarrow$ signed graphs $\rightsquigarrow$ $\overset{i \; - \; j}{\bullet\!\!-\!\!\!-\!\!\bullet} \to x_i - x_j$

$\overset{i \; + \; j}{\bullet\!\!-\!\!\!-\!\!\bullet} \to x_i + x_j$

$\to x_i$

- a description of Zaslavski of circuits in signed graphs enables an easy generalization of the above claim;

$\bar{\partial}_\ell \big|_{\mathcal{C}^{NC}_{\ell+1}(\mathcal{A})}$ = bij. , if $\mathcal{A}$ signed graphic arr.

- this claim does not hold in general. ($\bar{\partial}_\ell$ not inj.)

- we try to solve this pb. in general by replacing the subset of chordless circuits with a more suitable subset of circ.

- any permutation π on $[n]$ defines an order on $[n]$

$\pi^{-1}(1) <_\pi \pi^{-1}(2) <_\pi \cdots <_\pi \pi^{-1}(n)$, hence a monomial order

$$e_{\pi^{-1}(1)} \prec_{\pi} \cdots \prec_{\pi} e_{\pi^{-1}(n)}$$

- an element $i \in [n]$ is called <u>active</u> w.r.t. an independent set I of the matroid M if $I \cup \{i\}$ contains a circuit with minimal element $\{i\}$.

- Let $\mathcal{C}^{\pi}(M)$ be the subset of circuits of M s.th.

(1) $\inf_{\prec_{\pi}}(C) = \alpha_{\pi}(C)$

(2) $C \setminus \alpha_{\pi}(C)$ is inclusion minimal with props(1),

where $\alpha_{\pi}(C)$ is the smallest active element w.r.t. $C \setminus \{\inf_{\prec_{\pi}} C\}$.

<u>Thm</u> (Forge). $\mathcal{G}_{\pi} := \{\partial e_C \,/\, C \in \mathcal{C}^{\pi}(M)\}$ is a reduced G.b. for $I(M)$ w.r.t. the monomial order $\prec_{\pi}$.

<u>Prop</u>: Let π be such that $|\mathcal{C}^{\pi}_{\ell+1}(M)| = \min\{|\mathcal{C}^{\pi'}_{\ell+1}(M)| \,/\, \pi' \in S_n\}$ (perm. of $[n]$)

Then the map

$$\bar{\partial}_{\ell} : \langle e_C \,/\, C \in \mathcal{C}^{\pi}_{\ell+1}(M) \rangle_{\mathbb{k}} \longrightarrow \left(I(M) / \Lambda^{+} I(M) \right)^{\ell}_{\mathbb{k}} \text{ is bij.}$$

Sketch of proof.

$\mathcal{G}_\pi$ is a G.b. $\Rightarrow$ ∂_2 surj

∂_2 is injective $\Leftarrow$ $\mathcal{G}_\pi$ reduced.

$\sum_c \mu_c \partial e_c = 0$ $\Rightarrow$ $\mathrm{in}_{<_\pi}(e_c)$ divides (equals)

$\cap$

$\mathcal{C}^{\pi}_{\leq H}(M)$ same monomial in $\mathrm{supp}(c')$

$c' \neq c$

- any $\partial(e_c)$ has an initial monomial that does not cancel out with any other monomial on the left side of the equality:

$$\sum_{c \in \mathcal{C}^{\pi}_{\leq H}(M)} \mu_c \partial e_c = \sum_{\bar{c}, s} \xi_{\bar{c}, s}\, e_s\, \partial e_{\bar{c}}, \quad \xi_{c,s}, \mu_c \in k$$

$e_s \in \Lambda^+$,

$\bar{c} \in \mathcal{C}^{\pi}_{\leq 2}(M)$

$\Rightarrow \exists\, \bar{c}$ such that a monomial from cancels $\mathrm{in}_{<_\pi}(\partial e_c)$.

in. not in.

slightly modify $\pi \rightsquigarrow \tilde{\pi}$ - -

Example: $\mathcal{A} \subset \mathbb{C}^4$ $\quad xyzt\underbrace{(x+y+z+t)}_{H}\underbrace{(x-y-z+t)}_{P} = 0$

$$\mathcal{C}(\mathcal{A}) = \{\underbrace{(H, P, y, z)}_{c_1};\ \underbrace{(H, P, x, t)}_{c_2},\ \underbrace{(H, x, y, z, t)}_{c_3},\ \underbrace{(P, x, y, z, t)}_{c_4}\}$$

$$\Lambda^* = \Lambda(e_H, e_p, e_x, e_y, e_t, e_T)$$

$$\partial(e_{c_3}) - \partial(e_{c_4}) = (e_x - e_t)\partial e_{c_1} + (e_y - e_z)\partial e_{c_2}$$

- with the monomial order induced by

$$x < y < z < t < H < p$$

we get the reduced G.b $\{\partial e_{c_1}, \partial e_{c_2}, \partial e_{c_3}, \partial e_{c_4}\}$

- with the monomial order induced by

$$H < p < x < y < z < t$$

we get the reduced G.b $\{\partial e_{c_1}, \partial e_{c_2}, \partial e_{c_3}\}$

and $|G.b| = 3$ is the min. possible cardinal

$$\dim \left(I(A) / \Lambda^+ I(A) \right)_n^q = \begin{cases} 1, & q = 4 \\ 0, & q > 4 \\ \dim (I(A))^q, & q < 4 \end{cases}$$

Cor. The graded ab. gp. $I(M)/_{\Lambda^+ I(M)}$ is tors. free, in any deg q.

Thm: The decomp OS algebra $A_+^*(M)$ is torsion free, in any deg. q

$$0 \to \left(I(M)/_{\Lambda^+ I(M)} \right)^* \to A_+^*(M) \to A^*(M) \to 0.$$

Cor If A is hs & not ss and p is minimal with the property that $\pi_p(X(A)) \neq 0$, then

$gr_I^1 \pi_p(X(A))$ is torsion free.