Motion groupoids & mapping class groupoids

arXiv:2103.10377, with Paul Martin, João Faria Martins

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Brief overview

(I) Construction of the motion groupoid $\mathrm{Mot}_{\underline{M}}$ of a pair $\underline{M} = (M, A)$. Morphisms are equivalence classes of continuous flows of ambient space M which fix A, acting on $\mathcal{P}M$. Recover classical definition of the motion group associated to a manifold M and a submanifold $N \in \mathcal{P}M$, by looking at the morphism group at N. Obtain groups isomorphic to braid groups, loop braid groups.

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(II) Construction of mapping class groupoid $\mathrm{MCG}_{\underline{M}}$.
Morphisms are now equivalence classes of homeomorphisms of M, fixing A. The object set is again $\mathcal{P}M$. Again obtain groups isomorphic to braid groups, loop braid groups.

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(III) Construction of functor $\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$.
We prove that this is an isomorphism when π_0 and π_1 of space of homeomorphisms of M fixing A are trivial (With compact open topology). E.g. $\underline{M} = ([0,1]^n, \partial[0,1]^n)$.

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- Morphisms which do not start and end in the same configuration allowed.
- Expect interesting new algebraic structures

Motion Groupoid

Space self-homeomorphisms of a manifold M

Let **Top** denote the category of topological spaces and continuous maps.

$\mathbf{Top}(X,X)$	Set of continuous maps from X to X
$\mathbf{Top}^h(X,X)$	Subset of $\mathbf{Top}(X,X)$ of self-homeomorphisms. Note this is a group.
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Lemma

(Hatcher) Let X be a compact space and Y a metric topological space with metric d. Then
(i) the function

$$d'(f, g) := \sup_{x \in X} d(f(x), g(x))$$

is a metric on $\mathbf{Top}(X, Y)$; and
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$\mathbf{Top}^h_A(M, M), \mathbf{TOP}^h_A(M, M)$ versions with subset $A \subset M$ fixed pointwise

Definition
Fix a manifold, submanifold pair $\underline{M} = (M, A)$. A flow in $\underline{M}$ is a map $f \in \mathsf{Top}(\mathbb{I}, \mathsf{TOP}^h_A(M, M))$ with $f_0 = \mathrm{id}_M$. Define,

$$\mathrm{Flow}_{\underline{M}} = \{f \in \mathsf{Top}(\mathbb{I}, \mathsf{TOP}^h_A(M, M)) \mid f_0 = \mathrm{id}_M\}.$$

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For any manifold M the path $f_t = \mathrm{id}_M$ for all t, is a flow. We will denote this flow Id_M.

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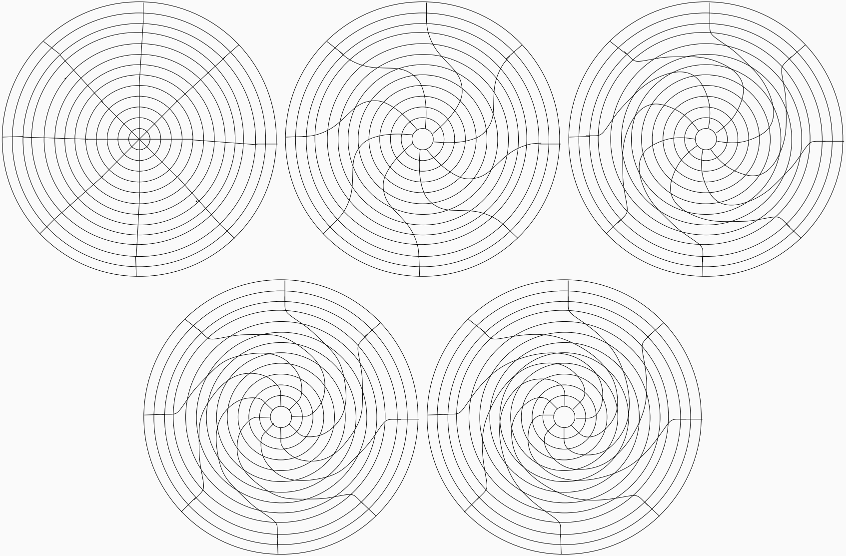
Example
For $M = S^1$ (the unit circle) we may parameterise by $\theta \in \mathbb{R}/2\pi$ in the usual way. Consider the functions $\tau_\phi : S^1 \to S^1$ ($\phi \in \mathbb{R}$) given by $\theta \mapsto \theta + \phi$, and note that these are homeomorphisms. Then consider the path $f_t = \tau_{t\pi}$ ('half-twist'). This is a flow.

Example $M = D^2$



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Lemma
Let M be a manifold. There exists a set map

$$\begin{aligned} \bar{\ } : \mathrm{Flow}_{\underline{M}} &\to \mathrm{Flow}_{\underline{M}} \\ f &\mapsto \bar{f} \end{aligned}$$

with

$$\bar{f}_t = f_{(1-t)} \circ f_1^{-1}. \tag{1}$$

Proposition
Let M be a manifold. There exists a composition

$$\begin{aligned} *\colon \mathrm{Flow}_{\underline{M}} \times \mathrm{Flow}_{\underline{M}} &\to \mathrm{Flow}_{\underline{M}} \\ (f, g) &\mapsto g * f \end{aligned}$$

where

$$(g * f)_t = \begin{cases} f_{2t} & 0 \leq t \leq 1/2, \\ g_{2(t-1/2)} \circ f_1 & 1/2 \leq t \leq 1. \end{cases} \tag{2}$$

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For a pair $\underline{M} = (M, A)$, $(\mathrm{Flow}_{\underline{M}}, *)$ is a magma.

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Lemma
For $f, g \in \mathrm{Flow}_{\underline{M}}$, $f^{-1} \overset{p}{\sim} \bar{f}$ and $g \cdot f \overset{p}{\sim} g * f$.

Motions

Definition
Fix a $\underline{M} = (M, A)$. A motion in M is a triple $(f, N, f_1(N))$ consisting of a flow $f \in \mathrm{Flow}_{\underline{M}}$, a subset $N \subseteq M$ and the image of N at the endpoint of f, $f_1(N)$.

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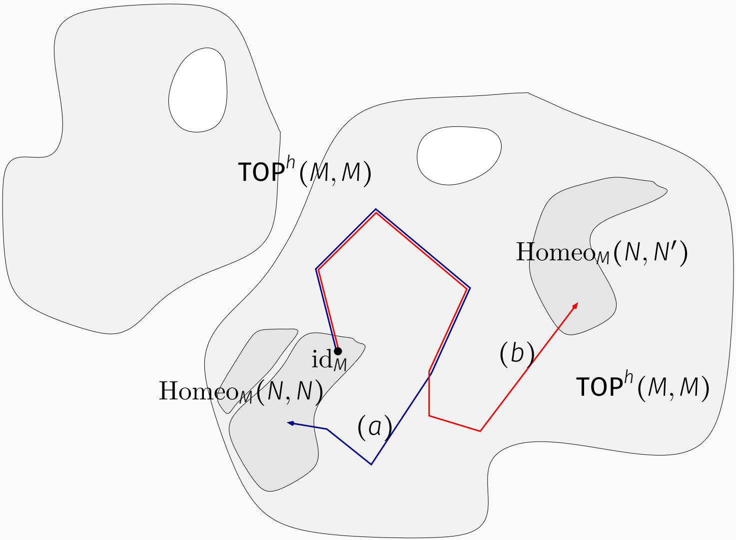
$$Mt_M(N, N') = \{\text{motions } f\colon N \rightarrowtail N'\}$$



$\mathsf{TOP}^h(M, M)$
$\mathrm{Homeo}_M(N, N')$
id_M
(b)
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Motions

For any $N \subset M$, $Id_M\colon N \rightsquigarrow N$ is a motion. Let $f\colon N \rightsquigarrow N'$ and $g\colon N' \rightsquigarrow N''$ be motions in M, then $g \cdot f\colon N \rightsquigarrow N''$ $\left((g \cdot f)_t = g_t \circ f_t\right)$ is a motion.

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Lemma
There is a group action of $(\mathrm{Flow}_{\underline{M}}, \cdot)$ on $\mathcal{P}M$, thus obtain an action groupoid

$$\mathrm{Mt}^{\cdot}_{\underline{M}} = (\mathcal{P}M, \mathrm{Mt}_{\underline{M}}(N, N'), \cdot, \mathrm{Id}_M, f^{-1}).$$

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$$\mathrm{Mt}^{*}_{\underline{M}} = (\mathcal{P}M, \mathrm{Mt}_{\underline{M}}(N, N'), *).$$

Definition

Let M be a manifold and $N, N' \subset M$. Let $\mathrm{Mt}^{hom}_{\underline{M}}(N, N') \subset \mathbf{Top}^h(M \times \mathbb{I}, M \times \mathbb{I})$ denote the subset of homeomorphisms $g \in \mathbf{Top}^h(M \times \mathbb{I}, M \times \mathbb{I})$ such that

(I) $g(m, 0) = (m, 0)$ for all $m \in M$,

(II) $g(M \times \{t\}) = M \times \{t\}$ for all $t \in \mathbb{I}$, and

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Theorem

Let M be a manifold and $N, N' \subset M$. There is a bijection

$$\begin{aligned} \Theta \colon \mathrm{Mt}_{\underline{M}}(N, N') &\to \mathrm{Mt}^{hom}_{\underline{M}}(N, N'), \\ f &\mapsto ((m, t) \mapsto (f_t(m), t)). \end{aligned}$$

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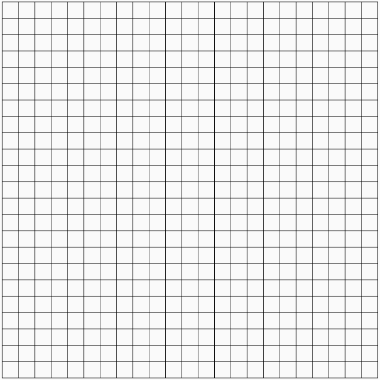
Idea of proof
(e.g. Hatcher) As M is locally compact, Hausdorff, there is a bijection

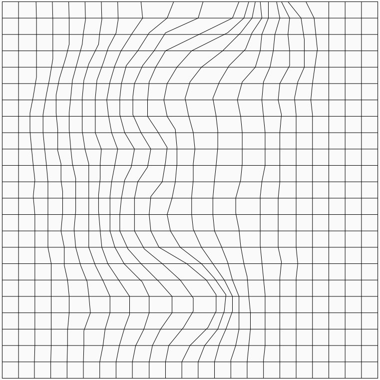
$$\Phi\colon \mathbf{Top}(\mathbb{I}, \mathbf{TOP}(M, M)) \to \mathbf{Top}(M \times \mathbb{I}, M).$$

(Coming from an adjunction between the product functor $M \times -$ and the hom functor $\mathbf{TOP}(M, -)$). It follows that the image is continuous. To show that the image is a homeomorphism we need that $\mathbf{TOP}^h(M, M)$ is a topological group.

$M = \mathbb{I}$



$\mapsto$



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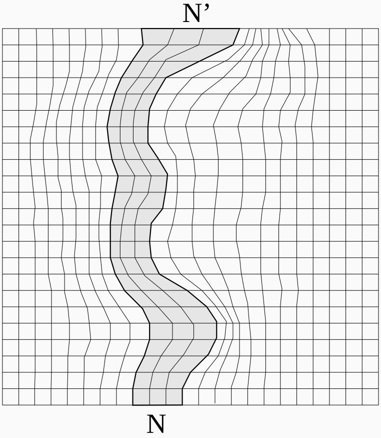
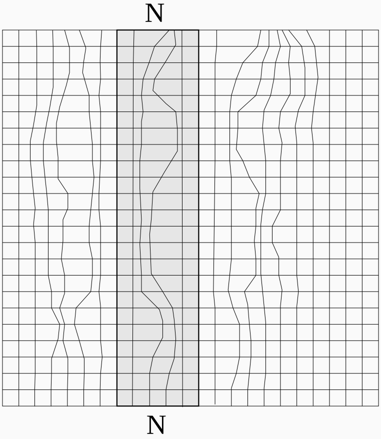
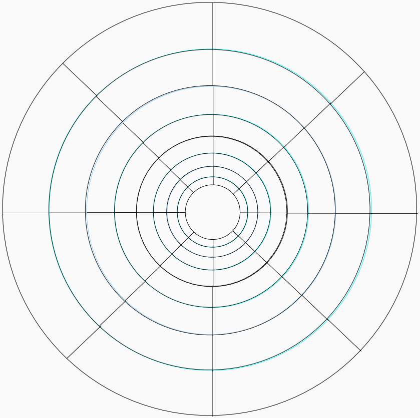




$M = S^1$



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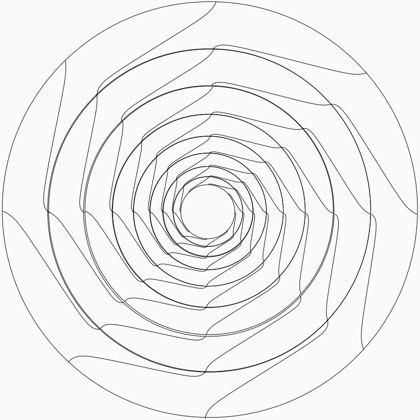
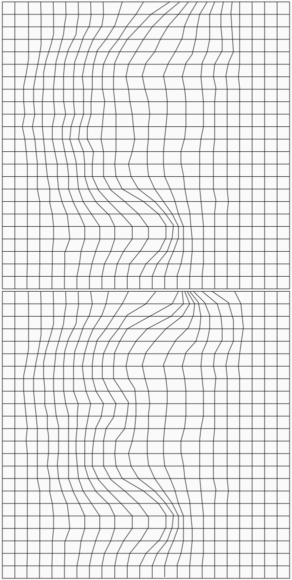
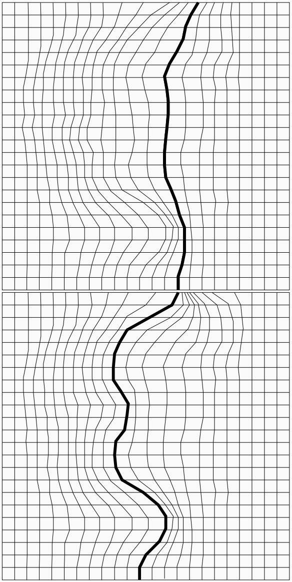


$\ast$ COMPOSITION WHEN $M = \mathbb{I}$





Definition
Let $\underline{M} = (M, A)$ be a manifold, subset pair and $N \subset M$ a subset. A motion $f\colon N \rightharpoondown N$ in $\underline{M}$ is said to be <u>N-stationary</u> if $f_t(N) = N$ for all $t \in \mathbb{I}$. Define

$$\mathrm{SetStat}_{\underline{M}}^{N} = \left\{ f\colon N \rightharpoondown N \in \mathrm{Mt}_{\underline{M}}(N, N) \;\middle|\; f_t(N) = N \text{ for all } t \in \mathbb{I} \right\}.$$

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Example
Let $M = D^2$ and let $\tau_{2\pi}$ denote a flow such that $(\tau_{2\pi})_t$ is a $2\pi t$ rotation of the disk. Now let N be a circle centred on the centre of the disk. Then $\tau_{2\pi} \colon N \rightarrowtail N$ is N-stationary.

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Example
Let $M = D^2$, the 2-disk and let $N \subset M$ be a finite set of points. Then a motion $f\colon N \rightsquigarrow N$ is N-stationary if and only if $f_t(x) = x$ for all $x \in N$ and $t \in \mathbb{I}$. More generally this holds if N is a totally disconnected subspace of M, e.g. $\mathbb{Q}$ in $\mathbb{R}$.

Congruence by set-stationary motions

Lemma
For $N, N' \subset M$, denote by $\overset{m}{\sim}$ the relation

$$f\colon N \rightarrowtail N' \overset{m}{\sim} g\colon N \rightarrowtail N' \quad \text{if} \quad \overline{g} * f \in [\mathrm{SetStat}_{\underline{M}}^{N}]_{\mathrm{p}}$$

on $\mathrm{Mt}_{\underline{M}}(N, N')$. This is an equivalence relation.
We call this motion-equivalence and denote by $[f\colon N \rightarrowtail N']_{\mathrm{m}}$ the motion-equivalence class of $f\colon N \rightarrowtail N'$.

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Idea of proof

Quotient first by path-homotopy. Then classes which intersect $\mathrm{SetStat}_{\underline{M}}^{N}(N, N)$ form a totally disconnected normal subgroupoid. Can be proved in general that for any totally disconnected, normal subgroupoid $\mathcal{H}$ of a groupoid $\mathcal{G}$ there is a congruence given by the relation $g_1 \sim g_2$ if $g_2^{-1} *_{\mathcal{G}} g_1 \in \mathcal{H}$. This leads to an equivalent relation to the given relation.

Theorem
Let $\underline{M} = (M, A)$ where M is a manifold and $A \subset M$ a subset. There is a groupoid

$$\mathrm{Mot}_{\underline{M}} = (\mathcal{P}M,\ \mathrm{Mt}_{\underline{M}}(N, N')/\overset{m}{\sim}, *, [\mathrm{Id}_M]_{\mathrm{m}},\ [f]_{\mathrm{m}} \mapsto [\bar{f}]_{\mathrm{m}})$$

where

(I) objects are subsets of M;

(II) morphisms between subsets N, N' are motion-equivalence classes $[f\colon N \rightarrowtail N']_{\mathrm{m}}$ of motions;

(III) composition of morphisms is given by

$$[g\colon N' \rightarrowtail N'']_{\mathrm{m}} * [f\colon N \rightarrowtail N']_{\mathrm{m}} = [g * f\colon N \rightarrowtail N'']_{\mathrm{m}}.$$

(IV) the identity at each object N is the motion-equivalence class of $\mathrm{Id}_M\colon N \rightarrowtail N$, $(\mathrm{Id}_M)_t(m) = m$ for all $m \in M$;

(V) the inverse for each morphism $[f\colon N \rightarrowtail N']_{\mathrm{m}}$ is the motion-equivalence class of $\bar{f}\colon N' \rightarrowtail N$ where $\bar{f}_t = f_{(1-t)} \circ f_1^{-1}$. $\square$

Proposition
Let $\underline{M} = (M, A)$ where M is a manifold and $A \subset M$ a subset, then

$$\mathrm{Mot}_{\underline{M}} = (\mathcal{P}M,\ \mathrm{Mt}_{\underline{M}}(N, N')/\overset{m}{\sim}, \cdot, [\mathrm{Id}_M]_m,\ [f]_m \mapsto [\bar{f}^{-1}]_m).$$

Proof
It is sufficient to observe that motions which are path equivalent are motion equivalent. Let g, f be flows satisfying $f \overset{p}{\sim} g$, then $\bar{g} * f \overset{p}{\sim} g^{-1} \cdot f \overset{p}{\sim} g^{-1} \cdot g$, using that $\bar{g} \overset{p}{\sim} g^{-1}$, and $g * f \overset{p}{\sim} g \cdot f$. Then for all $t \in \mathbb{I}$, $(g^{-1} \cdot g)_t(N) = N$, hence it is stationary.

Suppose $N \subset \mathbb{I} \setminus \{0,1\}$ is a compact subset with a finite number of connected components i.e. N is a union of points and closed intervals.

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We can assign a word in $\{a,b\}$ to N by representing each point in N by a and each interval by b, ordered in the obvious way using the natural ordering on $\mathbb{I}$.
Let $N' \subset \mathbb{I} \setminus \{0,1\}$ be another subset defined in the same way. If the word assigned to N and N' is the same, $|\mathrm{Mot}_{\mathbb{I}}(N,N')| = 1$. Otherwise $\mathrm{Mot}_{\mathbb{I}}(N,N') = \varnothing$.

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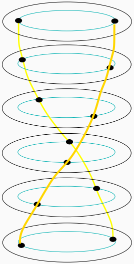
Let $N = \mathbb{I} \cap \mathbb{Q}$, then $\mathrm{Mot}_{\mathbb{I}}(N,N)$ is uncountably infinite.

Theorem (T., Faria Martins, Martin)
Let n be a positive integer. Consider $M = D^2$. Given any finite subset K, with n elements, in the interior of D^2, then $\mathrm{Mot}_{D^2}(K, K)$ is isomorphic to the braid group in n strands (as in Artin, Theory of Braids). In particular the image of the class of a motion which moves points as below is an elementary braid on two strands.



Also if $\underline{D^3} = (D^3, \partial D^3)$ and $L \subset D^3$ is an unlink in the interior with n components, then $\mathrm{Mot}_{\underline{D^3}}(L, L)$ is isomorphic to the extended loop braid group (as in Damiani, a journey through loop braid groups).

Lemma
Let (M, A) and (M', A') be pairs such that there exists a homeomorphism $\psi\colon M \to M'$ satisfying $\psi(A) = A'$. Then there is a isomorphism of categories

$$\Psi\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{Mot}_{\underline{M'}}$$

defined as follows. On objects $N \subset M$, $\Psi(N) = \psi(N)$. For a motion $f\colon N \rightharpoondown N'$ in M, let $(\psi \circ f \circ \psi^{-1})_t = \psi \circ f_t \circ \psi^{-1}$. Then Ψ sends the equivalence class $[f\colon N \rightharpoondown N']_m$ to the equivalence class $[\psi \circ f \circ \psi^{-1}\colon \psi(N) \to \psi(N')]_m$.

Proposition
For any pair (M, A) and subset $N \subseteq M$ there is an involutive endofunctor on $\mathrm{Mot}_{\underline{M}}$ defined by

$$\mathrm{Mot}_{\underline{M}}(N, N) \cong \mathrm{Mot}_{\underline{M}}(M \setminus N, M \setminus N),$$
$$f\colon N \rightarrowtail N' \mapsto f\colon M \setminus N \rightarrowtail M \setminus N'.$$

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Notice that generally these automorphism groups are not connected *in* the motion groupoid - this would imply N homeomorphic to $M \setminus N$.

Alternative equivalence relations on the motion groupoid

Worldlines of motions

Definition
The worldline of a motion $f\colon N \rightarrowtail N'$ in a manifold M is

$$\mathrm{W}\,(f\colon N \rightarrowtail N') := \bigcup_{t\in[0,1]} f_t(N) \times \{t\} \subseteq M \times \mathbb{I}.$$

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The worldline of a motion $f\colon N \rightsquigarrow N'$ in a manifold M is

$$\mathsf{W}\,(f\colon N \rightsquigarrow N') := \bigcup_{t\in[0,1]} f_t(N) \times \{t\} \subseteq M \times \mathbb{I}.$$

Proposition
Let $f, g\colon N \rightsquigarrow N'$ be motions with the same worldline, so we have

$$\mathsf{W}(f\colon N \rightsquigarrow N') = \mathsf{W}(g\colon N \rightsquigarrow N').$$

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Proof
For all $t \in \mathbb{I}$, $(g^{-1} \cdot f)_t(N) = g_t^{-1} \circ g_t(N) = N$. Thus $g^{-1} \cdot f$ is N-stationary, and hence $\bar{g} * f$ path-homotopic to a stationary motion.

Theorem (T., Faria Martins, Martin)
Let $\underline{M} = (M, A)$ where M is a manifold and $A \subset M$ a subset. Two motions $f, f' \colon N \rightsquigarrow N'$ in $\mathrm{Mt}_{\underline{M}}$ are motion equivalent if, and only if, their worldlines are level preserving ambient isotopic, relative to $(M \times (\{0,1\})) \cup (A \times \mathbb{I})$, pointwise.

Let M be a manifold and $A \subseteq M$ a subset.

Lemma

There is a (left) group action

$$\begin{aligned} \sigma^A \colon \mathsf{Top}_A^h(M, M) \times \mathcal{P}M &\to \mathcal{P}M \\ (\mathfrak{f}, N) &\mapsto \mathfrak{f}(N). \end{aligned}$$

Let M be a manifold and $A \subseteq M$ a subset.

Proposition

There is an action groupoid $\mathrm{Homeo}_{\underline{M}}$ with objects $\mathcal{P}M$ and morphisms
Explicitly the morphisms in $\mathrm{Homeo}_{\underline{M}}(N, N')$ are triples $(\mathfrak{f}, N, \mathfrak{f}(N))$ where

- $\mathfrak{f}$ is a homeomorphism $M \to M$,
- $\mathfrak{f}(N) = N'$,
- $\mathfrak{f}$ fixes A pointwise.

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We will denote triples $(\mathfrak{f}, N, \mathfrak{f}(N)) \in \mathrm{Homeo}_{\underline{M}}(N, N')$ as $\mathfrak{f}\colon N \rightarrowtail N'$.
Identity: $\mathrm{id}_M\colon N \rightarrowtail N$ Inverse: $\mathfrak{f}\colon N \rightarrowtail N' \mapsto \mathfrak{f}^{-1}\colon N' \rightarrowtail N$.

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Identity: $\mathrm{id}_M\colon N \rightarrowtail N$ Inverse: $\mathfrak{f}\colon N \rightarrowtail N' \mapsto \mathfrak{f}^{-1}\colon N' \rightarrowtail N$.
We will also sometimes consider $\mathrm{Homeo}_{\underline{M}}(N, N')$ as the projection to the first element of the triple. Then can equip morphism sets with a topology and $\mathsf{TOP}^h_A(M, M) = \mathrm{Homeo}_{\underline{M}}(\varnothing, \varnothing) = \mathrm{Homeo}_{\underline{M}}(M, M)$ and every $\mathrm{Homeo}_{\underline{M}}(N, N') \subseteq \mathsf{TOP}^h_A(M, M)$. Notice each self-homeomorphism $\mathfrak{f}$ of M will belong to many such $\mathrm{Homeo}_{\underline{M}}(N, N')$.

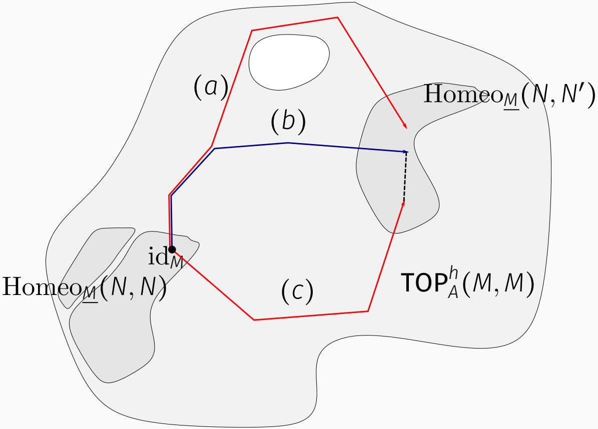
Definition
Fix a pair (M, A). Define a relation on $\mathrm{Mt}_{\underline{M}}(N, N')$ as follows. Let $f\colon N \rightarrowtail N' \overset{rp}{\sim} g\colon N \rightarrowtail N'$ if the motions $f\colon N \rightarrowtail N'$ and $g\colon N \rightarrowtail N'$ are relative path-homotopic. This means there exists a continuous map

$$H\colon \mathbb{I} \times \mathbb{I} \to \mathbf{TOP}^h(M, M)$$

such that

- for any fixed $s \in \mathbb{I}$, $t \mapsto H(t, s)$ is a motion from N to N',
- for all $t \in \mathbb{I}$, $H(t, 0) = f_t$, and
- for all $t \in \mathbb{I}$, $H(t, 1) = g_t$.

We call such a homotopy a relative path-homotopy.


(a)
(b)
$\mathrm{Homeo}_{\underline{M}}(N, N')$
id_M
$\mathrm{Homeo}_{\underline{M}}(N, N)$
(c)
$\mathsf{TOP}^h_A(M, M)$

Theorem (T. ,Faria Martins, Martin)
For a pair $\underline{M} = (M, A)$ and a motion $f\colon N \rightarrowtail N'$ in $\underline{M}$ we have

$$[f\colon N \rightarrowtail N']_{\mathrm{rp}} = [f\colon N \rightarrowtail N']_{\mathrm{m}}.$$

Key ingredients of proof
Direct construction of appropriate homotopies. Uses normality of stationary motions.

Theorem (T. ,Faria Martins, Martin)
For a pair $\underline{M} = (M, A)$ and a motion $f\colon N \rightsquigarrow N'$ in $\underline{M}$ we have

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Key ingredients of proof
Direct construction of appropriate homotopies. Uses normality of stationary motions.

Relative path equivalence is precisely the equivalence relation in the relative fundamental group, hence

$$\mathrm{Mot}_{\underline{M}}(N, N) = \pi_1(\mathrm{Homeo}_{\underline{M}}(\varnothing, \varnothing), \mathrm{Homeo}_{\underline{M}}(N, N), \mathrm{id}_M)$$

We will need this later!

Mapping class groupoids

Recall that for a pair $\underline{M} = (M, A)$ and for subsets $N, N' \subset M$, morphisms in $\mathrm{Homeo}_{\underline{M}}(N, N')$ are triples denoted $\mathfrak{f} \colon N \rightharpoondown N'$ where $\mathfrak{f} \in \mathsf{Top}^h(M, M)$ and $\mathfrak{f}(N) = N'$. We also think of the elements of $\mathrm{Homeo}_{\underline{M}}(N, N')$ as the projection to the first coordinate of each triple i.e. $\mathfrak{f} \in \mathsf{Top}^h_A(M, M)$ such that $\mathfrak{f}(N) = N'$.

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Definition

Let $N, N' \subset M$. For any $\mathfrak{f}\colon N \frown N'$ and $\mathfrak{g}\colon N \frown N'$ in $\mathrm{Homeo}_{\underline{M}}(N, N')$, $\mathfrak{f}\colon N \frown N'$ is said to be isotopic to $\mathfrak{g}\colon N \frown N'$, denoted by $\overset{i}{\sim}$, if there exists a continuous map

$$H\colon M \times \mathbb{I} \to M$$

such that

- for all fixed $s \in \mathbb{I}$, the map $m \mapsto H(m, s)$ is in $\mathrm{Homeo}_{\underline{M}}(N, N')$,
- for all $m \in M$, $H(m, 0) = \mathfrak{f}(m)$, and
- for all $m \in M$, $H(m, 1) = \mathfrak{g}(m)$.

We call such a map an isotopy from $\mathfrak{f}\colon N \frown N'$ to $\mathfrak{g}\colon N \frown N'$.

Lemma

The family of relations $(\mathrm{Homeo}_{\underline{M}}(N, N'), \overset{i}{\sim})$ for all pairs $N, N' \subseteq M$ are a congruence on $\mathrm{Homeo}_{\underline{M}}$.

Theorem

Let $\underline{M} = (M, A)$ be a manifold submanifold pair. There is a groupoid

$$\mathrm{MCG}_{\underline{M}} = (\mathcal{P}M, \mathrm{Homeo}_{\underline{M}}(N, N')/\overset{i}{\sim}, \circ, [\mathrm{id}_M]_i, [\mathfrak{f}]_i \mapsto [\mathfrak{f}^{-1}]_i).$$

We call this the <u>mapping class groupoid of M</u>.

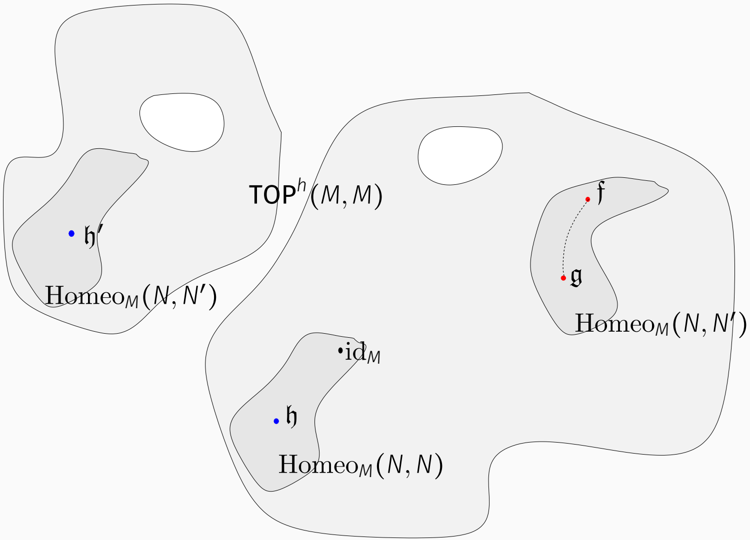
Using bijection

$$\Phi\colon \mathsf{Top}(\mathbb{I}, \mathsf{TOP}(M, M)) \to \mathsf{Top}(M \times \mathbb{I}, M),$$

a continuous map $M \times \mathbb{I} \to M$ which is an isotopy corresponds to a path $\mathbb{I} \to \mathrm{Homeo}_M(N, N')$ from $\mathfrak{f}$ to $\mathfrak{g}$. Hence

Lemma
Let M be a manifold. We have that as sets

$$\mathrm{MCG}_{\underline{M}}(N, N') = \pi_0(\mathrm{Homeo}_{\underline{M}}(N, N')).$$



$\mathrm{TOP}^h(M,M)$
$\mathfrak{h}'$
$\mathrm{Homeo}_M(N,N')$
$\mathfrak{f}$
$\mathfrak{g}$
$\mathrm{Homeo}_M(N,N')$
id_M
$\mathfrak{h}$
$\mathrm{Homeo}_M(N,N)$

Example
If $\underline{S^1} = (S^1, \varnothing)$, we have

$$\mathrm{MCG}_{\underline{S^1}}(\varnothing, \varnothing) = \mathbb{Z}/2\mathbb{Z}.$$

$\mathsf{TOP}^h(S^1, S^1)$ has two path-components, containing respectively the orientation preserving and the orientation reversing homeomorphisms from S^1 to itself. Each is homotopic to S^1 (Hamstrom). Therefore the homomorphism $\pi_0(\mathrm{Homeo}_{\underline{S^1}}(\varnothing, \varnothing)) \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ induced by the degree homomorphism $\deg\colon \mathsf{Top}^h(S^1, S^1) = \mathrm{Homeo}_{\underline{S^1}}(\varnothing, \varnothing) \to \{\pm 1\}$ is an isomorphism.

Proposition

Let $\underline{D^2} = (D^2, \partial D^2)$. The morphism group $\mathrm{MCG}_{\underline{D^2}}(\varnothing, \varnothing)$ is trivial.

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Proof
(This follows from the Alexander trick.) Suppose we have $\mathfrak{f} \colon \varnothing \rightsquigarrow \varnothing$ in $\underline{D^2}$. Define

$$f_t(x) = \begin{cases} t\,\mathfrak{f}(x/t) & 0 \le |x| \le t, \\ x & t \le |x| \le 1. \end{cases}$$

Notice that $f_0 = \mathrm{id}_{D^2}$ and $f_1 = \mathfrak{f}$ and each f_t is continuous. Moreover:

$$\begin{aligned} H \colon D^2 \times \mathbb{I} &\to D^2, \\ (x, t) &\mapsto f_t(x) \end{aligned}$$

is a continuous map. So we have constructed an isotopy from any boundary preserving self-homeomorphism of D^2 to id_{D^2}.

Functor from the motion groupoid to the mapping class groupoid

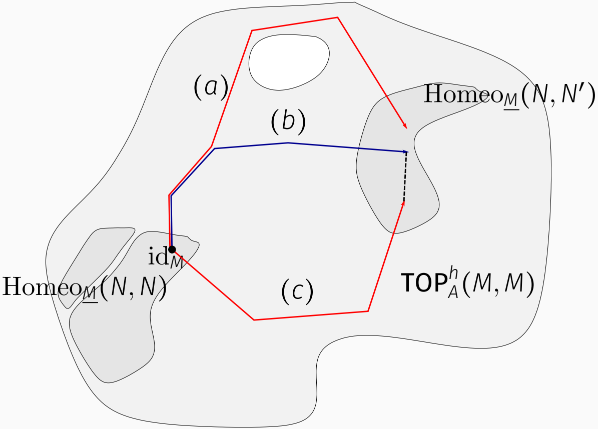
Theorem (T., Faria Martins, Martin)
Let $\underline{M} = (M, A)$. There is a functor

$$\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$$

which is the identity on objects and on morphisms we have

$$\mathsf{F}\left([f\colon N \rightsquigarrow N']_{\mathrm{m}}\right) = [f_1\colon N \rightsquigarrow N']_{\mathrm{i}}.$$

Well definedness of F

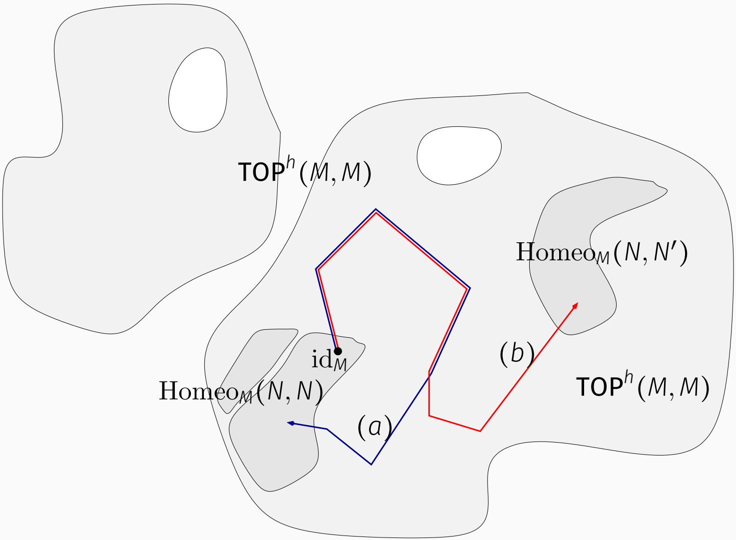


Lemma
The functor

$$\mathsf{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$$

is full if and only if $\pi_0(\mathsf{TOP}^h_A(M, M), \mathrm{id}_M)$ is trivial.

FUNCTOR $\mathrm{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$



Functor $\mathrm{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$

(Hatcher) Let X be a space, $Y \subset X$ a subspace and $x_0 \in Y$ a basepoint. There is a long exact sequence:

$$\ldots \to \pi_n(Y, \{x_0\}) \xrightarrow{i_*^n} \pi_n(X, \{x_0\}) \xrightarrow{j_*^n} \pi_n(X, Y, \{x_0\}) \xrightarrow{\partial^n} \pi_{n-1}(Y, \{x_0\}) \xrightarrow{i_*^{n-1}} \ldots \xrightarrow{i_*^0} \pi_0(X, \{x_0\}).$$

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Maps i and j are inclusions. Maps ∂ are restrictions to single face, in particular

$$\partial^1\colon \pi_1(X, A, \{x_0\}) \to \pi_0(A, \{x_0\}), \\ [\gamma]_{\mathrm{rp}} \mapsto [\gamma(1)]_{\mathrm{p}}.$$

Functor $\mathrm{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$

Recall $\mathrm{Mot}_{\underline{M}}(N,N) = \pi_1(\mathrm{Homeo}_M(\varnothing,\varnothing), \mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M)$ and $\mathrm{MCG}_{\underline{M}}(N,N) = \pi_0(\mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M)$.

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Lemma

Let $\underline{M} = (M, A)$ be a manifold, subset pair, and fix a subset $N \subset M$. Then we have a long exact sequence

$$\begin{aligned} \ldots \to \pi_n(\mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M) &\xrightarrow{i_*^n} \pi_n(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing), \mathrm{id}_M) \xrightarrow{j_*^n} \\ \pi_n(\mathrm{Homeo}_M(\varnothing,\varnothing), \mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M) &\xrightarrow{\partial^n} \pi_{n-1}(\mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M) \xrightarrow{i_*^{n-1}} \\ \ldots \xrightarrow{\partial^2} \pi_1(\mathrm{Homeo}_{\underline{M}}(N,N), \mathrm{id}_M) &\xrightarrow{i_*^1} \pi_1(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing), \mathrm{id}_M) \\ \xrightarrow{j_*^1} \mathrm{Mot}_{\underline{M}}(N,N) \xrightarrow{\mathrm{F}} \mathrm{MCG}_{\underline{M}}(N,N) &\xrightarrow{i_*^0} \pi_0(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing), \mathrm{id}_M) \end{aligned}$$

where all maps are group maps and F is the appropriate restriction of the functor $\mathrm{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}}$.

Lemma
Suppose

- $\pi_1(\mathrm{Homeo}_{\underline{M}}(\varnothing, \varnothing), \mathrm{id}_M)$ is trivial, and
- $\pi_0(\mathrm{Homeo}_{\underline{M}}(\varnothing, \varnothing), \mathrm{id}_M)$ is trivial.

Then there is a group isomorphism

$$\mathrm{F}\colon \mathrm{Mot}_{\underline{M}}(N, N) \xrightarrow{\sim} \mathrm{MCG}_{\underline{M}}(N, N).$$

Theorem (T., Faria Martins, Martin)
Let M be a manifold. If

- $\pi_1(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing),\mathrm{id}_M)$ is trivial, and
- $\pi_0(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing),\mathrm{id}_M)$ is trivial,

the functor

$$\mathrm{F}\colon \mathrm{Mot}_{\underline{M}} \to \mathrm{MCG}_{\underline{M}},$$

is an isomorphism of categories.

Proof

Suppose $\pi_1(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing),\mathrm{id}_M)$ and $\pi_0(\mathrm{Homeo}_{\underline{M}}(\varnothing,\varnothing),\mathrm{id}_M)$ are trivial. Already proved F is full. We check F is faithful. Let $[f\colon N \rightarrowtail N']_{\mathrm{m}}$ and $[f'\colon N \rightarrowtail N']_{\mathrm{m}}$ be in $\mathrm{Mot}_{\underline{M}}(N,N')$. If $\mathrm{F}([f\colon N \rightarrowtail N']_{\mathrm{m}}) = \mathrm{F}([f'\colon N \rightarrowtail N']_{\mathrm{m}})$, then

$$
\begin{aligned}
[\mathrm{id}_M\colon N \rightarrowtail N]_{\mathrm{i}} &= \mathrm{F}([f'\colon N \rightarrowtail N']_{\mathrm{m}})^{-1} \circ \mathrm{F}([f\colon N \rightarrowtail N']_{\mathrm{m}}) \\
&= \mathrm{F}([f'\colon N \rightarrowtail N']_{\mathrm{m}}^{-1} * [f\colon N \rightarrowtail N']_{\mathrm{m}}) \\
&= \mathrm{F}([\bar{f}' * f\colon N \rightarrowtail N]_{\mathrm{m}}).
\end{aligned}
$$

By group isomorphism this is true if and only if

$$[\bar{f}' * f\colon N \rightarrowtail N]_{\mathrm{m}} = [\mathrm{Id}_M\colon N \rightarrowtail N]_{\mathrm{m}}$$

which is equivalent to saying $\mathrm{Id}_M * (\bar{f}' * f)$ is path-equivalent to a stationary motion, and hence that $\bar{f}' * f$ is path-equivalent to the stationary motion (since $Id_M * (\bar{f}' * f) \overset{p}{\sim} \bar{f}' * f$). So we have $[f\colon N \rightarrowtail N']_{\mathrm{m}} = [f'\colon N \rightarrowtail N']_{\mathrm{m}}$.

Proposition
Let D^n be the n-disk, and $\underline{D^n} = (D^n, \partial D^n)$. Then we have an isomorphism

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Idea of proof

We proved that $\mathrm{MCG}_{\underline{D^2}}(\varnothing, \varnothing) = \pi_0(\mathrm{Homeo}_{\underline{D^2}}(\varnothing, \varnothing), \mathrm{id}_M)$ is trivial. Alexander trick gives same result for all n. Also $\mathrm{Homeo}_{\underline{D^n}}(\varnothing, \varnothing)$ is contractible (Hamstrom).

Examples: $M = D^2$

Suppose we don't fix the boundary.

Examples: $M = D^2$

Suppose we don't fix the boundary. Let $P_2 \subset D^2$ be a subset consisting of two points equidistant from the centre of the disk. Let τ_π be the path in $\mathsf{TOP}^h(D^2, D^2)$ such that $\tau_{\pi t}$ is a πt rotation of the disk.

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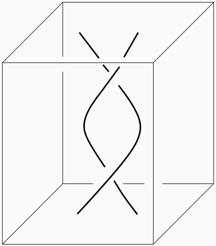
The motion $\tau_\pi \colon P_2 \rightarrowtail P_2$ represents a non-trivial equivalence class in Mot_{D^2}, and its end point also represents a non trivial element of MCG_{D^2}. Now consider the motion $\tau_\pi * \tau_\pi \colon P_2 \rightarrowtail P_2$.



Examples: $M = D^2$

In fact, the map $\mathrm{F}\colon \mathrm{Mot}_{D^2} \to \mathrm{MCG}_{D^2}$ is neither full nor faithful. The space Homeo_{D^2} is homotopy equivalent to $S^1 \sqcup S^1$, where the first connected component corresponds to orientation preserving homeomorphisms and the second orientation reversing (Hamstrom). Hence we have that $\pi_1(\mathrm{Homeo}_{D^2}(\varnothing,\varnothing),\mathrm{id}_{D^2}) = \mathbb{Z}$ where the single generating element corresponds to the 2π rotation. And $\pi_0(\mathrm{Homeo}_{D^2}(\varnothing,\varnothing),\mathrm{id}_{D^2}) = \mathbb{Z}/2\mathbb{Z}$. So we have an exact sequence:

$$\ldots \to \pi_1(\mathrm{Homeo}_{D^2}(N,N),\mathrm{id}_{D^2}) \xrightarrow{i^1_*} \mathbb{Z} \to \mathrm{Mot}_{D^2}(N,N) \to \mathrm{MCG}_{D^2}(N,N) \to \mathbb{Z}/2\mathbb{Z}.$$

Let $P \subset S^1$ be a subset containing a single point in S^1. Similarly to the disk, there is a non-trivial morphism in $\mathrm{Mot}_{\underline{S^1}}(P, P)$ represented by a 2π rotation of the circle.

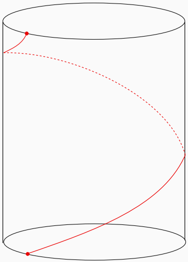


Figure 1: Example of motion of circle which is a 2π rotation carrying a point to itself.

Note that the connected component containing id_{S^1} of $\mathrm{Homeo}_{S^1}(P, P)$ is contractible, (Hamstrom). In particular $\pi_1(\mathrm{Homeo}_{S^1}(P, P), \mathrm{id}_{S^1})$ is trivial. We also have that $S^1 \sqcup S^1$ is a strong deformation retract of $\mathrm{Homeo}_{S^1}(\emptyset, \emptyset)$, with the first copy of S^1 corresponding to orientation preserving homeomorphisms and the second to orientation reversing. Hence the sequence becomes

$$\ldots \to \{1\} \to \mathbb{Z} \to \mathrm{Mot}_{S^1}(P, P) \to \mathrm{MCG}_{S^1}(P, P) \to \mathbb{Z}/2\mathbb{Z}.$$

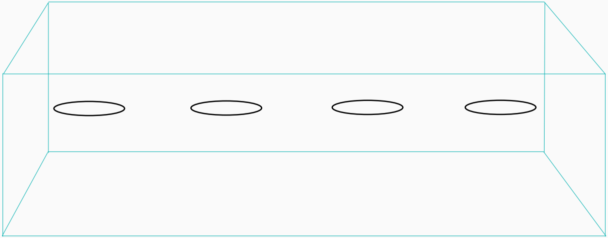
The exact sequence gives an injective map
$\mathbb{Z} \cong \pi_1(\mathrm{Homeo}_{\underline{S^1}}(\emptyset, \emptyset), \mathrm{id}_{S^1}) \to Mot_{S^1}(P, P)$, sending $n \in \mathbb{Z}$ to the equivalence class of the flow tracing a $2n\pi$ rotation of the circle S^1. The space $\mathrm{Homeo}_{\underline{S^1}}(P, P)$ only has two connected components, consisting of orientations preserving and orientation reversing homeomorphisms of S^1 fixing P. Hence the exact sequence becomes:

$$\ldots \to \{1\} \to \mathbb{Z} \xrightarrow{\cong} Mot_{S^1}(P, P) \xrightarrow{0} MCG_{S^1}(P, P) \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z}.$$

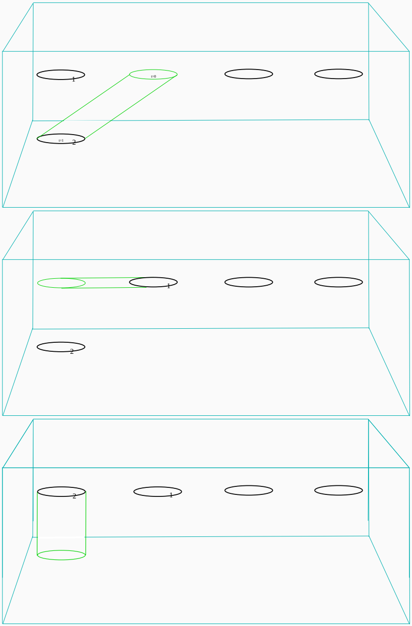
The loop braid category $\mathbb{L}$

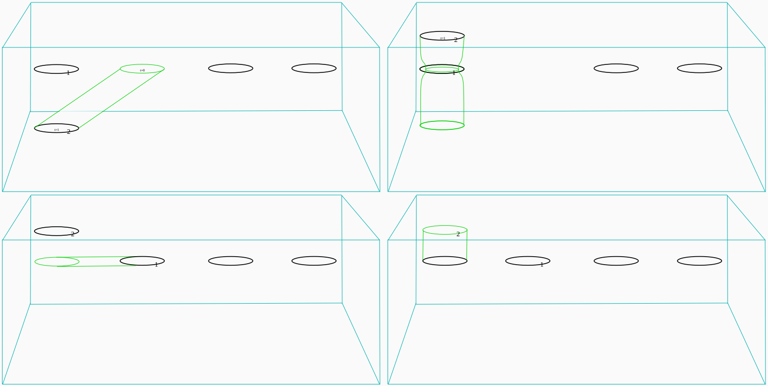
For each $n \in \mathbb{N}$, n evenly spaced circles in a plane in $[0,1]^3$.

For example for $n = 4$:



Morphisms in $\mathbb{L}$ - equivalence class of the swap motion ϱ_i




1
2
2
1
2
1
2
1

Composition in $\mathbb{L}$

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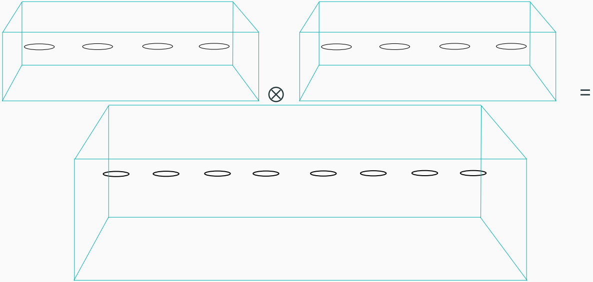
There is a function $\mathbb{I}^3 \sqcup \mathbb{I}^3$ to $\mathbb{I}^3$ that takes the corresponding $l_n \sqcup l_m$ to l_{n+m}:



This extends to morphisms to give monoidal composition.

The category L′ is the strict monoidal (diagonal, groupoid) category with object monoid the natural numbers, and two generating morphisms (and inverses) both in $L'(2,2)$, call them σ and s, obeying

$$s^2 = 1 \otimes 1$$

where (as a morphism) 1 denotes the unit morphism in rank one;

$$s_1 s_2 s_1 = s_2 s_1 s_2 \tag{3}$$

where $s_1 = s \otimes 1$ and $s_2 = 1 \otimes s$,

$$\text{(I) } \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \qquad \text{(II) } \sigma_1\sigma_2 s_1 = s_2\sigma_1\sigma_2, \qquad \text{(III) } \sigma_1 s_2 s_1 = s_2 s_1 \sigma_2. \tag{4}$$

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Proposition
The map on generators $s : 2 \to 2 \ \mapsto\ \varrho : 2 \to 2$ and $\sigma : 2 \to 2 \ \mapsto\ \varsigma : 2 \to 2$ is an isomorphism $L' \cong L$.

Definition
A monoidal loop braid representation is given by a monoidal functor

$$F \colon L \to \mathcal{C}$$

where $\mathcal{C}$ is a monoidal category.

Match^N CATEGORIES

Let Mat denote the category with objects $n \in \mathbb{N}$ and morphisms $f\colon i \to j$ are $j \times i$ matrices.

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Can relabel object N by 1, N^2 by 2 etc., so set of objects is $\mathbb{N}$, and we have $n \otimes m = n + m$. So Mat^N is a monoidal category with object monoid $(N^{\mathbb{N}}, x) \cong (\mathbb{N}, +)$.

Match^N CATEGORIES

Matrix in $\mathrm{Mat}^5(4,4)$ has rows and columns labelled by $|ijkl\rangle$ where $i, j, k, l \in \{1,2,3,4,5\}$.

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Definition

A matrix $M \in \mathrm{Mat}^N(n,n)$ is charge conserving if $M_{w,w'} = \langle w|M|w'\rangle \neq 0$ implies that w is a perm of w'. That is $w = \sigma w'$ for some $\sigma \in \Sigma_n$, where symmetric group Σ_n acts by place permutation.

Example in $\mathrm{Mat}^2(2,2)$

$$\begin{array}{cc} & \begin{array}{cccc} |11\rangle & |21\rangle & |12\rangle & |22\rangle \end{array} \\ \begin{array}{c} |11\rangle \\ |21\rangle \\ |12\rangle \\ |22\rangle \end{array} & \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \end{array}$$

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Charge conserving matrices form a monoidal subcategory of Mat^N - denote this Match^N.

Definition

A <u>charge conserving monoidal loop braid representation</u> is given by a strict monoidal functor

$$\mathsf{F}\colon \mathsf{L} \to \mathsf{Match}^{N}$$

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Since $\mathsf{L} \cong \mathsf{L}'$, such functors are given by giving the images of the generators of L':

$$F_* = (F(s), F(\sigma)) = (S, R)$$

such that $S, R \in \mathsf{Match}^N(2,2)$, and

$$S^2 = 1,$$

$$S_1 S_2 S_1 = S_2 S_1 S_2$$

where $S_1 = S \otimes 1$ and $S_2 = 1 \otimes S$ (where $\otimes$ is Kronecker product),

$$\text{(I) } R_1 R_2 R_1 = R_2 R_1 R_2, \qquad \text{(II) } R_1 R_2 S_1 = S_2 R_1 R_2, \qquad \text{(III) } R_1 S_2 S_1 = S_2 S_1 R_2.$$

Let $J_N^{\pm}$ be the set of signed multisets of compositions with at most two parts, of total rank N.

Example

$$J_2^{\pm} = \{(\ydiagram{1}^2,), (\ydiagram{2}^1,), (\ydiagram{1,1}^1,), (\ydiagram{1}^1, \ydiagram{1}^1), (, \ydiagram{1}^2), (, \ydiagram{2}^1), (, \ydiagram{1,1}^1)\}$$

Example

$$\lambda = \left(\ydiagram{1} \; \ydiagram{1} \; \ydiagram{1} \; \ydiagram{2,1} \; \ydiagram{1,2} \; \ydiagram{2,2} \; , \; \ydiagram{3} \; \ydiagram{2,1} \; \ydiagram{2,1} \; \ydiagram{1,3} \right)$$

is in $J_{26}^{\pm}$.

Theorem (Martin, Rowell, T.)
The set of all varieties of charge-conserving loop braid representations from the loop braid category L to the category Match^N of charge conserving matrices

$$F : \mathrm{L} \to \mathrm{Match}^N$$

may be indexed by $J_N^{\pm}$.

Motion groupoids & mapping class groupoids

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