An operadic approach to the homology of Hurwitz spaces

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<u>Goal</u>: understand $H_*^{sing}(\mathcal{M}(\mathbb{C}), \mathbb{Q})$, where $\mathcal{M}$ is a moduli scheme parameterizing some arithmetically interesting objects (living over a curve)

<u>Why</u>: Study and enumerate $\# \mathcal{M}(\mathbb{F}_q)$

Can approach via the Grothendieck-Lefschetz trace formula:

$$\# \mathcal{M}(\mathbb{F}_q) = \sum_{i=0}^{2\dim \mathcal{M}} (-1)^i \, tr(\mathrm{Frob}_q \curvearrowright \underbrace{H_{et}^i(\mathcal{M}_{\overline{\mathbb{F}_q}})}_{\cong H^*_{sing}(\mathcal{M}(\mathbb{C}))})$$

(Frob_q: generator of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$)

Specifically

$\mathcal{M} = \mathrm{Hur}^c_{G,n}$: moduli of G-Galois branched covers of $\mathbb{A}^1$ w/:

- n branch points over $\overline{k}$
- monodromy @ ∞ to lie in $c \subseteq G$ (closed under conj.)
- trivialization of the fibre over a rational tangential pt.

A rational tangential point of X/k is a map

$$\begin{array}{ccc} \text{generic pt} \quad \operatorname{Spec}(k((t))) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{two points } (t=0) \quad \operatorname{Spec}(k[[t]]) & \longrightarrow & \overline{X} \end{array}$$

(doesn't need to occur)

so: think of a rational tangential point as defining a point on a compactification of X + a tangent vector pointing into X.

Fact: $\mathrm{Hur}^{c}_{G,n}(\mathbb{C})$ (w/ analytic topology) is the covering space of $\mathrm{Conf}_n(\mathbb{C})$ w/ fibre given by

$$c^{\times n} \subseteq G^{\times n} = \mathrm{Hom}(\pi_1(\underbrace{\mathbb{A}^1_{\mathbb{C}} \setminus n\text{ pts}}_{= D^2 \setminus n}), G)$$

w/ action of $\pi_1 \mathrm{Conf}_n \mathbb{C} = B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 1,\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$

$$\sigma_i (g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, g_{i+1}, g_i^{g_{i+1}}, g_{i+2}, \dots, g_n)$$

(here $g^h = h^{-1} g h$)

Computing $H_*(\mathrm{Hur}^c_{G,n}, k)$

$$H_*(\mathrm{Hur}^c_{G,n}, k) = H_*(\mathrm{Conf}_n, kc^{\times n})$$
$$= H^{gp}_*(B_n, kc^{\times n})$$

← local system of fibre (G: finite)

Hurwitz action:

$$1 \to N \xrightarrow[\ker]{} B_n \longrightarrow \mathrm{Sym}(c^{\times n})$$

Notice: N: finite index, normal.

$Q = B_n / N$ finite gp

Assume: char $k = 0$

$$H^{gp}_*(B_n, kc^{\times n}) = \left(H^{gp}_*(N, kc^{\times n})\right)_Q$$
$$= \left(H^{gp}_*(N, k) \otimes kc^{\times n}\right)_Q$$
$$= H^{gp}_*(N, k) \otimes_Q (kc^{\times n})$$
$$= H^{gp}_*(N, k) \otimes_{B_n} (kc^{\times n})$$
$$= H^{gp}_*(N, k) \otimes_{B_n} V^{\otimes n}$$

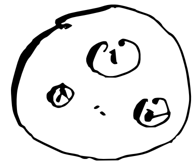
Set: $V = kc$, $V^{\otimes n} = kc^{\times n}$

So: enough to understand

- $H^{gp}_*(N, k)$, N: ranges over finite index, normal as a B_n-rep
- $V^{\otimes n}$ as a B_n-rep
- "Littlewood-Richardson" for B_n

Notice: replace N w/ finite index s.g. ⇒ same

Operads

Def'n: $E_2(n) = \Big\{$  $\Big\} = \Big\{$ n non-overlapping little disks in unit disk $\Big\}$

disk → center $\Big\downarrow \simeq$

$P\mathrm{Conf}_n \mathbb{C}$

is the 2D little disks operad.

Def'n: $E_2^{fin}(n)$ = inverse system of finite sheeted characteristic covers of $E_2(n)$.

in pro(Top)

Notice: $N \leq B_n$ is π_1 (term in tower)

Goal rephrased: compute $H_* E_2^{fin}(n)$ as a B_n-rep.

Def'n: A braided operad $\mathcal{O} = \{\mathcal{O}(n)\}$ in a symm. mon. cat $(\mathcal{D}, \otimes)$

- B_n action on $\mathcal{O}(n)$
- maps $\mathcal{O}(n) \otimes \mathcal{O}(m) \xrightarrow{\circ_i} \mathcal{O}(n+m-1)$
- axioms.

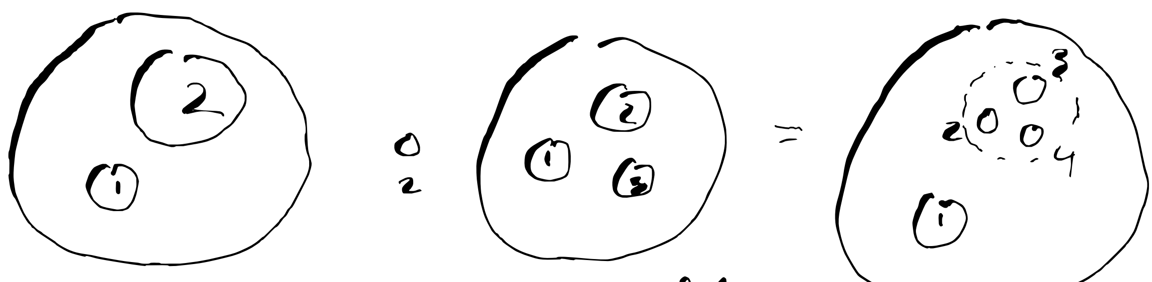
ex: If $(\mathcal{C}, \otimes)$ is a braided mon. cat (tensored over D, enriched), then the endomorphism operad of an object X in $\mathcal{C}$ is

$$\mathrm{End}_X(n) := \mathrm{Hom}_{\mathcal{C}}(X^{\otimes n}, X)$$

- B_n-action: via structure of cat.

$$(f \circ_i g)(x_1, \dots, x_{m+n-1}) = f(x_1, \dots, x_{i-1}, g(x_i, \dots, x_{i+n-1}), x_{i+n}, \dots, x_{m+n-1})$$

ex: $E_2 = \{E_2(n)\}$ is a symmetric operad (B_n-action factors through S_n)



Prop: Structure maps of E_2 lift to a braided operad structure on E_2^{fm}. (pro-Top)

Goal (take 3): compute $H_* E_2^{fin}$ as a braided operad in pro(Vect$_k$)

An algebra A for an operad $\mathcal{O}$ $\longleftrightarrow$ is a morphism ~~of operads~~ $\mathcal{O} \to End_A$

$$\longleftrightarrow \quad \mathcal{O}(n) \otimes_{B_n} (A^{\otimes n}) \longrightarrow A$$

If V is an obj. in a br. mon. cat $(\mathcal{C}, \otimes)$, $\mathcal{O}$ is a braided operad, the free $\mathcal{O}$-algebra on V is

Yetter-Drinfeld modules ↓

$$\mathcal{O}[V] = \bigoplus_n \mathcal{O}(n) \otimes_{B_n} V^{\otimes n}$$

Then: If $V = kC$: this is an object in $\mathcal{YD}_G^G$

Can form $(H_* E_2^{fin})[V] = \bigoplus_n H_*^e(HUr_{G,n}, k)$

Symmetric setting:

There are symmetric operads Com, Lie, Assoc such that

- $\mathrm{Com}[V] = \mathrm{Sym}[V]$
- $\mathrm{Assoc}[V] = T(V)$
- $\mathrm{Lie}[V] = \mathrm{Lie}(V) \subseteq T(V)$

$$\mathrm{Lie}(V) = P(T(V))$$

Thm (F. Cohen)

$$(H_* E_2)[V] = \mathrm{Com}(\Sigma^{-1}\mathrm{Lie}(\Sigma V))$$

Braided Hopf algebras

A braided Hopf algebra is a Hopf algebra object in a braided monoidal category

- unital, assoc. alg.
- counital, coassoc. coalg
- antipode
- $\Delta: A \to A \otimes A$ is an alg. map

mult on $A \otimes A$:

$$(A \otimes A) \otimes (A \otimes A) \xrightarrow{1 \otimes \sigma \otimes 1} (A \otimes A) \otimes (A \otimes A) \xrightarrow{\mu \otimes \mu} A \otimes A$$

Classical fact: If A is a Hopf alg. in $Vect_k$, then the primitives

$$P(A) = \{x \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

form a Lie-subalgebra of A w/ $[x,y] = xy - yx$ $\;xy - \sigma(x,y)$

ex: If V is a v.s., $T(V)$ is a Hopf alg. where $V \subseteq P(T(V))$. In fact

$$P(T(V)) = Lie[V]$$

why (Milnor-Moore): $T(V) = \mathcal{U}(P(T(V)))$

compare universal properties

Problem: If is a braided H.A., then $P(A)$ need not be a Lie algebra.

Question: what structure does $P(A)$ support?

Thm (W.)

There are braided operads (in pro-Vect) BrCom, BrPrim such that:

1. Braided comm. algebras ($x \cdot y = \mu \circ (x \otimes y)$) are the same as BrCom-alg.

Free: $BrCom[V] = \bigoplus_n (V^{\otimes n})_{B_n} =: Sym_{br}(V)$

2. If A is a braided Hopf alg. $P(A)$ are a BrPrim-algebra.

Free: $BrPrim[V] \cong P(T(V))$

3. $\exists$ isomorphisms of operads

$H_0 E_2^{fin} \cong BrCom$ $\qquad$ $H_{top} E_2^{fin} \cong BrPrim$

$H_{n-1} E_2^{fin}(n) = BrPrim(n)$.

Hope: (false) that this is everything:

$(H_* E_2^{fin})[V] = BrCom(BrPrim[V])$

$\exists$ map, not an iso.

Galois theory

Hence: $\pi_0 h\,\mathrm{Aut}(\hat{E}_2) \cong \widehat{GT} \hookleftarrow \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

Reflects: spaces $E_2(n) \simeq \mathrm{PConf}_n$ are alg. varieties $/\mathbb{Q}$

$\rightsquigarrow$ $\acute{e}t(\mathrm{PConf}_n)$ carries a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action.

characteristic

$E_2^{fin}(n)$: tower of finite-sheeted covers of $E_2(n)$

Then $\widehat{GT}$ acts on $H_*(E_2^{fin}(n))$

Thm (W., in progress)

$\widehat{GT}$ acts on $H_*(E_2^{fin}(n))$ through (skew) operad maps. This action is faithful, even faithful for $\mathrm{BrPrim}(3) \cong H_2 E^{fin}(3)$

faithful: Belyi: $H_2 E^{fin}(3) \cong k[\hat{F}_2]$

There is a "Lie bracket" in BrPrim(2):

$$\{x,y\} = \sum_{i=0}^{n-1} (-1)^i \mu \circ \sigma^i (x \otimes y)$$

where order of σ on $x \otimes y$ is n

$H_1 E_2^{fin}(2) = BrPrim(2) = k(\{x,y\})$ is the sign rep of B_2

Fact: Q(BrPrim)(n) is a nonzero, pro-cyclic B_n-rep. for every n.

Koszul duality

Can make sense of a bar construction for braided operads.

- BO is a braided co-operad
- $DO = (BO)^*$ is a divided power braided operad

$(D\,Br\text{-}Com)[V] \cong P(T(V))$
fixed pt version

BrCom is Koszul:

$H_* B(BrCom)(n) = \begin{cases} 0 & * \neq n-1 \\ \end{cases}$

$H_{n-1} B(BrCom)(n) =$ dualizing module for B_n.